Math 3321

Homogeneous Systems of Linear Differential Equations. Part I

University of Houston

Lecture 23

Outline

Systems of Linear Differential Equations

Let $a_{11}(t),\ a_{12}(t),\ \ldots,\ a_{nn}(t)$, and $b_1(t),\ b_2(t),\ \ldots,\ b_n(t)$ be continuous functions on the interval I.

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The system of n first-order linear differential equations

$$\begin{array}{rcl} x_1' & = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' & = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots & & \qquad\qquad\qquad\qquad \vdots \\ x_n' & = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{array}$$

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is called a **first-order linear differential system**.

The system is **homogeneous** if

$$b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0 \quad \text{on } I.$$

It is **nonhomogeneous** if the functions $b_i(t)$ are not all identically zero on I.

Systems of Linear Differential Equations

Set

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

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$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

The first-order linear differential system can be written in the **vector-matrix form**

$$x' = A(t)\,x + b(t). \tag{S}$$

Homogeneous Systems of Linear Differential Equations

A homogeneous systems of linear differential equations has the form

$$\begin{array}{rcl} x_1' & = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t) \\ x_2' & = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t) \\ \vdots & & \vdots \\ x_n' & = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t) \end{array}$$

or, in matrix form,

$$x' = A(t)x. \tag{H}$$

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or, in matrix form,

$$x' = A(t)x. \tag{H}$$

Note: The zero vector $z(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution of (H).

This solution is called the **trivial solution**.

Homogeneous Systems of Linear Differential Equations

The following results generalize properties of standard homogeneous linear differential equations

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Theorem

If x_1 and x_2 are solutions of (H), then $u = x_1 + x_2$ is also a solution of (H); the sum of any two solutions of (H) is a solution of (H).

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Theorem

If x is a solution of (H) and a is any real number, then $u = a\,x$ is also a solution of (H); any constant multiple of a solution of (H) is a solution of (H).

More generally, we have the following result.

Theorem

If $x_1, x_2, \ldots, x_k$ are solutions of (H), and if $C_1, C_2, \ldots, C_k$ are real numbers, then

$$C_1x_1 + C_2x_2 + \cdots + C_kx_k$$

is a solution of (H); any linear combination of solutions of (H) is also a solution of (H).

Homogeneous Systems of Linear Differential Equations

Recall that a set of vectors $w_1(t), w_2(t), \ldots, w_m(t)$ is linearly dependent on I if there exist m real numbers $c_1,\ c_2,\ \ldots,\ c_m,$ not all zero, such that

$$c_1w_1(t) + c_2w_2(t) + \cdots + c_mw_m(t) \equiv 0 \quad \text{on} \quad I.$$

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$$c_1 w_1(t) + c_2 w_2(t) + \cdots + c_m w_m(t) \equiv 0 \quad \text{on} \quad I.$$

Theorem

Let $v_1(t), v_2(t), \ldots, v_k(t)$ be k, k-component vector functions defined on an interval I. If the vectors are **linearly dependent**, then the determinant

$$W(t) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix} \equiv 0 \quad \text{on} \quad I.$$

That is, the determinant is 0 **for all** $t \in I$.

Equivalently, we have the following result.

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Theorem

Let $v_1(t),\ v_2(t),\ \ldots,\ v_k(t)$ be k, k-component vector functions defined on an interval I. The vectors are **linearly independent** if the determinant

$$W(t) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix} \neq 0$$

for at least one $t \in I$.

Definition

The determinant

$$W(t) = \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1k} \\ v_{21} & v_{22} & \cdots & v_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kk} \end{vmatrix}$$

is called the **Wronskian** of the vector functions $v_1, v_2, \ldots, v_k$.

Homogeneous Systems of Linear Differential Equations

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Proposition

Let $x_1, x_2, \ldots, x_n$ be n solutions system of n equations (H). Exactly one of the following holds:

1. $W(x_1, x_2, \ \ldots, x_n)(t) \equiv 0$ on I and the solutions are linearly dependent.
2. $W(x_1, x_2, \ \ldots, x_n)(t) \neq 0$ for all $t \in I$ and the solutions are linearly independent.

The last Proposition implies the following theorem

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Theorem

Let $x_1, x_2, \ldots, x_n$ be n linearly independent solutions of (H). Let u be *any* solution of (H). Then there exists a unique set of constants $C_1, C_2, \ldots, C_n$ such that

$$u = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n.$$

That is, every solution of (H) can be written as a unique linear combination of $x_1, x_2, \ldots, x_n$.

A set of n linearly independent solutions of the Homogeneous Systems of Linear Differential Equations (H)

$$x_1, x_2, \ldots, x_n$$

is called a **fundamental set of solutions**.

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In this case,

$$x = C_1x_1 + C_2x_2 + \cdots + C_nx_n,$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants, is the **general solution** of (H).

