Noname manuscript No.
(will be inserted by the editor)

Detection of singularities by discrete multiscale directional representations

Kanghui Guo · Demetrio Labate

Received: date / Accepted: date

Abstract One of the most remarkable properties of the continuous curvelet and shearlet transforms is their sensitivity to the directional regularity of functions and distributions. As a consequence of this property, these transforms can be used to characterize the geometry of edge singularities of functions and distributions by their *asymptotic* decay at fine scales. This ability is a major extension of the conventional continuous wavelet transform which can only describe pointwise regularity properties. However, while in the case of wavelets it is relatively easy to relate the asymptotic properties of the continuous transform to properties of discrete wavelet coefficients, this problem is surprisingly challenging in the case of discrete curvelets and shearlets where one wants to handle also the geometry of the singularity. No result for the the discrete case

K. Guo
Department Mathematics, Missouri State University, Springfield, MO
Tel.: +1-417-8366712
Fax: +1-417-8366966
E-mail: KanghuiGuo@MissouriState.edu

D. Labate
Department Mathematics, University of Houston, Houston, TX
Tel.: +1-713-7433492
Fax: +1-713-7433505
E-mail: dlabate@math.uh.edu

was known so far. In this paper, we derive *non-asymptotic* estimates showing that discrete shearlet coefficients can detect, in a precise sense, the location and orientation of curvilinear edges. We discuss connections and implications of this result to sparse approximations and other applications.

Keywords Analysis of singularities · continuous wavelets · curvelets · edge detection · shearlets · wavelets

Mathematics Subject Classification (2000) 42C15 · 42C40 · 65T60

1 Introduction

Shearlets and curvelets are representation systems consisting of oscillatory well-localized functions defined not only over a range of scales and locations, like classical wavelets, but also over multiple orientations and with anisotropic shapes associated to parabolic scaling. They were introduced about a decade ago to overcome the limitations of wavelets in dealing with edges and other distributed singularities of multivariate functions [1,21]. By combining multiscale and directional sensitivity, such systems yield (nearly) optimally sparse representations for *cartoon-like images*, a class of piecewise C^2 functions containing edges along C^2 curves that is useful for applications in image analysis, outperforming wavelet representations [1,8,19]. The critical feature needed to ensure such sparsity property is that, given a cartoon-like image f, the 'large' representation coefficients of f are relatively small in number as they can be identified among those whose location is associated with an edge and whose orientation is aligned with the edge; away from the edge all fine scale coefficients are 'small'. Denoting as $\beta_{j,\ell,k} = \langle f, \psi_{j,\ell,k} \rangle$ the (discrete) shearlet coefficients of f, where j, ℓ, k are the discrete indices associated with scales, orientations and locations, respectively, it was proved in [8, Thm 1.3] that there is a constant $C > 0$ independent of j such that $|\beta_{j,\ell,k}| \leq C\, 2^{-\frac{3}{2}j}$ and that, for j sufficiently large, for any $C_1 > 0$ there is a constant $C_2 > 0$ independent of j such that

$$\#\{(j,\ell,k) : |\beta_{j,\ell,k}| > C_1\, 2^{-\frac{3}{2}j}\} \leq C_2\, 2^j. \tag{1}$$

This estimate directly implies the sparse approximation result (cf. [8, 19, 24]).

A complementary viewpoint to the analysis of images with edges is provided by the *continuous* wavelet transform and its generalizations. For an appropriate well-localized function $\psi \in L^2(\mathbb{R}^2)$, the continuous wavelet transform is the map

$$f \to \mathcal{W}_\psi f(a,t) = \langle f, \psi_{a,t} \rangle, \quad a > 0, t \in \mathbb{R}^2$$

where the analyzing functions $\psi_{a,t}(x) = a^{-1}\psi(a^{-1}(x-t))$ are waveforms ranging over various scales and locations, controlled by the variables a and t, respectively. It is known that the continuous wavelet transform has a special ability to characterize pointwise regularity properties of functions [15, 16] and can detect the singular support. In the case of multivariate functions however, pointwise regularity alone is not sufficient to capture the geometry of edges. It turns out that the continuous shearlet and curvelet transforms overcome this limitation and are able to describe directional regularity properties of functions [2, 6, 17]. The *continuous shearlet transform* is defined as the mapping

$$f \to \mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{a,s,t} \rangle, \quad a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2,$$

