Affine, Quasi-Affine and Co-Affine Wavelets

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Abstract. "Classical" wavelets are obtained by the action of a particular countable subset of operators associated with the affine group on a function $\psi \in L^2(\mathbb{R})$. More precisely, this set is the collection $\{D_{2^j} T_k : j,k \in \mathbb{Z}\}$, where T_k is the translation by the integer k and D_{2^j} is the (unitary) dilation by 2^j. We thus obtain the discrete wavelet system. Ron and Shen [4] have shown that by interchanging and renormalizing "half" of the operators in this set one obtains an important collection of systems that can be considered "equivalent" to this affine system. In this paper we show that, in a precise sense, the choice of Ron and Shen is optimal.

1. Introduction

We begin with some observations about the "classical" *discrete wavelets*. These are those functions $\psi \in L^2(\mathbb{R}^n)$ for which the system $\{\psi_{j,k}(x)\} = \{2^{-j/2}\,\psi(2^{-j}x - k)\}$, $j,k \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{R})$.

In order to understand these and related systems, it is useful to consider the *affine group* associated with $\mathbb{R}$, which is the group generated by the dilations $x \to ax$, $a > 0$, and the translations $x \to x+b$, $b \in \mathbb{R}$. We can also consider this group as the collection of operators (say, on $L^2(\mathbb{R})$) generated by the dilation operators D_a, $a > 0$, and the translation operators T_b, $b \in \mathbb{R}$, where $(D_a\, f)(x) = a^{1/2}\, f(ax)$ and $(T_b\, f)(x) = f(x+b)$. All these operators are unitary. The discrete wavelets introduced above are obtained by the action on the functions ψ of a very special subset of this affine group, the elements of the form $D_{2^{-j}}\, T_{-k}$, $j,k \in \mathbb{Z}$. That is, $\psi_{j,k} = D_{2^{-j}}\, T_{-k}\, \psi$; observe that the translations by $k \in \mathbb{Z}$ are applied first to ψ and, then, the dilations $D_{2^{-j}}$ are applied to $T_{-k}\,\psi$. This set of operators $\{D_{2^{-j}}\, T_{-k} : j,k \in \mathbb{Z}\}$ is not a subgroup of the affine group.

If $\|\psi\|_2 \geq 1$, the fact that $\{\psi_{j,k}\}$, $j,k \in \mathbb{Z}$, is an orthonormal basis is equivalent to the reproducing formula, valid for all $f \in L^2(\mathbb{R})$,

$$f \;=\; \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle\, \psi_{j,k}, \tag{1.1}$$

with convergence in $L^2(\mathbb{R})$. Moreover, (1.1) is equivalent to

$$\|f\|_2^2 \;=\; \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2, \tag{1.2}$$

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for all $f \in L^2(\mathbb{R})$. If either (1.1) or (1.2) hold and $\|\psi\|_2 \geq 1$, then, clearly, we must have $\|\psi\|_2 = 1$; if $0 < \|\psi\|_2 < 1$, then either of these two equalities assert that the system $\{\psi_{j,k}\}$ is a *normalized tight frame* (a tight frame with constant 1). See Chapter 7 of [2] for proofs of these claims and related questions.

It is natural to ask if other subsets of the affine group can be used to obtain similar normalized tight frames (or, more generally, frames) for $L^2(\mathbb{R})$. A significant discovery in this direction was made by A.Ron and Z.Shen [4]. They showed that, if the system $\{D_{2^{-j}}\, T_{-k}\, \psi : j,k \in \mathbb{Z}\}$ is changed so that, for $j > 0$, $D_{2^{-j}}\, T_{-k}\, \psi$ is replaced by $2^{-j/2}\, T_{-k}\, D_{2^{-j}}$ for all $k \in \mathbb{Z}$, we do obtain a normalized tight frame whenever the original system has this property (and vice versa). In fact, this equivalence is true more broadly in the sense that one of these systems is a frame for $L^2(\mathbb{R})$ if and only if the same is true for the other. One of the goals of this paper is to gain a better understanding of this and related matters.