Homogeneous Systems of Linear Differential Equations

Example: $x_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $x_2 = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$ are solutions of the homogeneous systems of linear differential equations

$$x' = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} x \quad \text{(Verify)}$$

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Since $W(x_1 . x_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & e^{3t} \end{vmatrix} = -e^{5t} \neq 0$, then $\left\{ \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} \right\}$ is a fundamental set of solutions and

$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

is the **general solution of the system**.

Homogeneous Systems with Constant Coefficients

Here we consider **homogeneous systems of linear differential equations with constant coefficients**

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
- \quad & - \qquad - \qquad - \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}
$$

where a_{11}, a_{12}, $\ldots, a_{nn}$ are constants.

Homogeneous Systems with Constant Coefficients

The same system is written in vector-matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or as

$$x' = Ax.$$

Theorem

Consider the homogeneous system with constant coefficients

$$x' = Ax$$

If λ is an eigenvalue of A and v is a corresponding eigenvector, then

$$x = e^{\lambda t} v$$

is a solution.

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Proof:

Let λ be an eigenvalue of A with corresponding eigenvector v.
Set $x = e^{\lambda t} v$

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If $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k$ are distinct eigenvalues of A with corresponding eigenvectors $v_1,\ v_2,\ \cdots,\ v_k,$ then

$$x_1 = e^{\lambda_1 t} v_1,\ x_2 = e^{\lambda_2 t} v_2,\ \cdots, x_k = e^{\lambda_k t} v_k$$

are linearly independent solutions of the system.

Homogeneous Systems with Constant Coefficients

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$$x_1 = e^{\lambda_1 t} v_1, \ x_2 = e^{\lambda_2 t} v_2, \ \cdots , x_n = e^{\lambda_n t} v_n$$

is a fundamental set of solutions of the system and

$$x(t) = C_1 x_1 + C_2 x_2 + \ \cdots \ + C_n x_n$$

is the general solution.

Example 1: Find the general solution of

$$x' = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} x.$$

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Step 1. Find the eigenvalues of A:

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$$= \lambda^2 - \lambda - 6.$$

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Eigenvalues: $\lambda_1 = 3,\ \lambda_2 = -2.$

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For $\lambda_1 = 3$, solve $(A - \lambda_1 I)u_1 = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} u_1 = 0$

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For $\lambda_2 = -2$ solve $(A - \lambda_2 I)u_2 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} u_2 = 0$

We find the family of eigenvectors $u_2 = \beta \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Homogeneous Systems with Constant Coefficients

Hence we find the fundamental set of solution vectors:

$$\left\{ x_1 = e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad x_2 = e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

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The general solution of the system is:

$$x = C_1 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

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$$= -\lambda^3 + 4\lambda^2 - \lambda - 6.$$

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Characteristic equation:

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda - 3)(\lambda - 2)(\lambda + 1) = 0.$$

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Eigenvalues:

$$\lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = -1.$$

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For $\lambda_1 = 3$, we solve $(A - \lambda_1 I)v_1 = \begin{pmatrix} 0 & -1 & -1 \\ -2 & 0 & 2 \\ 4 & -1 & -5 \end{pmatrix} v_1 = 0$

This gives the family of eigenvectors $v_1 = \alpha \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Similarly,

For $\lambda_2 = 2$, we solve $(A - \lambda_2 I)v_2 = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & 2 \\ 4 & -1 & -4 \end{pmatrix} v_2 = 0$

This gives the family of eigenvectors $v_2 = \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Similarly,

For $\lambda_2 = 2$, we solve $(A - \lambda_2 I)v_2 = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & 2 \\ 4 & -1 & -4 \end{pmatrix} v_2 = 0$

This gives the family of eigenvectors $v_2 = \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

For $\lambda_3 = -1$, we solve $(A - \lambda_3 I)v_3 = \begin{pmatrix} 4 & -1 & -1 \\ -2 & 4 & 2 \\ 4 & -1 & -1 \end{pmatrix} v_3 = 0$

This gives the family of eigenvectors $v_3 = \gamma \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$.

Hence we find the fundamental set of solution vectors:

$$x_1 = e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad x_2 = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$x_3 = e^{-t}\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

Homogeneous Systems with Constant Coefficients

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The general solution of the system is:

$$x = C_1 e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

Example 3: Find the solution of the initial-value problem

$$x' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

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We found above that the general solution of the system is:

$$x = C_1 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

To find the solution satisfying the initial condition, we set $t = 0$ and solve

$$C_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

Homogeneous Systems with Constant Coefficients

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Hence we need to solve an algebraic linear system

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$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 1 & 1 & 7 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

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We write the augmented matrix:

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -1 & 0 & -3 & -3 \\ 1 & 1 & 7 & 1 \end{array}\right)$$

Homogeneous Systems with Constant Coefficients

By Gaussian elimination we get

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -1 & 0 & -3 & -3 \\ 1 & 1 & 7 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 6 & 0 \end{array}\right)$$

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This gives the solution
$C_3 = 0, C_2 = -2, C_1 = 3.$

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This gives the solution
$C_3 = 0, C_2 = -2, C_1 = 3.$

Hence the IVP solution is

$$x = 3e^{3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 2e^{2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$