where the analyzing elements $\psi_{a,s,t}$ are well-localized waveforms defined not only over a range of scales and locations, indexed by the continuous variables a and t, respectively, but also at various orientations controlled by s. It was proved that this transform can be used to precisely characterize the geometry of edges through its asymptotic decay at fine scales [9–11, 20]. The key observation here is that the continuous transform of a cartoon-like image f, denoted as $\mathcal{SH}_\psi f(a,s,t)$, exhibits rapid asymptotic decay as $a \to 0$ for all values of s, t unless $t = t_0$ is located at an edge point and $s = s_0$ corresponds to the normal orientation to the edge at t. In this last case, one has

$$\lim_{a \to 0} a^{3/4} \mathcal{SH}_\psi f(a, s_0, t_0) = c > 0. \tag{2}$$

These observations about the continuous shearlet transform are consistent with sparsity, as we remarked that discrete shearlet coefficients away from edges have negligible size, for j sufficiently large. However, the asymptotic estimate (2) showing that the continuous shearlet transform *detects* edge points has no proper correspondence in the sparsity result as sparsity does not directly imply that discrete shearlet coefficients associated with edge points satisfy a lower bound condition.

The goal of this paper is to analyze the properties of discrete shearlet coefficients $\beta_{j,\ell,k} = \langle f, \psi_{j,\ell,k} \rangle$ associated with edge points and to prove the existence of a lower bound for discrete locations k corresponding to edge points. The derivation of this result is rather challenging as it is not possible to directly transfer the microlocal viewpoint of the continuous shearlet transform into the realm of discrete representations. Specifically, the techniques developed for the analysis of the continuous shearlet transform and leading to asymptotic estimates, as $a \to 0$, do not carry over to the analysis of discrete shearlet coefficients and some important features of the continuous analysis are lost. In fact, it turns out that discrete shearlet coefficients lose the exact directional sensitivity of the continuous shearlet transform and can only detect the direction with an uncertainty.

Using our new discrete lower bound result, we will be able to refine (1) and show that, for cartoon-like images containing curvilinear edges, we have the new estimate

$$\#\{(j,\ell,k) : |\beta_{j,\ell,k}| > C 2^{-\frac{3}{2}j}\} \simeq 2^j.$$

Our method also applies to other types of singularities such as delta-type singularities supported along curves and to higher dimensions. For simplicity of presentation, the paper will focus mostly on the 2-dimensional case. In the last section of the paper, we discuss how to extend our results to the 3-dimensional setting. We also remark that the same techniques developed in this paper can be adapted to analyze discrete curvelets and derive similar results. Again, for brevity we do not discuss this analysis in this paper.

We finally recall that discrete shearlets have been already employed in a number of highly competitive numerical algorithms of edge detection and feature extraction in image processing applications [4, 22, 25]. The current study provides a theoretical justification that the microlocal properties of the continuous shearlet transform carry over to the discrete and numerical settings.

The rest of the paper is organized as follows. In the following subsections, we recall the construction and basic properties of shearlets. In Sec. 2, we present the main result of this paper, dealing with characteristic functions of compact regions having a smooth boundary. In Sec. 3, we present the proof of this new result. In Sec. 4, we discuss the extension of our main result to other functions classes and to the three-dimensional setting.

1.1 Notation.

In the following, we adopt the convention that $x \in \mathbb{R}^2$ is a column vector, i.e., $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and that $\xi \in \widehat{\mathbb{R}}^2$ (in the frequency domain) is a row vector, i.e., $\xi = (\xi_1, \xi_2)$. A vector x multiplying a matrix $A \in GL_2(\mathbb{R})$ on the right is understood to be a column vector, while a vector ξ multiplying A on the left is a row vector. Thus, $Ax \in \mathbb{R}^2$ and $\xi A \in \widehat{\mathbb{R}}^2$. The *Fourier transform* of $f \in L^1(\mathbb{R}^2)$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)\, e^{-2\pi i \xi x}\, dx,$$

where $\xi \in \widehat{\mathbb{R}}^2$, and the inverse Fourier transform is

$$\check{f}(x) = \int_{\widehat{\mathbb{R}}^2} f(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

We will use the notation $f(x) \simeq g(x)$ if there exist constants $0 < C_1 \le C_2 < \infty$, independent of x, such that $C_1\, g(x) \le f(x) \le C_2\, g(x)$. Similarly, given two sequences $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$, we write $a_j \simeq b_j$ if there are $C_1 \neq 0$, $C_2 \neq 0$ such that $C_1\, b_j \le a_j \le C_2\, b_j$ for all large j.

