Sparse Multidimensional Representations using Anisotropic Dilation and Shear Operators

Kanghui Guo, Gitta Kutyniok, and Demetrio Labate

Dedicated to Professor Charles K. Chui on the occasion of his 65th birthday.

Abstract. Recent advances in applied mathematics and signal processing have shown that, in order to obtain sparse representations of multi-dimensional functions and signals, one has to use representation elements distributed not only at various scales and locations – as in classical wavelet theory – but also at various directions. In this paper, we show that we obtain a construction having exactly these properties by using the framework of affine systems. The representation elements that we obtain are generated by translations, dilations, and shear transformations of a single mother function, and are called *shearlets*. The shearlets provide optimally sparse representations for 2-D functions that are smooth away from discontinuities along curves. Another benefit of this approach is that, thanks to their mathematical structure, these systems provide a Multiresolution analysis similar to the one associated with classical wavelets, which is very useful for the development of fast algorithmic implementations.

§1. Introduction

One of the most important features of wavelets is their ability to efficiently represent smooth functions with pointwise discontinuities. Indeed, if f is a one-variable function that is smooth away from point discontinuities, the rate of convergence of the best n-term wavelet approximation is optimal, hence, in particular, it is significantly better than the corresponding Fourier approximation [17]. On the other hand, it is well-known

Conference Title
Editors pp. 1–6.

ISBN 0-0-9728482-x-x

that wavelets do not perform as well in dimensions larger than one. This situation is illustrated, for example, by the problem of approximating a function of two variables containing a discontinuity along a curve. Because the discontinuity is spatially distributed, it interacts extensively with the elements of the wavelet basis, and, as a consequence, the wavelet representation is not sparse, that is, "many" wavelet coefficients are needed to accurately represent the discontinuous function.

This limitation has stimulated an active research both in the mathematical and the engineering literature. The problem of representing edges is of basic importance in image processing, and several variations of the wavelet scheme have been proposed to address this issue, including the *directional filter banks* [2], the *directional wavelets* [1], the *complex wavelets* [13], and the *contourlets* [8]. In the mathematical literature, this problem has been extensively examined by Candès and Donoho, who introduced the *ridgelets* [4] and then the *curvelets* [5] in order to overcome the limitations of traditional wavelets. The curvelets, in particular, are able to achieve an (almost) optimal approximation property for 2-D smooth functions with discontinuities along C^2-curves [5].

In a number of recent papers, the authors of the present paper and their collaborators have developed an alternative approach to the construction of an efficient representation of multivariable functions [10, 11, 16]. The elements of this representation, which we call discrete **shearlets**, are obtained by applying dilations, shear transformations, and translations to an appropriate mother function. For example, in dimension 2, the system of discrete shearlets is of the form

$$\{\psi_{i,j,k} = |\det A|^{i/2}\, \psi(S^j A^i x - k) :\; i, j \in \mathbb{Z}, k \in \mathbb{Z}^2\}, \tag{1}$$

where A is the anisotropic expanding matrix $\begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, S is the shear matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and ψ is an appropriate band-limited function in $L^2(\mathbb{R}^2)$. As a result, the shearlets form a Parseval frame of well-localized oscillatory waveforms which have many scales, shapes, and directions, and are increasingly elongated at finer scales. From this point of view, the shearlets have many similarities to the curvelets, but with the additional advantage of a simplified mathematical structure. In fact, the shearlets are an *affine-like system* of the form (1), while the curvelets, whose construction involves translation, rotation, and dilation operators, are not generated by the action of these operators on a single function. Another important feature of the shearlets and their generalizations, is the existence of a Multiresolution analysis (MRA) associated with these systems. This is very important for the development of fast algorithmic implementations of these systems, as we will discuss further in Section 3.4. Finally, it is interesting to notice that the recent digital implementation of curvelets [3] uses shear transformation rather than the rotations that are employed in the definition of the continuous curvelet transform and its discretization [6, 7].

In this paper, our main goal is to show that, using a framework very close to the classical theory of affine systems, one can construct a representation system of elements distributed not only across various scales and locations (as the classical wavelets) but also across various orientations. Our point of view is to start from a special class of continuous affine systems associated with a 2-parameter dilation group, and then to derive a Parseval frame of discrete shearlets by sampling the continuous system appropriately.

