Math 3321

Homogeneous Systems of Linear Differential Equations. Part II

University of Houston

Lecture 24

Outline

Homogeneous Systems of Linear Differential Equations

Recall that a homogeneous systems of linear differential equations has the form

$$\begin{array}{rcl} x_1' & = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t) \\ x_2' & = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t) \\ \vdots & & \vdots \\ x_n' & = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t) \end{array}$$

or, in matrix form,

$$x' = A(t)x. \tag{H}$$

where

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

We stated the following results

Theorem

Consider the homogeneous system with constant coefficients

$$x' = Ax$$

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If $\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_k$ are **distinct** eigenvalues of A with corresponding eigenvectors $v_1, \ v_2, \ \cdots, \ v_k,$ then

$$x_1 = e^{\lambda_1 t} v_1, \ x_2 = e^{\lambda_2 t} v_2, \ \cdots, x_k = e^{\lambda_k t} v_k$$

are **linearly independent solutions** of the system.

Homogeneous Systems with Constant Coefficients

Corollary

Consider the homogeneous system of n equations with constant coefficients

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If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n **distinct eigenvalues** of A with corresponding eigenvectors $v_1, v_2, \cdots, v_n,$ then

$$x_1 = e^{\lambda_1 t} v_1,\ x_2 = e^{\lambda_2 t} v_2,\ \cdots, x_n = e^{\lambda_n t} v_n$$

is a fundamental set of solutions of the system and

$$x(t) = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n$$

is the general solution.

Homogeneous Systems with Constant Coefficients

Corollary

Consider the homogeneous system of n equations with constant coefficients

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If $\lambda_1, \ \lambda_2, \ \cdots, \ \lambda_n$ are n **distinct eigenvalues** of A with corresponding eigenvectors $v_1, \ v_2, \ \cdots, \ v_n,$ then

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$$x(t) = C_1 x_1 + C_2 x_2 + \ \cdots \ + C_n x_n$$

is the general solution.

This holds for both real and complex distinct eigenvalues.

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Homogeneous Systems with Constant Coefficients

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In this lecture, I illustrated the case of n **real or complex distinct eigenvalues**.

In the next lecture, I will illustrated the case where there are **no** n **distinct eigenvalues.**

Example 1: Find the general solution of

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$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5.$$

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$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5.$$

Solving the characteristic equation, we find the eigenvalues:

$$\lambda_1 = -1 + 2i, \ \lambda_2 = -1 - 2i.$$

Homogeneous Systems - Complex Eigenvalues

To compute the eigenvectors, we examine the matrix

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For $\lambda_1 = -1 + 2i$: Solve

$$\left(\begin{array}{cc|c} -2-2i & -2 & 0 \\ 4 & 2-2i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

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The solution set is:

$$x_2 = -(1+i)x_1, \quad x_1 \quad \text{arbitrary}$$

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Set $x_1 = 1$. Then, for $\lambda_1 = -1+2i$:

$$v_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Homogeneous Systems - Complex Eigenvalues

For $\lambda_2 = -1 - 2i$: Solve

$$\left(\begin{array}{cc|c} -2+2i & -2 & 0 \\ 4 & 2+2i & 0 \end{array} \right) \to \left(\begin{array}{cc|c} -1+i & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

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The solution set is:

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Set $x_1 = 1$. Then, for $\lambda_1 = -1 - 2i$:

$$v_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Note that $v_2 = v_1^*$

Homogeneous Systems - Complex Eigenvalues

Hence we find the **solutions:**

$$u_1(t) = e^{(-1+2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

$$u_2(t) = e^{(-1-2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]$$

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Problem: Complex-valued!

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Problem: Complex-valued!

We want real-valued solutions. We will show that we can apply some algebra to derive a real-valued solution set.

Using Euler's formula

$$
\begin{aligned}
u_1 = e^{\lambda_1 t} v_1 =& \\
&= e^{(-1+2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&= e^{-t}(\cos\, 2t + i\, \sin\, 2t)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&= e^{-t}\left[\cos\, 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin\, 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&+ i\, e^{-t}\left[\cos\, 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin\, 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].
\end{aligned}
$$

Homogeneous Systems - Complex Eigenvalues

Similarly,

$$
\begin{aligned}
u_2 &= e^{\lambda_2 t} v_2 \\
&= e^{(-1-2i)t}\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&= e^{-t}(\cos\, 2t - i\,\sin\, 2t)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&= e^{-t}\left[\cos\, 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin\, 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\
&\quad - i\, e^{-t}\left[\cos\, 2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin\, 2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].
\end{aligned}
$$

Homogeneous Systems - Complex Eigenvalues

Fundamental set:

$$\begin{aligned} x_1 &= \tfrac{u_1+u_2}{2} \\ &= e^{-t}\left[\cos\,2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin\,2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \end{aligned}$$

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General solution:

$$x = C_1\,e^{-t}\left[\cos\,2t\begin{pmatrix} 1 \\ -1 \end{pmatrix} - \sin\,2t\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] \\ \quad +C_2\,e^{-t}\left[\cos\,2t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin\,2t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

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Eigenvalues:

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The solution set is:

$$x_2 = \tfrac{1}{5}(-1-3i)x_1, \quad x_1 \quad \text{arbitrary}$$

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We have the eigenvectors:

$$\lambda_1 = 2 + 3i, \quad v_1 = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\lambda_2 = 2 - 3i, \quad v_2 = \begin{pmatrix} 5 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Homogeneous Systems - Complex Eigenvalues