We use the convention that the same symbol C or c can be used to denote a different generic constants in different expressions.

1.2 Discrete shearlets

We briefly recall the construction of discrete shearlets (cf. [7,18]).

Let $\psi^{(1)}, \psi^{(2)} \in L^2(\mathbb{R}^2)$ and

$$A_{(1)} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad B_{(2)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The *horizontal* and *vertical shearlets* generated by $\psi^{(1)}$ and $\psi^{(2)}$, respectively, are the collections of functions

$$\Psi^{(\nu)} = \{\psi^{(\nu)}_{j,\ell,k} = \psi^{(\nu)}(B^{\ell}_{(\nu)} A^{j}_{(\nu)}(\cdot - k)) : \, j \geq 0, -2^j \leq \ell \leq 2^j, k \in \mathbb{Z}^2\}, \quad \nu = 1, 2. \tag{3}$$

The reason for the 'horizontal' and 'vertical' adjectives in the name is due to the fact that the anisotropic dilation matrix $A_{(1)}$ dilates along the horizontal direction twice as much as the vertical direction; the opposite is true for the dilation matrix $A_{(2)}$.

For appropriate choices of the generator functions $\psi^{(1)}$ and $\psi^{(2)}$, the shearlet systems are Parseval frames of the subspaces of $L^2(\mathbb{R}^2)$ associated the following cone-shaped regions in the Fourier domain $\widehat{\mathbb{R}}^2$:

$$\mathcal{C}_1 = \left\{(\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \, |\xi_1| > \frac{1}{4}, |\frac{\xi_2}{\xi_1}| \leq 1\right\},$$
$$\mathcal{C}_2 = \left\{(\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \, |\xi_2| > \frac{1}{4}, |\frac{\xi_2}{\xi_1}| > 1\right\}.$$

Namely, for $\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2$, $\nu \in \{1, 2\}$, we say that $\psi^{(\nu)}$ is an *admissible shearlet* if

$$\hat{\psi}^{(\nu)}(\xi_1, \xi_2) = W(\xi_\nu)\, G_{(\nu)}(\xi_1, \xi_2), \tag{4}$$

where $W \in C_0^\infty(\mathbb{R})$ is a function with support $\operatorname{supp}(W) \subset [-\frac{1}{2}, \frac{1}{2}] \setminus [-\frac{1}{16}, \frac{1}{16}]$ and satisfying $\sum_{j \in \mathbb{Z}} |W(2^{-2j}\omega)|^2 = 1$, for a.e. $\omega \in \mathbb{R}$; $G_{(1)}(\xi_1, \xi_2) = V(\frac{\xi_2}{\xi_1})$ and $G_{(2)}(\xi_1, \xi_2) = V(\frac{\xi_1}{\xi_2})$, where $V \in C_0^\infty(\mathbb{R})$ is a function satisfying $\operatorname{supp} V \subset [-1, 1]$ and

$$|V(u-1)|^2 + |V(u)|^2 + |V(u+1)|^2 = 1 \quad \text{for } |u| \leq 1. \tag{5}$$

Then, we have the following result from [7].

Theorem 1 *Let $\psi^{(\nu)}$, $\nu \in \{1,2\}$, be admissible shearlets. Then each corresponding shearlet system $\Psi^{(\nu)}$ is a Parseval frame of $L^2(\mathcal{C}_\nu)^\vee = \{f \in L^2(\mathbb{R}^2)\colon \operatorname{supp} \hat{f} \subset \mathcal{C}_\nu\}$.*

A *Parseval frame of shearlets* of the entire space $L^2(\mathbb{R}^2)$ is obtained by combining the horizontal and vertical shearlet systems together with an appropriate coarse scale system. We refer to [7,18] for more details. However, for the analysis of edge points carried out in this paper, we will only be concerned with the cone-adapted shearlet systems (3) as the coarse scale shearlets play no role in this analysis.