The paper is organized as follows. In Section 2 we introduce the continuous shearlet transform and we show that, unlike traditional continuous "isotropic" wavelets, this transform is able to identify both the location and the direction of discontinuities along curves. In Section 3, we obtain Parseval frames of discrete shearlets by sampling the corresponding continuous shearlet transform, and we show that these systems provide a multiscale, directional representation that is efficient in handling 2-dimensional functions with discontinuities along curves.

§2. Continuous Shearlets

A continuous **affine system** in $L^2(\mathbb{R}^n)$ is a collection of functions of the form

$$\{T_t\, D_M\, \psi : \, t \in \mathbb{R}^n, \, M \in G\}, \tag{2}$$

where $\psi \in L^2(\mathbb{R}^n)$, T_t are the **translations**, defined by $T_t\, f(x) = f(x-t)$, D_M are the **dilations**, defined by $D_M\, f(x) = |\det M|^{-1/2}\, f(M^{-1}x)$, and G is a subset of $GL_n(\mathbb{R})$. We are interested in the special case where $n = 2$ and G is the 2-parameter dilation group

$$G = \{M_{as} = \begin{pmatrix} a & \sqrt{a}\, s \\ 0 & \sqrt{a} \end{pmatrix} : \, (a,s) \in \mathbb{R}^+ \times \mathbb{R}\}. \tag{3}$$

Observe that $M_{as} = S_s\, A_a$ is the composition of the **anisotropic dilation** $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ and the **shear transformation** $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

In addition, for any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1 \neq 0$, we assume that ψ is given by

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2\Big(\frac{\xi_2}{\xi_1}\Big), \tag{4}$$

where ψ_1 satisfies

$$\int_0^\infty |\hat{\psi}_1(a\,\omega)|^2\, \frac{da}{a} = 1 \qquad \text{for a.e. } \omega \in \mathbb{R}, \tag{5}$$

and $\|\psi_2\|_2 = 1$. Observe that (5) is the standard *admissibility condition* satisfied by real-valued continuous wavelets [19]. We will show that, under these assumptions, the affine system

$$\{\psi_{ast}(x) = a^{-3/4}\,\psi(M_{as}^{-1}(x-t)) : \, a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2\} \tag{6}$$

is in fact a reproducing system for $L^2(\mathbb{R}^2)$, that is,

$$\|f\|^2 = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^\infty |\langle f, \psi_{ast}\rangle|^2 \, \frac{da}{a^3}\, ds\, dt \quad \text{for all } f \in L^2(\mathbb{R}^2). \tag{7}$$

In addition, we choose ψ_1 such that $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and ψ_2 such that $\hat{\psi}_2 \in C^\infty(\mathbb{R})$, $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$, cf. [11]. We refer to the systems (6) satisfying these assumptions as **continuous shearlets**, and define the **continuous shearlet transform** of $f \in L^2(\mathbb{R})$ as the function

$$C_f(a, s, t) = \langle f, \psi_{ast}\rangle, \quad a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2.$$

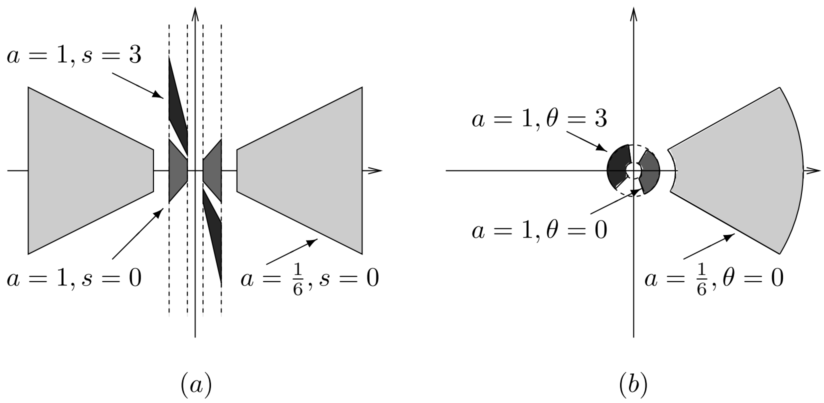

Fig. 1. (a) Support of the shearlets $\hat{\psi}_{ast}$ (in the frequency domain) for different values of a and s. (b) Support of the curvelets $\hat{\gamma}_{ab\theta}$ (in the frequency domain) for different values of a and θ.