Most of what we consider applies to more general situations: higher dimensions, "multi-systems" that are obtained by applying these translations and dilations to a finite family $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and "dual" systems in which one family, $\Psi = \{\psi^1, \dots, \psi^L\}$, is used to "analyze" a function and another, $\Phi = \{\phi^1, \dots, \phi^L\}$, to "synthesize", or reproduce, the given function from the data obtained from the "analysis". That is, (1.1) is extended to equalities of the form

$$f \;=\; \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \langle f, \psi^\ell_{j,k} \rangle \, \phi^\ell_{j,k}. \tag{1.3}$$

In order to focus on the properties of appropriate subsets of the affine group and the systems they generate in the manner described above, however, we will restrict our attention to the 1−dimensional case and to systems obtained from a single function. We break this resolve, however, in the following few paragraphs devoted to the description of continuous wavelets. It is our opinion that the more general setting will present a better perspective of the questions we are addressing.

Let D be a closed subgroup of $GL(x,\mathbb{R})$, the general linear group acting on $\mathbb{R}^n$. Let us form the semi-direct product

$$G = D \times \mathbb{R}^n \equiv \{g = (a,b) : a \in D, b \in \mathbb{R}^n\}$$

and endow this set with the product

$$g_1 \circ g_2 = (a_1, b_1) \circ (a_2, b_2) = (a_1 a_2, a_2^{-1} b_1 + b_2). \tag{1.4}$$

This operation corresponds to the action $(a,b)(x) = a(x+b)$ on the points of $\mathbb{R}^n$. A simple calculation shows that $d\lambda(a,b) = d\mu(a)\, db$ is a left Haar measure on D. Moreover, we have $(a,b)^{-1} = (a^{-1}, -ab)$, so that the action, T, of G on functions $\psi \in L^2(\mathbb{R}^n)$ defined by

$$(T_g\, \psi)(x) = |\det a|^{-1/2}\, \psi(g^{-1}(x)) = |\det a|^{-1/2}\, \psi(a^{-1}x - b),$$

$g = (a,b) \in G$, produces a "continuous" system that is a natural analog to the discrete system $\{\psi_{j,k}\}$ we introduced in the first paragraph of this section. We write

$$\psi_{a,b}(x) = |\det a|^{-1/2}\, \psi(a^{-1}x - b), \tag{1.5}$$

for $(a,b) \in G$, in order to complete this analogy.

It is natural to find a condition on ψ that guarantees the reproducing property

$$\|f\|_2^2 \;=\; \int_G |\langle f, \psi_{a,b}\rangle|^2 \, d\lambda(a,b) \;=\; \int_D \Big(\int_{\mathbb{R}^n} |\langle f, \psi_{a,b}\rangle|^2 \, db \Big) d\mu(a) \tag{1.6}$$

for all $f \in L^2(\mathbb{R}^n)$, which is clearly an analog of (1.2). The following extension of the "Calderón condition" provides us with a characterization of those ψ for which (1.6) is true.

Theorem 1. *Equality (1.6) is valid for all $f \in L^2(\mathbb{R}^n)$ if and only if for a.e. $\xi \in \mathbb{R}$*

$$\Delta_\psi(\xi) = \int_D |\hat\psi(\xi a)|^2 \, d\mu(a) = 1. \tag{1.7}$$

In [6] one can find a rather complete discussion of equality (1.7) and its relation to the original Calderón condition; in particular, a proof of this theorem is presented in the cited article (see Theorem (2.1)). We shall refer to the functions ψ satisfying (1.7) (or, equivalently, (1.6)) as the *continuous wavelet on $\mathbb{R}^n$ associated with the dilation group D.* Let us examine, in view of the result of Ron and Shen that involved the interchange of the order in which translations and dilations are applied to ψ, what happens in the case of continuous wavelets when this order is interchanged.