We remark that there is a different construction of discrete shearlets introduced more recently by the authors in [13], which has the advantage of generating a Parseval frame of $L^2(\mathbb{R}^2)$ consisting of smooth shearlets. This property is not true for the construction above where each cone-adapted shearlet system is smooth but the combined shearlet system used to obtain a Parseval frame of $L^2(\mathbb{R}^2)$ is not since the shearlet functions whose frequency supports intersect the lines $\xi_1 = \pm\xi_2$ are truncated. For this alternative construction, the function W in (4) is chosen to be $W \in C_0^\infty(\mathbb{R}^2)$ with support $\operatorname{supp}(W) \subset [-\frac{1}{2}, \frac{1}{2}]^2 \setminus [-\frac{1}{16}, \frac{1}{16}]^2$ and satisfying $\sum_{j\in\mathbb{Z}} |W(2^{-2j}\xi)|^2 = 1$, for a.e. $\xi \in \widehat{\mathbb{R}}^2$. In addition, one needs to include an appropriate set of *boundary* shearlets that are obtained by modifying the functions $\psi^{(\nu)}_{j,\ell,k}$, for $\ell = \pm 2^j$, in order to ensure that all the elements of the system are C_0^∞ in the Fourier domain. We refer to [13] for additional details about this construction. The drawback of this construction is that one cannot assume W to be odd and this assumption is needed in our proof of Theorem 2.

2 Main result

We consider the problem of detecting singularities of functions or distributions on $\mathbb{R}^2$. Our idealized functional model consists of functions of the form $\mathcal{T} = \chi_S$, where $S \subset \mathbb{R}^2$ is a bounded region having a smooth boundary curve ∂S.

In Sec. 4, we will discuss how to extend this analysis to higher dimensional singularities and to delta-type curvilinear singularities.

As indicated in Sec. 1, the continuous shearlet transform is able to describe the location and orientation of the singularity curve ∂S through its asymptotic decay at fine scales. The following result shows that the discrete shearlet coefficients can be similarly used to detect ∂S. Namely, one can find indices k and ℓ such that the discrete shearlet coefficients $\langle \psi^{(\nu)}_{j,\ell,k}, \mathcal{T} \rangle$ satisfy a lower bound condition corresponding to the location and orientation of the singularity curve ∂S.

For the proof of this new result, we will make the following additional assumptions on the shearlet generators (4):

$$(i)\ W \text{ is real, odd;} \tag{6}$$

$$(ii)\ V \text{is real, increasing on } [-1,0], \text{decreasing on } [0,1]; V(x) = 0 \text{ if } x \leq -\frac{3}{4}; \tag{7}$$
$$V(x) = \sqrt{1 - e^{-\frac{1}{x^2}}} \text{ if } x \in [-\frac{5}{8}, 0), V(0) = 1; V(x) = e^{-\frac{1}{2(x-1)^2}} \text{ if } x \in [\frac{3}{8}, 1).$$

The existence of a function V satisfying the assumptions above is proved in the Appendix. We remark that similar (but slightly weaker) assumptions hold for the result valid in the case of continuous shearlets [9].

Theorem 2 *Let $\mathcal{T} = \chi_S$ where $S \subset \mathbb{R}^2$ is a bounded region having a smooth boundary ∂S with non-vanishing curvature. Select admissible shearlets $\psi^{(\nu)}$, $\nu \in \{1,2\}$ satisfying the assumptions* (6)-(7)*. For a large j and each ℓ satisfying $|\ell| \leq \epsilon\, 2^j$ with sufficiently small $\epsilon > 0$, one can find $k_\ell = (k_{1,\ell}, k_{2,\ell}) \in \mathbb{Z}^2$ such that the shearlet coefficients satisfy*

$$|\langle \mathcal{T}, \psi^{(\nu)}_{j,\ell,k_\ell} \rangle| \geq C\, 2^{-\frac{3}{2}j},$$

where $C > 0$ is independent of ν, j, ℓ and k_ℓ.

Remark 1 The lower bound found in Theorem 2 cannot be improved as one can show that, for the curvilinear singularities considered in the theorem, there is a constant $C > 0$, independent of j, ℓ and k, such that $|\beta_{j,\ell,k}| \leq C 2^{-\frac{3}{2}j}$ for all j, ℓ and k.

Remark 2 Theorem 2 refines the classical sparse approximation results of cartoon-like images using shearlets (or curvelets) [1,8]. The most critical step in that argument is that, denoting as $\beta_{j,\ell,k}$ the shearlet coefficients of a cartoon-like image on $\mathbb{R}^2$, then the following estimate holds [8, Thm 1.3]: for j sufficiently large and any $C_1 > 0$, there exists $C_2 > 0$ independent of j such that $\#\{(j,\ell,k) : |\beta_{j,\ell,k}| > C_1 2^{-\frac{3}{2}j}\} \leq C_2 2^j$. One can also show that there is another constant $C > 0$ independent of j such that $|\beta_{j,\ell,k}| \leq C 2^{-\frac{3}{2}j}$. Therefore, our new result implies that, for cartoon-like images with smooth boundaries and nonvanishing Gaussian curvature, for j sufficiently large, there is a constant $C > 0$ such that

$$\#\{(j,\ell,k) : |\beta_{j,\ell,k}| > C 2^{-\frac{3}{2}j}\} \simeq 2^j.$$

Remark 3 The result of Theorem 2 justifies the application of curvelet and shearlet methods to the discrete analysis of edges as done, for instance in [4, 22,25]. So far, the only theoretical results available in the literature to support the application of shearlets and curvelets to the analysis of edges in digital images were asymptotic results based on continuous transforms.