The geometrical properties of the continuous shearlets are more evident in the frequency domain. Using (4), a direct calculation shows that

$$\begin{aligned}\hat{\psi}_{ast}(\xi) &= \Big(T_t\, D_{M_{as}}\psi\Big)^{\wedge}(\xi) = a^{\frac{3}{4}}\, e^{-2\pi i \xi t}\, \hat{\psi}(M_{as}^T \xi) \\ &= a^{\frac{3}{4}}\, e^{-2\pi i \xi t}\, \hat{\psi}_1(a\,\xi_1)\, \hat{\psi}_2(a^{-1/2}(s + \tfrac{\xi_2}{\xi_1})).\end{aligned}$$

It follows, by the properties of ψ_1 and ψ_2, that the function $\hat{\psi}_{ast}$ has frequency support contained in the set

$$\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}] \text{ and } |s + \tfrac{\xi_2}{\xi_1}| \le \sqrt{a}\}.$$

This is illustrated in Figure 1(a) for some particular values of s and a. Thus, the shearlets are oriented waveforms, whose orientation is controlled by the shear parameter s, and they become increasingly elongated at fine scales (i.e., as $a \to 0$). This observation also illustrates the role played by the matrices A_a and S_s in their construction. The anisotropic dilation A_a controls the 'scale' of the shearlets, by applying a different dilation factor along the two axes. This ensures that the frequency support of the shearlets becomes increasingly elongated at finer scales. The shear matrix S_s, on the other hand, is not expansive, and determines the orientation of the shearlets.

Some properties of the continuous shearlets are similar to the recently introduced continuous curvelet transform of Candès and Donoho [6]. Let us recall that the continuous curvelet transform is defined as $\Gamma_f(a, \theta, t) = \langle f, \gamma_{a\theta t}\rangle$, where $\gamma_{a\theta t}$ is obtained by applying translations by t and rotations by θ to appropriate functions γ_a, $a \in \mathbb{R}^+$, where a is a scale parameter. However, unlike the shearlets, the curvelets are not generated by an affine transformation acting on a single function γ. Figure 1(b) shows the tiling of the frequency plane induced by the continuous curvelets [6, 7].

We will now show that the shearlets $\{\psi_{ast} : a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2\}$ are a reproducing system for $L^2(\mathbb{R}^2)$.

Theorem 1. *Let $\psi \in L^2(\mathbb{R}^2)$ be given by (4), where ψ_1 is a real-valued continuous wavelet and $\psi_2 \in L^2(\mathbb{R})$. Then (7) holds for all $f \in L^2(\mathbb{R}^2)$.*

Proof: In order to prove this theorem, we recall a well-known fact about continuous wavelets that can be found for instance in [19, Thm.2.1]: the equality

$$\|f\|^2 = \int_{\mathbb{R}^2} \int_G |\langle f, T_t\, D_M \psi\rangle|^2 \, d\lambda(M)\, dt,$$

where $G \subset GL_n(\mathbb{R})$ and λ is a measure on G, holds for all $f \in L^2(\mathbb{R}^2)$ if and only if

$$\Delta(\psi)(\xi) = \int_G |\hat{\psi}(M^T \xi)|^2 \, |\det M| \, d\lambda(M) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^2. \tag{8}$$

Now we focus on the special case that G is given by (3). Then, using the measure $d\lambda(M) = \frac{da}{a^3}\, ds$, the admissibility condition (8) is:

$$\Delta(\psi)(\xi) = \int_{\mathbb{R}} \int_0^\infty |\hat{\psi}(M_{as}^T \xi)|^2 \, a^{-3/2} \, da\, ds = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^2. \tag{9}$$