Perhaps a good way of seeing the effect of such interchange is to endow the set $\{(a,b) : a \in D, b \in \mathbb{R}^n\}$ with the product

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 a_2, b_1 + a_1 b_2) \tag{1.8}$$

that corresponds to the action $(a,b)(x) = ax + b$ on the points of $\mathbb{R}^n$. Let us denote the group having this operation by G^*. We distinguish G and G^* by calling G the *affine group* and G^* the *co-affine group* (associated with the dilation group D). The system $\{\psi_{a,b}\}$ will be referred to as the *affine system* and

$$\psi^*_{a,b}(x) = |\det a|^{-1/2}\, \psi(a^{-1}(x - b)) \tag{1.9}$$

is defined to be the corresponding (continuous) *co-affine system.* [1] We say that ψ is a (continuous) *affine wavelet* if (1.6) is true for all $f \in L^2(\mathbb{R}^n)$; ψ is a

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[1] With respect to the operation (1.8), $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$. Thus $\psi^*_{a,b}(x) = |\det a|^{-1/2}\, \psi((a,b)^{-1}(x))$.

(continuous) *co-affine wavelet* provided the reproducing formula

$$\|f\|_2^2 = \int_{G^*} |\langle f, \psi^*_{a,b}\rangle|^2 \, d\lambda^*(a,b) = \int_D \Big(\int_{\mathbb{R}^n} |\langle f, \psi^*_{a,b}\rangle|^2 \, db \Big) \frac{d\mu(a)}{|\det a|} \tag{1.10}$$

is valid for all $f \in L^2(\mathbb{R}^n)$, where λ^* is the left Haar measure for G^*. A simple calculation shows that $d\lambda^*(a,b) = |\det a|^{-1} \, d\mu(a) \, db$.

Theorem 2. *ψ is a continuous affine wavelet if and only if it is a continuous co-affine wavelet. Moreover, either of these two properties is equivalent to (1.7).*

Let us examine the proof of this result, so we can compare it with the situation in the discrete case. In doing this we also provide a proof of Theorem 1. We have

$$\hat{\psi}_{a,b}(\xi) = |\det a|^{1/2} \, \hat{\psi}(\xi a) \, e^{-2\pi i \xi a \cdot b}, \quad \hat{\psi}^*_{a,b}(\xi) = |\det a|^{1/2} \, \hat{\psi}(\xi a) \, e^{-2\pi i \xi \cdot b}, \tag{1.11}$$

where ξa is the product of the row vector ξ with the matrix a, and the Fourier transform we are using has the form

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) \, e^{-2\pi i \xi \cdot x} \, dx.$$

Using the equalities (1.11) and the Plancherel theorem, we have

$$\begin{aligned}
\int_G |\langle f, \psi_{a,b}\rangle|^2 \, d\lambda(a,b) &= \int_D \int_{\mathbb{R}^n} |\det a| \left| \int_{\mathbb{R}^n} \hat{f}(\xi) \, \overline{\hat{\psi}(\xi a)} \, e^{2\pi i \xi a \cdot b} \, d\xi \right|^2 db \, d\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \hat{f}(\xi) \, \overline{\hat{\psi}(\xi a)} \, e^{2\pi i \xi \cdot b} \, d\xi \right|^2 db \, d\mu(a) \\
&= \int_D \int_{\mathbb{R}^n} |\det a| \left| \int_{\mathbb{R}^n} \hat{f}(\xi) \, \overline{\hat{\psi}(\xi a)} \, e^{2\pi i \xi \cdot b} \, d\xi \right|^2 \frac{d\mu(a)}{|\det a|} \, db \\
&= \int_{G^*} |\langle f, \psi^*_{a,b}\rangle|^2 \, d\lambda^*(a,b).
\end{aligned}$$

This shows that each expression $\int_G |\langle f, \psi_{a,b}\rangle|^2 \, d\lambda(a,b)$ and $\int_{G^*} |\langle f, \psi^*_{a,b}\rangle|^2 \, d\lambda^*(a,b)$ is equal to

$$\begin{aligned}
\int_D \int_{\mathbb{R}^n} \left| (\hat{f} \, \overline{\hat{\psi}(\cdot a)})^\vee(b) \right|^2 db \, d\mu(a) &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \int_D |\hat{\psi}(\xi a)|^2 \, d\mu(a) \, d\xi \\
&= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, \Delta_\psi(\xi) \, d\xi.
\end{aligned}$$

But this last expression equals $\|f\|_2^2$ for all $f \in L^2(\mathbb{R}^n)$ (see (1.6) and (1.10)) if and only if $\Delta_\psi(\xi) = 1$ a.e. (see (1.7). The detailed (easy) proof of this is given in [6](Theorem 2.1).