3 Proof of main theorem

To prove our main result, we need first some preparation.

3.1 Localization lemmata and other useful results

In this section, we establish a number of lemmata providing the analytical tools needed to prove Theorem 2. As indicated below, some of these preparatory results, e.g., the localization lemma, are similar to results proved by the authors in the case of continuous shearlets [9]

We start by writing the Fourier transform of $\mathcal{T}$. For $\xi \in \mathbb{R}^2$, let $\rho = |\xi|$, $\Theta(\theta) = (\cos\theta, \sin\theta)$ so that $\xi = \rho\,\Theta(\theta)$. Using the divergence theorem (cf [5, Sec.5.11]), we can express the Fourier transform of $\mathcal{T}$,

$$\widehat{\mathcal{T}}(\xi) = \int_{\mathbb{R}^2} \chi_S(x)\, e^{-2\pi i \langle \xi, x\rangle} dx = \int_S e^{-2\pi i \langle \xi, x\rangle}\, dx,$$

as the curvilinear integral

$$\widehat{\mathcal{T}}(\xi) = -\frac{1}{2\pi i |\xi|} \int_{\partial S} e^{-2\pi i \langle \xi, x\rangle} \Theta(\theta) \cdot \mathbf{n}(x)\, d\sigma(x),$$

or, after converting into polar coordinates,

$$\widehat{\mathcal{T}}(\rho, \theta) = -\frac{1}{2\pi i \rho} \int_{\partial S} e^{-2\pi i \rho \Theta(\theta)\cdot x}\, \Theta(\theta) \cdot \mathbf{n}(x)\, d\sigma(x),$$

where $\mathbf{n}(x)$ is the normal to the boundary curve ∂S. We remark that we used the same idea in [9], where the interested reader will find additional details.

By a smooth partition of unit for ∂S, we can decompose the boundary curve as $\partial S = \bigcup_{m=1}^{M} \alpha_m \partial S_m$, where for each m, $\alpha_m \in C_0^\infty(\partial S_m)$ and ∂S_m can be parametrized locally either as a vertical curve $(f(u), u)$ or a horizontal curve $(u, f(u))$ for $u \in (a, b)$. Here we clarify that for a vertical curve we mean that the slope of the tangent lines of the curve greater then or equal to 2, while for a horizontal curve we mean that the slope of the tangent lines of the curve smaller then or equal to 2. In this sense, $y = x,\ x \in (-1, 1)$ is a horizontal curve while $y = 2x,\ x \in (-1, 1)$ is a vertical curve and should be written as $(\frac{1}{2}y, y)$, $y \in (-2, 2)$. Without loss of generality, in the following we will only consider the case of a vertical curve $(f(u), u)$, for $u \in (a, b)$, since the other case can be handled very similarly. For simplicity of notation, we will denote the boundary curve ∂S even though this might be just a subsection ∂S_m (and similarly we will denote α for α_m).

Additionally, in the following, for all our arguments it will be sufficient to consider the horizontal shearlet system Ψ^1 since the shearlet coefficients associated with the vertical shearlet system Ψ^2 will always yield very rapid decay as a function of j, as the normal vector to the vertical curve is oriented away from the orientation of vertical shearlets (the proof below shows that non-rapid decay only occurs when the shearlet orientation is aligned with the singularity curve).

In Fourier domain, $\psi^{(1)}_{j,\ell,k}$ of Ψ^1, given by (3), can be written as

$$\widehat{\psi}^{(1)}_{j,\ell,k}(\xi) = 2^{-\frac{3j}{2}}\, W(2^{-2j}\xi_1)\, V(2^j\frac{\xi_2}{\xi_1} - \ell)\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}, \tag{8}$$

for $j \geq 0, |\ell| \leq 2^j, k \in \mathbb{Z}^2$. Note that $A_{(1)}^{-j} B_{(1)}^{-\ell} k = (2^{-2j}(k_1 - \ell k_2), 2^{-j}k_2)$.