Thus, it only remains to show that (9) is satisfied. Since ψ_1 is a real-valued continuous wavelet, (5) holds. Thus, using (4) and (5), a direct calculation shows that, for a.e. $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1 \neq 0$, we have:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty |\hat{\psi}(M_{as}^T \xi)|^2 \, a^{-3/2} \, da \, ds \\
&\quad = \int_{\mathbb{R}} \int_0^\infty |\hat{\psi}_1(a\xi_1)|^2 \, |\hat{\psi}_2(a^{-1/2}(s + \tfrac{\xi_2}{\xi_1}))|^2 a^{-3/2} \, da \, ds \\
&\quad = \int_0^\infty |\hat{\psi}_1(a\xi_1)|^2 \int_{\mathbb{R}} |\hat{\psi}_2(s + a^{-1/2} \tfrac{\xi_2}{\xi_1})|^2 \, ds \, \frac{da}{a} \\
&\quad = \int_0^\infty |\hat{\psi}_1(a\xi_1)|^2 \, \frac{da}{a} = 1. \qquad \square
\end{aligned}$$

2.1. Resolution of edges using the continuous shearlets

It is known [18] that, if ψ is a 'nice' continuous wavelet, then the continuous wavelet transform $\mathcal{W}_f(a,t)\langle f, \psi_{at}\rangle$, where $\psi_{at} = T_t \, D_a \, \psi$, is able to localize the singularities of f in the following sense. For $a \to 0$, the function $\mathcal{W}_f(a,t)$ tends rapidly to zero, when t is outside the singularity, and $\mathcal{W}_f(a,t)$ tends to zero slowly when t is on the singularity.

One major property of continuous shearlets is their ability to resolve the discontinuities of 2-D functions by identifying not only the location, but also the *orientation* of the discontinuity.

More precisely, let f be a 2-D function that is smooth away from a discontinuity along a curve $\mathcal{C}$. Then, for $a \to 0$ the continuous curvelet transform satisfies

$$|C_f(a,s,t)| \leq C \, a^N, \qquad \text{for each } N = 1, 2, \ldots,$$

unless t is on $\mathcal{C}$ and s describes the orientation perpendicular to $\mathcal{C}$ at t (we refer to [14] for more details). The following example is a special case of this general property :

Example 1. *Let $f = \chi_D$, where D is the unit disc in $\mathbb{R}^2$, then, for $a \to 0$,*

- *if $t \in \partial D$ and s corresponds to the direction normal to ∂D, then $|C_f(a,s,t)| \sim a^{3/4}$;*
- *otherwise, for each $N = 1, 2, \ldots$, $|C_f(a,s,t)| \leq C \, a^N$.*

§3. Discrete Shearlets

In this section we present a general framework for the discretization of the continuous shearlet transform, that leads to a variety of shearlet frames.

We recall the following simple facts from the theory of frames. A countable family $\{e_j : j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathcal{H}$

is a **frame** if there exist constants $0 < A \le B < \infty$ satisfying $A \|f\|^2 \le \sum_{j\in\mathcal{J}} |\langle f, e_j\rangle|^2 \le B \|f\|^2$ for all $f \in \mathcal{H}$. A frame is **tight** if A and B can be chosen so that $A = B$, and is a **Parseval frame** if $A = B = 1$. Thus, if $\{e_j : j \in \mathcal{J}\}$ is a Parseval frame for $\mathcal{H}$, then $\|f\|^2 = \sum_{j\in\mathcal{J}} |\langle f, e_j\rangle|^2$ for each $f \in \mathcal{H}$. This is equivalent to the reproducing formula $f = \sum_{j\in\mathcal{J}} \langle f, e_j\rangle\, e_j$ for all $f \in \mathcal{H}$, where the series converges in the norm of $\mathcal{H}$. This shows that a Parseval frame provides a basis-like representation. In general, however, a Parseval frame need not be a basis.

3.1. General method

By sampling the continuous shearlet transform $C_f(a, s, t) = \langle f, \psi_{ast}\rangle$ on an appropriate discrete set of the scaling, shear, and translation parameters $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, it is possible to obtain a frame or even a Parseval frame for $L^2(\mathbb{R}^2)$. By keeping the greatest generality, we can choose an arbitrary set of scales $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$; next we choose the shear parameters $\{s_{jk}\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ dependent on j, so that the directionality of the representation is allowed to change with the scale. Finally, in order to provide a "uniform covering", we allow the location parameter to describe a different grid depending on j, and hence, on k. We let $t_{jkm} S_{s_{jk}} A_{a_j} bm$, $m \in \mathbb{Z}^2$, $b > 0$. Thus, observing that $T_{\{S_{s_{jk}} A_{a_j} bm\}} D_{S_{s_{jk}} A_{a_j}} = D_{S_{s_{jk}} A_{a_j}} T_{bm}$, we obtain the discrete set

$$\{\psi_{jkm} = D_{S_{s_{jk}} A_{a_j}}\, T_{bm}\, \psi : j, k \in \mathbb{Z},\ m \in \mathbb{Z}^2\}.$$

In [14], we derive estimates for the frame bounds of such a general discrete shearlet system, thereby providing a very general strategy to design specially adapted discrete directional representations.