This establishes Theorem 2. Consequently, in the continuous case, the affine wavelets are the same as the co-affine wavelets. This situation is completely different in the discrete case. This is undoubtedly the case since the countinuous wavelets involve the systems $\{\psi_{a,b}\}$ and $\{\psi^*_{a,b}\}$, $(a,b) \in D \times \mathbb{R}^n$, which, as sets, are equal. In the discrete case, however, the sets

$$\{\psi_{j,k} = 2^{-j/2}\, D_{2^{-j}}\, T_{-k}\, \psi : j,k \in \mathbb{Z}\}, \quad \{\psi_{j,k}{}^* = 2^{-j/2}\, T_{-k}\, D_{2^{-j}}\, \psi : j,k \in \mathbb{Z}\} \tag{1.12}$$

are not equal. Let us call the first set the *discrete affine system* generated by ψ and the second set the *discrete co-affine system* generated by ψ. Let $X(\psi)$ denote the discrete affine system and $X^*(\psi)$ the corresponding co-affine system. The system

$$\widetilde{X}(\psi) = \{\tilde{\psi}_{j,k} = \psi_{j,k} \text{ if } j < 0, k \in \mathbb{Z};\ \tilde{\psi}_{j,k} = 2^{-j/2}\, \psi_{j,k}{}^* \text{ if } j \geq 0, k \in \mathbb{Z}\}, \tag{1.13}$$

studied by Ron and Shen, is a sort of hybrid of these two. Ron and Shen called this system the *quasi-affine system* generated by ψ. In this paper we examine the properties of these three systems, as well as other related systems. We end this first section with a trivial observation about the difference between the affine and co-affine systems in the discrete case.

Suppose ψ is an orthonormal (discrete) wavelet on $\mathbb{R}$. Then a simple calculation shows that

$$\langle \psi^*_{j,k}, \psi_{-1,-1} \rangle \;=\; \langle \psi_{j,0}, \psi_{-1,-(2k+1)} \rangle \;=\; 0$$

for all $j,k \in \mathbb{Z}$. This shows that if $X(\psi)$ is an orthonormal basis for $L^2(\mathbb{R})$, then $X^*(\psi)$ cannot generate a complete system. More precisely, the closure of the algebraic span of $X^*(\psi)$ has a non-empty orthogonal complement (containing $\psi_{-1,-1}$ as well as many other elements of $X(\psi)$). In the next section we will show other properties of $X^*(\psi)$ that show that, unlike $X(\psi)$, it cannot be easily modified in order to obtain even a frame or a Bessel system. This will give us further insight into the Ron and Shen system $\widetilde{X}(\psi)$.

2. Frames and the three systems $X(\psi), X^*(\psi)$, and $\widetilde{X}(\psi)$

As promised in the introduction, we restrict our analysis to one dimension. For the most part of this section, we replace the dilations by powers of 2 to powers of a (fixed) real number $a > 1$. When no confusion is likely, we keep the notation we introduced when we defined these various discrete affine systems. Thus, for example, $\psi_{j,k}(x)$ now denotes the function $a^{-j/2}\, \psi(a^{-j}x - k)$, and $\psi^*_{j,k}(x)$ denotes the function $a^{-j/2}\, \psi(a^{-j}(x-k))$. Associated with these discrete systems are the continuous wavelets produced by the groups G and G^*, associated with the dilation group D, where $D = \{a^j : j \in \mathbb{Z}\} \subset GL(1,\mathbb{R})$. Then, the system corresponding to the one defined by (1.5) is the collection of functions

$$\psi_{j,b}(x) = a^{-j/2}\, \psi(a^{-j}x - b), \qquad j \in \mathbb{Z}, b \in \mathbb{R}.$$