Let $\beta_{j,\ell,k} = \langle \mathcal{T}, \psi^{(1)}_{j,\ell,k}\rangle$, for $j \geq 0$, $|\ell| \leq 2^j$, $k = (k_1, k_2) \in \mathbb{Z}^2$ where $\psi^{(1)}$ is an admissible shearlet. Using Plancherel Theorem, expression (8) and then converting Cartesian to polar coordinates, we have that

$$\begin{aligned}
\beta_{j,\ell,k} &= \langle \widehat{\mathcal{T}}, \hat{\psi}^{(1)}_{j,l,k}\rangle \\
&= 2^{-\frac{3}{2}j} \int_{\mathbb{R}^2} W(2^{-2j}\xi_1)\, V(2^j\tfrac{\xi_2}{\xi_1} - \ell)\, e^{-2\pi i \xi\cdot 2^{-2j}(k_1-\ell k_2, 2^j k_2)}\, \widehat{\mathcal{T}}(\xi)\, d\xi \\
&= -\frac{2^{-\frac{3}{2}j}}{2\pi i} \int_0^\infty \int_0^{2\pi} \Gamma_{j,\ell}(\rho,\theta) e^{-2\pi i \rho \Theta(\theta)\cdot\left(2^{-2j}(k_1-\ell k_2), 2^{-j}k_2\right)} \\
&\times \int_{\partial S} e^{-2\pi i \rho \Theta(\theta)\cdot x}\, \Theta(\theta)\cdot \mathbf{n}(x)\, d\sigma(x)\, d\theta\, d\rho,
\end{aligned}$$

where $\Gamma_{j,\ell}(\rho,\theta) = W(2^{-2j}\rho\cos\theta)\, V(2^j\tan\theta - \ell)$. We denote by $\Omega_{j,\ell}$ the support of $\Gamma_{j,\ell}$. It is easy to verify that its measure satisfies $|\Omega_{j,\ell}| \leq c\, 2^{3j}$, for some constant $c > 0$ independent of j, ℓ. Next, assuming that ∂S is a vertical curve $(f(u), u)$, $u \in (a, b)$, observing that $\mathbf{n}(x) = (-1, f'(u))$, we have:

$$\begin{aligned}
\beta_{j,\ell,k} &= -\frac{2^{-\frac{3}{2}j}}{2\pi i} \int_0^\infty \int_0^{2\pi} \Gamma_{j,\ell}(\rho,\theta) e^{-2\pi i \rho \Theta(\theta)\cdot\left(2^{-2j}(k_1-\ell k_2), 2^{-j}k_2\right)} \\
&\times \int_a^b e^{-2\pi i \rho\Theta(\theta)\cdot(f(u),u)} \Theta(\theta)\cdot(-1, f'(u))\, \alpha(u)\, du\, d\theta\, d\rho, \\
&= -\frac{2^{-\frac{3}{2}j}}{2\pi i} \int_0^\infty \int_0^{2\pi} \int_a^b \Gamma_{j,\ell}(\rho,\theta) e^{-2\pi i \rho \Theta(\theta)\cdot\left(2^{-2j}(k_1-\ell k_2) - f(u), 2^{-j}k_2 - u\right)} \\
&\times \Theta(\theta)\cdot(-1, f'(u))\, \alpha(u)\, du\, d\theta\, d\rho.
\end{aligned}$$

We fix j, ℓ and $k = (k_1, k_2)$. For $\epsilon > 0$, let

$$U_\epsilon = \{u \in (a,b) : |2^{-j}k_2 - u| < \epsilon\}, \quad V_\epsilon = \{u \in (a,b) : |2^{-j}k_2 - u| \geq \epsilon\}.$$

Using this notation we can decompose the integral $\beta_{j,\ell,k}$ into two integrals, where one is defined for u near $2^{-j}k_2$ and the other one on its complement.