3.2. Discrete shearlets on $\mathbb{R}^2$

In this section, we apply the general machinery described above to construct a Parseval frame of shearlets for $L^2(\mathbb{R}^2)$. For the scaling parameter we use the dyadic sampling $a_j = 2^j$, $j \in \mathbb{Z}$. In order to cover the frequency plane "uniformly", we need a larger number of directions as j is getting smaller, and, thus, we set the shear parameter to be $s_{jk} = k\sqrt{a_j} = k2^{\frac{j}{2}}$, $k \in \mathbb{Z}$. Finally, the location parameter is determined by adjusting the canonical grid $\mathbb{Z}^2$ to the particular scaling and shear parameter, i.e., we choose $t_{jkm} S_{s_{jk}} A_{a_j} m$, $m \in \mathbb{Z}^2$. Combining all this and observing that $T_{\{S_{k2^{j/2}} A_2^j m\}} D_{S_k A_{2^j}} D_{A_2^j S_k} T_m$, we obtain the discrete system

$$\{\psi_{jkm} = D_{A_2^j S_k}\, T_m\, \psi : j, k \in \mathbb{Z},\ m \in \mathbb{Z}^2\}. \tag{10}$$

It is interesting to observe that this choice of sampling points gives a special case of the affine systems with composite dilations introduced in [11].

We will now choose $\psi \in L^2(\mathbb{R}^2)$ in a way similar to the continuous case. We define $\psi \in L^2(\mathbb{R}^2)$ by (4), where $\psi_1 \in L^2(\mathbb{R})$ with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and $\operatorname{supp} \hat{\psi}_1 \subset [-\frac{1}{2}, -\frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}]$ satisfying

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}_1(2^j\omega)|^2 = 1 \quad \text{for } \omega \in \mathbb{R}, \tag{11}$$

and $\psi_2 \in L^2(\mathbb{R})$ with $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_2 \subset [-1, 1]$ satisfying

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}_2(k+\omega)|^2 = 1 \quad \text{for } \omega \in \mathbb{R}. \tag{12}$$

Notice that $\hat{\psi}_\ell \in C^\infty(\mathbb{R})$, $\ell = 1, 2$, implies that $|\psi_\ell(x)| \le K_N\,(1+|x|)^{-N}$, $K_N > 0$, for any $N \ge 0$, and thus the function ψ is well-localized.

We point out that there exist several choices of functions ψ_1 and ψ_2 satisfying these properties. For example, we can choose ψ_1 to be the Lemariè–Meyer wavelet ψ_{LM} (see [12, Sec.1.4]), defined by $\hat{\psi}_{LM}(\omega) = e^{i\pi\omega}\, b(\omega)$, where

$$b(\omega) = \begin{cases} \sin(\frac{\pi}{2}\,(3|\omega|-1)) & : \ \frac{1}{3} \le |\omega| \le \frac{2}{3}, \\ \sin(\frac{3\pi}{4}\,(\frac{4}{3}-|\omega|)) & : \ \frac{2}{3} < |\omega| \le \frac{4}{3}, \\ 0 & : \ \text{otherwise}. \end{cases} \tag{13}$$

In order to construct ψ_2, let ϕ be a compactly supported C^∞ bump function supported in $[-1, 1]$, and define ψ_2 by

$$\hat{\psi}_2(\omega) = \frac{\phi(\omega)}{\sqrt{\sum_{k\in\mathbb{Z}} |\phi(\omega+k)|^2}}. \tag{14}$$

We have the following result from [11].