As we saw in the example above, we can write a real solution where the real part of the eigenvalue determines the exponent of the exponential and the imaginary part gives the periodicity of the trig function:

$$\lambda = a \pm ib \Rightarrow e^{at}, \cos(bt), \sin(bt)$$

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Hence we have the **Fundamental set:**

$$x_1 = e^{2t}\left[\cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right]$$

$$x_2 = e^{2t}\left[\cos 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right]$$

Homogeneous Systems - Complex Eigenvalues

The **General Solution** of the homogeneous system of linear differential equations is:

$$\begin{aligned} x &= C_1\, e^{2t} \left[\cos\, 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin\, 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right] \\ &+ C_2\, e^{2t} \left[\cos\, 3t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \sin\, 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right] \end{aligned}$$

Homogeneous Systems - Complex Eigenvalues

Summary: Method to find the solution of $x' = Ax$, where A $n \times n$ is a constant matrix, when $a + ib, \quad a - ib$ are complex eigenvalues.

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1. Compute the eigenvectors $\vec{\alpha} + i\,\vec{\beta}, \quad \vec{\alpha} - i\,\vec{\beta}$.
2. Write the independent (complex-valued) solutions:

$$u_1 = e^{(a+ib)t}\left(\vec{\alpha} + i\,\vec{\beta}\right) \quad u_2 = e^{(a-ib)t}\left(\vec{\alpha} - i\,\vec{\beta}\right)$$

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3. Write corresponding real-valued solutions:

$$x_1 = e^{at}\left[\cos\, bt\,\vec{\alpha} - \sin\, bt\,\vec{\beta}\right] \quad x_2 = e^{at}\left[\cos\, bt\,\vec{\beta} + \sin\, bt\,\vec{\alpha}\right]$$

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4. Write the general solution:

$$x = C_1\,e^{at}\left[\cos\,bt\,\vec{\alpha} - \sin\,bt\,\vec{\beta}\right] + C_2 e^{at}\left[\cos\,bt\,\vec{\beta} + \sin\,bt\,\vec{\alpha}\right]$$

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$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 & -1 \\ 3 & 2-\lambda & 3 \\ 1 & 1 & 3-\lambda \end{vmatrix} =$$

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The eigenvalues are:

$$\lambda_1 = 2,\ \lambda_2 = 2 + 3i,\ \lambda_3 = 2 - 3i.$$

Homogeneous Systems - Complex Eigenvalues

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$$\left(\begin{array}{ccc|c} -1 & -4 & -1 & 0 \\ 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & -4 & -1 & 0 \\ 0 & -12 & 0 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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Hence we have the solution $x_2 = 0$, $x_1 = -x_2$.
Choosing $x_2 = 1$, we obtain the eigenvector:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The eigenvalue $\lambda_1 = 2$ and the eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are associated with solution vector:

$$x_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

Homogeneous Systems - Complex Eigenvalues

For $\lambda_2 = 2 + 3i$ we solve

$$\left(\begin{array}{ccc|c} -1-3i & -4 & -1 & 0 \\ 3 & -3i & 3 & 0 \\ 1 & 1 & 1-3i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 1 & 1 & 1-3i & 0 \\ -1-3i & -4 & -1 & 0 \end{array}\right) \to$$

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$$\left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 0 & 1+i & -3i & 0 \\ 0 & -1-i & 3i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 0 & 1+i & -3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 0 & 1 & -\frac{3}{2}(1+i) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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For $\lambda_2 = 2 + 3i$ we solve

$$\left(\begin{array}{ccc|c} -1-3i & -4 & -1 & 0 \\ 3 & -3i & 3 & 0 \\ 1 & 1 & 1-3i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 1 & 1 & 1-3i & 0 \\ -1-3i & -4 & -1 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 0 & 1+i & -3i & 0 \\ 0 & -1-i & 3i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 0 & 1+i & -3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{ccc|c} 1 & -i & 1 & 0 \\ 0 & 1 & -\frac{3}{2}(1+i) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, the solution set is:

$$u_1 = \left(-\frac{5}{2} + \frac{3}{2}\, i\right) u_3,\ \ u_2 = \left(\frac{3}{2} + \frac{3}{2}\, i\right) u_3, u_3 \text{ arbitrary}$$

Homogeneous Systems - Complex Eigenvalues

Hence the eigenvectors v_2 and v_3 corresponding to the eigenvalues λ_2 and λ_3 are

$$v_2 = \begin{pmatrix} -5+3i \\ 3+3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$

and

$$v_3 = \begin{pmatrix} -5-3i \\ 3-3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$

Homogeneous Systems - Complex Eigenvalues

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and

$$v_3 = \begin{pmatrix} -5-3i \\ 3-3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$

Therefore the solutions associated with the eigenvectors v_2 and v_3 are

$$z_1 = e^{(2+3i)t} \left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \right]$$

and

$$z_2 = e^{(2-3i)t} \left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \right]$$

Homogeneous Systems - Complex Eigenvalues

As we saw above, we can convert the complex solutions into the real solutions :

$$x_2 = e^{2t}\left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}\right]$$

and

$$x_3 = e^{2t}\left[\cos 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}\right]$$

Homogeneous Systems - Complex Eigenvalues

Combining all solution vectors, we have the fundamental set of solution vectors:

$$x_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$x_2 = e^{2t} \left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \right],$$

$$x_3 = e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \right].$$

Homogeneous Systems - Complex Eigenvalues

Combining all solution vectors, we have the fundamental set of solution vectors:

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$$x_3 = e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \right].$$

General solution:

$$x = C_1\, x_1 + C_2\, x_2 + C_3\, x_3$$