That is, $\beta_{j,\ell,k} = \frac{-1}{2\pi i}(I_{j,\ell,k} + J_{j,\ell,k})$, where

$$\begin{aligned} I_{j,\ell,k} &= 2^{-\frac{3}{2}j} \int_0^\infty \int_0^{2\pi} \int_{U_\epsilon} \Gamma_{j,\ell}(\rho,\theta)\, e^{-2\pi i \rho \Theta(\theta)\cdot \left(2^{-2j}(k_1-\ell k_2)-f(u),2^{-j}k_2-u\right)} \\ &\quad \times \Theta(\theta)\cdot(-1,f'(u))\,\alpha(u)\,du\,d\theta\,d\rho, \\ J_{j,\ell,k} &= 2^{-\frac{3}{2}j} \int_0^\infty \int_0^{2\pi} \int_{V_\epsilon} \Gamma_{j,\ell}(\rho,\theta)\, e^{-2\pi i \rho \Theta(\theta)\cdot \left(2^{-2j}(k_1-\ell k_2)-f(u),2^{-j}k_2-u\right)} \\ &\quad \times \Theta(\theta)\cdot(-1,f'(u))\,\alpha(u)\,du\,d\theta\,d\rho. \end{aligned}$$

Using exactly the same argument as in the proof of Lemma 4.1 in [9], we have the following localization result.

Lemma 1 *For any $N > 0$, there is a constant $C_N > 0$, independent of j, ℓ, k such that*

$$|J_{j,\ell,k}| \le C_N\, 2^{-Nj}.$$

It follows from Lemma 1 that the lower bound of $\beta_{j,\ell,k}$ is only determined by the integral $I_{j,\ell,k}$. Thus in the following we only need to analyze the term $I_{j,\ell,k}$.

Next we introduce a local quadratic approximation of the curve ∂S. Assuming that near the location of interest the curve can be parametrized as $(f(u),u)$, we use the following quadratic approximation near $(f(2^{-j}k_2), 2^{-j}k_2)$: for $u_0 = 2^{-j}k_2$, we set $g(u) = f(u_0) + f'(u_0)(u-u_0) + \frac{f''(u_0)}{2}(u-u_0)^2$. We denote by $\widetilde{\mathcal{T}}$ the modified version of $\mathcal{T}$ and by $\widetilde{S}$ the corresponding modified version of S obtained by replacing the curve L with the curve $G = \{(g(u),u) : \ u \in (a,b)\}$ and let $\widetilde{\beta}_{j,\ell,k} = \langle \widetilde{\mathcal{T}}, \psi^{(1)}_{j,\ell,k}\rangle$.

The following lemma shows that in order to find the lower bound of $\beta_{j,\ell,k}$ it is sufficient to analyze $\widetilde{\beta}_{j,\ell,k}$. A similar idea was originally introduced by the authors in [9] for continuous shearlets. The argument below is more involved.

Lemma 2 *There exists a constant C, independent of j, ℓ, k and $u_0 = 2^{-j}k_2$ such that*

$$|\beta_{j,\ell,k} - \widetilde{\beta}_{j,\ell,k}| \le C 2^{-2j}.$$

Proof: Let $U_j = \{(x_1, x_2),\, |x_2 - u_0| \le 2^{-\frac{7}{8}j}\}$. We have

$$\begin{aligned}\beta_{j,\ell,k} - \widetilde{\beta}_{j,\ell,k} &= \int_{\mathbb{R}^2} \overline{\psi^{(1)}_{j,\ell,k}(x)}(\chi_S(x) - \chi_{\widetilde{S}}(x))\,dx\\ &= D^{(1)}_{j,\ell,k} + D^{(2)}_{j,\ell,k},\end{aligned}$$

where

$$\begin{aligned}D^{(1)}_{j,\ell,k} &= \int_{U_j} \overline{\psi^{(1)}_{j,\ell,k}(x)}(\chi_S(x) - \chi_{\widetilde{S}}(x))\,dx\\ D^{(2)}_{j,\ell,k} &= \int_{U_j^c} \overline{\psi^{(1)}_{j,\ell,k}(x)}(\chi_S(x) - \chi_{\widetilde{S}}(x))\,dx\end{aligned}$$

We first estimate $D^{(1)}_{j,\ell,k}$. Since $\psi^{(1)}$ is an admissible shearlet, it has rapid decay as $|x| \to \infty$. Hence, since $\psi^{(1)}_{j,\ell,k}(x) = 2^{\frac{3j}{2}}\psi(B^{\ell}_{(1)}A^{j}_{(1)}x - k)$, it follows that, for any $N = 0, 1, 2, \cdots$, there is a constant C_N, independent of j, ℓ, k and $x \in \mathbb{R}^2$, such that

$$|\psi_{j,\ell,k}(x)| \le C_N\, 2^{\frac{3j}{2}}(1 + |B^{\ell}_{(1)}A^{j}_{(1)}x - k|)^{-N}.$$