Theorem 2. *Let $\psi \in L^2(\mathbb{R}^2)$ be given by (4), where ψ_1 and ψ_2 satisfy (11) and (12), respectively. Then the system (10) is a Parseval frame for $L^2(\mathbb{R}^2)$.*

Thus, with the choice of ψ that we described in this section, the system (10) is a Parseval frame of well-localized waveforms, with frequency support increasingly elongated at finer scale ($j \to -\infty$) and with the directions depending on k and j. We will refer to the elements of this system as **discrete shearlets**. The induced tiling of the frequency plane of the discrete shearlets is illustrated in Figure 2(a).

3.3. Discrete shearlets on the cone

By modifying the construction of the previous section, we can obtain a Parseval frame for functions whose Fourier transform is supported in the cone

$$C = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geq \tfrac{1}{4}, |\tfrac{\xi_2}{\xi_1}| \leq 1\}. \tag{15}$$

We sample the scaling parameter of the continuous curvelet transform by choosing $a_j = 2^{-2j}$, $j \geq 0$. Next we set the shear parameter to be $s_{jk} = k\sqrt{a_j} = k2^{-j}$, $-2^j \leq k \leq 2^j$, and the location parameter $t_{jkm} S_{s_{jk}} A_{a_j} m$, $m \in \mathbb{Z}^2$. Hence, using a calculation similar to the previous section, we obtain the discrete set

$$\{\psi_{jkm} = D_{A_2^{-2j} S_k} T_m \, \psi : j \geq 0, -2^j \leq k \leq 2^j, \, m \in \mathbb{Z}^2\}. \tag{16}$$

Again, define $\psi \in L^2(\mathbb{R}^2)$ by (4), where $\psi_1 \in L^2(\mathbb{R})$, $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and satisfies

$$\sum_{j\geq 0} |\hat{\psi}_1(2^{-2j}\xi)|^2 = 1 \quad \text{for } |\xi| \geq \frac{1}{4}, \tag{17}$$

and $\psi_2 \in L^2(\mathbb{R})$ is a band-limited function with $\operatorname{supp} \hat{\psi}_2 \subset [-1, 1]$, $\hat{\psi}_2 \in C^\infty(\mathbb{R})$ and satisfies

$$|\hat{\psi}_2(\xi - 1)|^2 + |\hat{\psi}_2(\xi)|^2 + |\hat{\psi}_2(\xi + 1)|^2 = 1 \quad \text{for } |\xi| \leq 1.$$

It follows that, for any $j \geq 0$,

$$\sum_{k=-2^j}^{2^j} |\hat{\psi}_2(2^j \, \xi + k)|^2 = 1 \quad \text{for } |\xi| \leq 1. \tag{18}$$

As mentioned in the previous section, $\hat{\psi}_1, \hat{\psi}_2 \in C^\infty(\mathbb{R})$ yield a well-localized function ψ.

Let us mention that there exist an abundance of functions ψ_1 and ψ_2 satisfying those conditions. Let ψ_{LM} denote the Lemariè–Meyer wavelet, and let the corresponding window function b be defined as in (13). Then it is easy to see that

$$\sum_{j=-1}^{\infty} |b(2^{-j}\xi)|^2 = 1 \quad \text{for } |\xi| \geq \frac{1}{4}.$$

Now letting $|\hat{\psi}_1(\xi)|^2 = |b(2\xi)|^2 + |b(\xi)|^2$, it follows that

$$\sum_{j\geq 0}^{\infty} |\hat{\psi}_1(2^{-2j}\xi)|^2 = \sum_{j=-1}^{\infty} |b(2^{-j}\xi)|^2 = 1 \quad \text{for } |\xi| \geq \frac{1}{4}.$$

The function ψ_2 can be chosen as in (14).

We have the following result.

Theorem 3. *Let $\psi \in L^2(\mathbb{R}^2)$ be given by (4), where ψ_1 and ψ_2 satisfy (17) and (18), respectively. Then the system (16) is a Parseval frame for $L^2(C)^\vee$.*

Proof: Using (17) and (18) it is easy to see that, for $\xi = (\xi_1, \xi_2) \in C$,

$$\begin{aligned}
&\sum_{j\geq 0} \sum_{k=-2^j}^{2^j} |\hat{\psi}(S_k^T A_2^{-2j} \xi)|^2 \\
&= \sum_{j\geq 0} \sum_{k=-2^j}^{2^j} |\hat{\psi}_1(2^{-2j}\, \xi_1)|^2\, |\hat{\psi}_2(2^j\, \tfrac{\xi_2}{\xi_1} + k)|^2 \\
&= \sum_{j\geq 0} |\hat{\psi}_1(2^{-2j}\, \xi_1)|^2 \sum_{k=-2^j}^{2^j} |\hat{\psi}_2(2^j\, \tfrac{\xi_2}{\xi_1} + k)|^2 = 1.
\end{aligned}$$