The reproducing property (1.6) is, then,

$$\|f\|_2^2 \;=\; \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\langle f, \psi_{j,b}\rangle|^2 \, db, \tag{2.1}$$

for all $f \in L^2(\mathbb{R})$. If we use the group G^* in this case, then the formula (1.10) reduces to

$$\|f\|_2^2 \;=\; \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\langle f, \psi^*_{a,b}\rangle|^2 \, db. \tag{2.2}$$

In this case, D is an abelian group (isomorphic to $(\mathbb{Z},+)$), and μ is the counting measure. G and G^*, however, are not unimodular and $d\lambda^*(a,b) = a^{-j}\, d\mu(j)\, db$. Equality (1.7) has the form

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(\xi a^{-j})|^2 \;=\; 1 \text{ a.e.} \tag{2.3}$$

The factor a^{-j} in (2.2), that arises from the form of the left Haar measure λ^*, could be incorporated in the definition of the co-affine system thus giving us a re-normalization of the elements $\psi^*_{j,k}$. In fact, the quasi-affine system $\widetilde{X}(\psi)$ does this for "half" the system: for $j > 0$, we let $\tilde{\psi}_{j,k} = a^{-j/2}\, T_{-k}\, D_{a^{-j}}\, \psi$, while $\tilde{\psi}_{j,k} = \psi_{j,k}$ if $j \le 0$.

In order to clarify the situation, we are now going to study the discrete affine systems $X(\psi) = \{\psi_{j,k} : j,k \in \mathbb{Z}\}$, the discrete quasi-affine systems $\widetilde{X}(\psi) = \{\tilde{\psi}_{j,k} : j,k \in \mathbb{Z}\}$, and the discrete co-affine systems $X^*(\psi) = \{\psi^*_{j,k} : j,k \in \mathbb{Z}\}$, where the dilations are integral powers of $a > 1$ and $\psi \in L^2(\mathbb{R})$. The following observations will present further evidence for the discovery of Ron and Shen to be of importance.

We showed, at the end of the first section, that $X^*(\psi)$ cannot be an orthonormal basis for $L^2(\mathbb{R})$ when this is the case for $X(\psi)$. In view of the "equivalence" between the systems $X(\psi)$ and $\widetilde{X}(\psi)$, and the fact that $\widetilde{X}(\psi)$ consists of a specific renormalization of "half" the system $X^*(\psi)$, it is reasonable to inquire if there are renormalizations of $X^*(\psi)$ that provide a frame (or even a Bessel system). More precisely, does there exist a real sequence $\{c_j\}$, $j \in \mathbb{Z}$, such that $\{c_j\, \psi^*_{j,k}\}$, $j,k \in \mathbb{Z}$, is a frame for $L^2(\mathbb{R})$? That is, are there constants A, B such that $0 < A \le B < \infty$, for which

$$A\,\|f\|_2^2 \le \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, c_j\, \psi^*_{j,k}\rangle|^2 \le B\,\|f\|_2^2 \tag{2.4}$$

for all $f \in L^2(\mathbb{R})$?

Let us suppose that such a function ψ and sequence $\{c_j\}$ exist. Let $f = c_{j_0}\, \psi^*_{j_0,k_0}$. Then, using the second inequality in (2.4), we have

$$|c_{j_0}|^4\, \|\psi\|_2^4 \;=\; |c_{j_0}|^4\, \|\psi^*_{j_0,k_0}\|_2^4 \;\le\; \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle c_{j_0}\, \psi^*_{j_0,k_0}, c_j\, \psi^*_{j,k}\rangle|^2$$

$$\leq \; B\,|c_{j_0}|^2\,\|\psi^*_{j_0,k_0}\|_2^2 \;=\; B\,|c_{j_0}|^2\,\|\psi\|_2^2.$$

It follows that

$$|c_{j_0}|^2 \;\leq\; B\,\|\psi\|_2^{-2}, \tag{2.5}$$

for all $j_0 \in \mathbb{Z}$.

Let N be the functional on $L^2(\mathbb{R})$ defined by

$$N^2(f) \;=\; \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, c_j\,\psi^*_{j,k}\rangle|^2,$$

and let $w(x) = N^2(T_x\,f)$ for $x \in \mathbb{R}$. Then w is clearly a 1-periodic function [2] since $\langle T_x\,f, c_j\,\psi^*_{j,k}\rangle = \langle f, c_j\,T_{-x-k}\,D_{a^{-j}}\,\psi\rangle$ and, thus,

$$\begin{aligned} w(x+1) \;&=\; \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, c_j\,T_{-x-(k+1)}\,D_{a^{-j}}\,\psi\rangle|^2 \\ &=\; \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f, c_j\,T_{-x-k}\,D_{a^{-j}}\,\psi\rangle|^2 \;=\; w(x). \end{aligned}$$

We claim that

$$\int_0^1 w(x)\,dx \;=\; \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |c_j|^2\,a^j\,|\hat{\psi}(a^j\,\xi)|^2\,d\xi. \tag{2.6}$$

To see this, we use the Plancherel theorem (after a change of variables):

$$\begin{aligned} \int_0^1 w(x)\,dx \;&=\; \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \int_0^1 |\langle f, c_j\,T_{-(x+k)}\,D_{a^{-j}}\,\psi\rangle|^2\,dx \\ &=\; \sum_{j\in\mathbb{Z}} |c_j|^2 \sum_{k\in\mathbb{Z}} \int_k^{k+1} |\langle f,\,T_{-y}\,D_{a^{-j}}\,\psi\rangle|^2\,dy \\ &=\; \sum_{j\in\mathbb{Z}} |c_j|^2 \int_{\mathbb{R}} |\langle f,\,T_{-y}\,D_{a^{-j}}\,\psi\rangle|^2\,dy \\ &=\; \sum_{j\in\mathbb{Z}} |c_j|^2\,a^j \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{\psi}(a^j\xi)}\,e^{2\pi i\xi y}\,d\xi \right|^2 dy \\ &=\; \sum_{j\in\mathbb{Z}} |c_j|^2\,a^j \int_{\mathbb{R}} \left| (\hat{f}\,\overline{\hat{\psi}(a^j\cdot)})^\vee(y) \right|^2 dy \\ &=\; \sum_{j\in\mathbb{Z}} |c_j|^2\,a^j \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\psi}(a^j\,\xi)|^2\,d\xi \end{aligned}$$

[2] This kind of periodicity is shared with the quasi-affine systems and it is at the root of the Ron-Shen approach of reducing wavelet problems to those in shift-invariant spaces.

and this establishes (2.6).

Since $\|T_x f\|_2 = \|f\|_2$ for all $x \in \mathbb{R}$, it follows from (2.4) that $A\,\|f\|_2^2 \le w(x) \le B\,\|f\|_2^2$. This, together with (2.6), yields

$$A\,\|f\|_2^2 \le \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j\in\mathbb{Z}} |c_j|^2\, a^j\, |\hat{\psi}(a^j\,\xi)|^2\, d\xi \le B\,\|f\|_2^2 \tag{2.7}$$

for all $f \in L^2(\mathbb{R})$. An easy consequence of this (obtained by making an appropriate choice of f) is that

$$A \;\le\; \sum_{j\in\mathbb{Z}} |c_j|^2\, a^j\, |\hat{\psi}(a^j\,\xi)|^2 \;\le\; B \tag{2.8}$$

for a.e. $\xi \in \mathbb{R}$. If we replace ξ by $a^n\eta$, for any fixed $n \in \mathbb{Z}$ and, then, integrate over the interval $[1, a]$, we have

$$\begin{aligned} A(a-1) \quad &\le \quad \int_1^a \sum_{j\in\mathbb{Z}} |c_j|^2\, a^j\, |\hat{\psi}(a^{j+n}\,\eta)|^2\, d\eta \\ &= \quad \int_1^a \sum_{\ell\in\mathbb{Z}} |c_{\ell-n}|^2\, a^{\ell-n}\, |\hat{\psi}(a^\ell\,\eta)|^2\, d\eta \\ &= \quad a^{-n} \sum_{\ell\in\mathbb{Z}} |c_{\ell-n}|^2 \int_{a^\ell}^{a^{\ell+1}} |\hat{\psi}(\xi)|^2\, d\xi. \end{aligned}$$