The claim follows immediately from this observation. □

Similarly to the previous section, the system (16) is a Parseval frame of well-localized oscillatory waveforms, with the number of directions increasing with j, dependent on k and j, and increasingly elongated for $j \to \infty$. We will refer to the elements of this system as **discrete shearlets on** C.

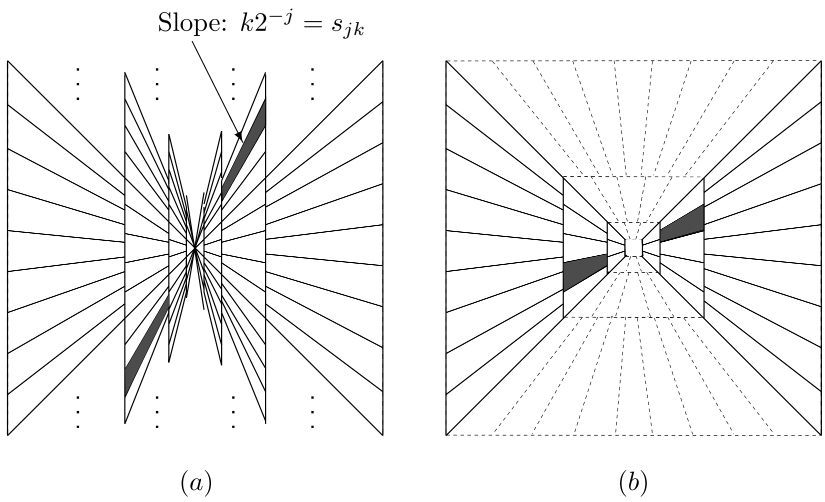

Fig. 2. (a) The tiling of the frequency domain induced by the discrete shearlets. (b) The tiling of the frequency domain induced by the discrete shearlets on the cone.

In order to obtain a reproducing system for the larger space $L^2(\mathbb{R}^2)$, we can construct in a similar way a Parseval frame for $L^2(\widetilde{C})^\vee$, where

$$\widetilde{C} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| > 1 \text{ and } |\tfrac{\xi_2}{\xi_1}| > 1\};$$

next we can construct a Parseval frame for $L^2(T)^\vee$, where

$$T = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1|, |\xi_2| \leq 1\}$$

by taking translates of an appropriate smooth window function. Finally, we obtain a reproducing system for $L^2(\mathbb{R}^2)$ by appropriately combining the three complementary Parseval systems that we have described. We refer to the elements of this system as **discrete shearlets on the cone**. Figure 2(b) illustrates the tiling of the frequency plane induced by this system. The benefit of this construction, compared to (10), is that, for each j, the shear parameter k ranges over a finite set. This is a clear advantage for their numerical implementation. We observe that the tiling of frequency plane using concentric squares also appears in [8] and [3].

3.4. Further results

The discrete shearlets provide (essentially) optimally efficient representations of 2-D objects with edges. More precisely, let f be a 2-D function that is C^2 apart from discontinuities along C^2-curves. Then, denoting by f_m the approximation obtained by taking the best m terms in the discrete shearlets expansion of f, the L^2-error $\|f - f_m\|^2$ of such approximation decays asymptotically as $O(m^{-2}(\log m)^3)$, as $m \to \infty$ [9]. This result is essentially optimal and is exactly the rate obtained using the curvelets [5].

As we mentioned before, a very important feature in the theory of shearlets and their generalizations is the existence of a Multiresolution analysis associated with these systems, called AB–MRA. This theory has been developed by the authors in [10, 11], where it is shown that the AB–MRA generalizes to a great extent the classical MRA associated with wavelets. One of the consequences of this approach is that there is a simple recursive algorithm for decomposing multi-dimensional functions and signals, that generalizes the so-called *cascade algorithm*. As in the classical wavelet theory, this is very useful for the development of a fast algorithmic implementation of shearlets [16, 15].