Applying (2.5) to the last expression, we obtain

$$a^{-n} \sum_{\ell\in\mathbb{Z}} |c_{\ell-n}|^2 \int_{a^\ell}^{a^{\ell+1}} |\hat{\psi}(\xi)|^2\, d\xi \le a^{-n} \sum_{\ell\in\mathbb{Z}} B\,\|\psi\|_2^{-2} \int_{a^\ell}^{a^{\ell+1}} |\hat{\psi}(\xi)|^2\, d\xi \le a^{-n}\, B.$$

We have shown

$$(a-1)A \;\le\; a^{-n}B \qquad \text{for all } n \in \mathbb{Z}. \tag{2.9}$$

From this we see that there cannot exist a pair (A, B) such that $0 < A \le B < \infty$ for which the frame property (2.4) is true. We have proved

Theorem 3. *If $\psi \in L^2(\mathbb{R})$ and $\{c_j\}, j \in \mathbb{Z}$, is any numerical sequence, then $\{c_j\,\psi^*_{j,k}\}, j, k \in \mathbb{Z}$, cannot be a frame for $L^2(\mathbb{R})$.*

This result, and the more elementary observation made at the end of Section 1, show that while the selection of operators that produce the affine systems and the quasi-affine systems provide us with complete discrete systems for analyzing and reproducing functions, this is not the case for the co-affine systems. In this connection, it is relevant that equality (2.3) is one of the two equations that characterize those ψ such that $X(\psi)$ and $\widetilde{X}(\psi)$ are normalized tight frames for $L^2(\mathbb{R})$. The other equation, when $a = 2$, is

$$t_q(\xi) = \sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi+q))} = 0 \quad \text{a.e.} \tag{2.10}$$

whenever q is an odd integer (see Chapter 7 in [2]). There are good reasons to consider (2.3) to represent the completeness of the system $X(\psi)$ (or $\widetilde{X}(\psi)$). For example, if ψ satisfies (2.3) and, say, $X(\psi)$ is an orthonormal system, then it is an orthonormal basis (see [1], [5], [3] for this and more general results). We have just seen that this completeness fails for $X^*(\psi)$.

Let us clarify a few points. The "equivalence" between the systems $X(\psi)$ and $\widetilde{X}(\psi)$ is, indeed, true when the dilation a is an integer. In this case, if $X(\psi)$ is a normalized tight frame, then so is $\widetilde{X}(\psi)$ and vice versa. For general dilation, if $\widetilde{X}(\psi)$ is such a frame, so is $X(\psi)$. The converse may be false in general.

We finish this note with an observation that gives us further insight into the affine and co-affine systems. Let $\widetilde{X}^N(\psi)$ be obtained by "cutting off" the affine system at $N \geq 0$, $N \in \mathbb{N}$. More precisely, $\tilde{\psi}_{j,k} = a^{-j/2}\,T_{-k}\,D_{a^{-j}}\,\psi$ if $j > N$, and $\tilde{\psi}_{j,k} = \psi_{j,k}$ otherwise. Then $\widetilde{X}^N(\psi)$ can be a normalized tight frame for appropriate ψ; in this case, $X(\psi)$ is such a frame as well. There are, however, ψ such that $X(\psi)$ is a normalized tight frame and, yet, this fails to be the case for $\widetilde{X}^N(\psi)$. A precise result when $N = 1$ and $a = 2$ that explains this situation is the following fact:

Theorem 4. *$\widetilde{X}^1(\psi)$ is a normalized tight frame if and only if*

(i) $$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \quad \text{a.e.}$$

(ii) $$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi+q))} = 0 \quad \text{a.e. when } q \text{ is odd}$$

(iii) $$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi+\tfrac{1}{2}q))} = 0 \quad \text{a.e. when } q \text{ is odd.}$$

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