The shearlet transform associated with the discrete shearlets on the cone is currently being implemented. For details about numerical implementations of our results we refer to [15].

Acknowledgment

The research for this paper was performed while GK was visiting the Department of Mathematics at Washington University in St. Louis and the School of Mathematics at the Georgia Institute of Technology. This author thanks these departments for their hospitality and support during these visits. The authors thank C. Heil, W.-Q. Lim, G. Weiss, and E. Wilson for useful discussions. GK acknowledges support from DFG Research Fellowship KU 1446/5, and DL acknowledges support from a FR&PD grant from NCSU.

References

1. Antoine, J. P., R. Murenzi, and P. Vandergheynst, Directional wavelets revisited: Cauchy wavelets and symmetry detection in patterns, Appl. Computat. Harmon. Anal. **6** (1999), 314-345.
2. Bamberger R. H., and M. J. T. Smith, A filter bank for directional decomposition of images: theory and design, IEEE Trans. Signal Process. **40** (1992), 882–893.
3. Candès, E. J., L. Demanet, D. L. Donoho, and L. Ying, Fast discrete curvelet transforms, preprint (2005).
4. Candès, E. J., and D. L. Donoho, Ridgelets: a key to higher–dimensional intermittency?, Phil. Trans. Royal Soc. London A **357** (1999), 2495–2509.
5. Candès, E. J., and D. L. Donoho, New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities, Comm. Pure and Appl. Math. **56** (2004), 216–266.
6. Candès, E. J., and D. L. Donoho, Continuous curvelet transform: I. Resolution of the wavefront set, Appl. Comput. Harmon. Anal. **19** (2005), 162–197.
7. Candès, E. J., and D. L. Donoho, Continuous curvelet transform: II. Discretization and frames, Appl. Comput. Harmon. Anal. **19** (2005), 198–222.
8. Do, M. N., and M. Vetterli, The contourlet transform: an efficient directional multiresolution image representation, IEEE Trans. Image Process. **14** (2005), 2091–2106.
9. Guo, K., and D. Labate, Optimally sparse multidimensional representations using shearlets, preprint (2006).
10. Guo, K., W.-Q. Lim, D. Labate, G. Weiss, and E. Wilson, Wavelets with composite dilations, Electr. Res. Announc. of AMS **10** (2004), 78–87.
11. Guo, K., W.-Q. Lim, D. Labate, G. Weiss, and E. Wilson, Wavelets with composite dilations and their MRA properties, Appl. Comput. Harmon. Anal., to appear.
12. Hernández, E., and G. Weiss, A first course on wavelets, CRC Press, Boca Raton, 1996.

13. Kingsbury, N., Complex wavelets for shift invariant analysis and filtering of signals, Appl. Computat. Harmon. Anal. **10** (2001), 234–253.
14. Kutyniok, G., and D. Labate, Resolution of the wavefront set using shearlets, preprint (2006).
15. Lim, W.-Q., Wavelets with Composite Dilations, Ph.D. Thesis, Washington University in St. Louis, 2006.
16. Labate, D., W.-Q. Lim, G. Kutyniok, and G. Weiss, Sparse multidimensional representation using shearlets, Wavelets XI (San Diego, CA, 2005), 254–262, SPIE Proc. **5914**, SPIE, Bellingham, WA, 2005.
17. Mallat, S., A Wavelet Tour of Signal Processing, Academic Press, San Diego (1998).
18. Meyer, Y., Ondelettes et Operateurs, I, Hermann, Paris (1990).
19. Weiss, G., and E. Wilson, The mathematical theory of wavelets, Proceeding of the NATO–ASI Meeting. Harmonic Analysis 2000. A Celebration. Kluwer Publisher, 2001.

Kanghui Guo
Department of Mathematics
Southwest Missouri State University
Springfield, MO 65804, USA
`kag026f@smsu.edu`

Gitta Kutyniok
Institute of Mathematics
Justus–Liebig–University Giessen
35392 Giessen, Germany
`gitta.kutyniok@math.uni-giessen.de`
`http://www.math.uni-giessen.de/Numerik/gittak`

Demetrio Labate
Department of Mathematics
North Carolina State University
Raleigh, NC 27695, USA
`dlabate@math.ncsu.edu`
`http://www.math.ncsu.edu/~dlabate`