It follows that

$$\begin{aligned}|D^{(1)}_{j,\ell,k}| &\le C\,2^{\frac{3}{2}j}\int_{|u-u_0|<2^{-\frac{7}{8}j}} |f(u) - g(u)|\,du\\ &\le C\,2^{\frac{3}{2}j}\int_{|u-u_0|<2^{-\frac{7}{8}j}} |u - u_0|^3\,du\\ &\le C\,2^{\frac{3}{2}j}2^{-\frac{7}{2}j}\\ &= C\,2^{-2j}.\end{aligned}$$

For $D^{(2)}_{j,\ell,k}$, we observe that the set $S\bigcup\widetilde{S}$ is compact, so we may assume that there is some $M > 0$ such that

$$U_j\bigcap(S\bigcup\widetilde{S}) \subset \{(x_1, x_2),\, |x_2 - u_0| \ge 2^{-\frac{7}{8}j},\, x_1 \in [-M, M]\}.$$

Also we recall that $B^{\ell}A^{j}x - k = \big(2^{2j}x_1 + \ell 2^{j}x_2 - k_1,\, 2^{j}(x_2 - 2^{-j}k_2)\big)$. Thus

$$\begin{aligned}|D^{(2)}_{j,\ell,k}| &\le C_N\, 2^{\frac{3}{2}j}2M\int_{|x_2-u_0|\ge 2^{-\frac{7}{8}j}} (1 + |B^{\ell}_{(1)}A^{j}_{(1)}x - k|)^{-N}\,dx_2\\ &\le C\,2^{\frac{3}{2}j}\int_{|x_2-2^{-jk_2}|\ge 2^{-\frac{7}{8}j}} (2^{j}|x_2 - 2^{-jk_2}|)^{-N}\,dx_2\\ &\le C\,2^{\frac{3}{2}j}\,2^{-Nj}\,2^{\frac{7}{8}(N-1)j}\\ &= 2^{-(\frac{1}{8}N-\frac{5}{8})j}.\end{aligned}$$

The statement follows if one chooses a large $N \geq 21$ so that $\frac{1}{8}N - \frac{5}{8} \geq 2$. □

The following lemma is a special case of the classical method of stationary phase (cf. Proposition 8.3 in [23]).

Lemma 3 *Let ϕ and ψ be smooth functions. Suppose $\phi(x_0) = 0$, $\phi'(x_0) = 0$ and $\phi''(x_0) \neq 0$. If ψ is supported in a sufficiently small neighborhood of x_0, then*

$$I(\lambda) = \int_{\mathbb{R}^2} e^{i\,\lambda\,\phi(x)}\,\psi(x)\,dx \sim \lambda^{-1/2} \sum_{m=0}^{\infty} a_m \lambda^{-\frac{m}{2}}, \tag{9}$$

in the sense that

$$(\frac{d}{d\lambda})^r [I(\lambda) - \lambda^{-\frac{1}{2}} \sum_{m=0}^{M} a_m \lambda^{-\frac{m}{2}}] = O(\lambda^{-r-\frac{M+2}{2}}), \tag{10}$$

as $\lambda \to \infty$.

In particular when $r = 0$, $M = 0$, we have

$$I(\lambda) - a_0\,\lambda^{-\frac{1}{2}} = O(\lambda^{-1}), \tag{11}$$

Remark: As in the remarks at page 337 in [23], we see that $a_0 = \left(\frac{2\pi i}{\phi''(x_0)}\right)^{\frac{1}{2}}\,\psi(x_0)$. Also each coefficient a_m appears in the asymptotic expansion (9) and the bounds occurring in the error term of (10) and hence the error term of (11) only depend on upper bounds of finitely many derivatives of ϕ and ψ in the support of ψ as well as the size of the support of ψ.

Finally we will need the following lemma whose proof is the same as Lemma 4.4 in [9].

Lemma 4 *all $\alpha > 0$, we have*

$$\int_0^B f(u)\,\cos(\pi\alpha u^2)\,du > 0, \quad \int_0^B f(u)\,\sin(\pi\alpha u^2)\,du > 0.$$

3.2 Proof of Theorem 2

By Lemma 1, we only need to examine the integral $I_{j,\ell,k}$ valid when the boundary curve ∂S is localized near the location corresponding to $u = 2^{-j}k_2$.

We can locally approximate f as $f(u) = A_m u^m + O(u^{m+1})$ near $u = 0$, where $A_m \neq 0$ for some $m \geq 1$. By Lemma 1, we only need to consider the curve near $u = 0$ that is, $\{(f(u), u),\ |u| < \epsilon\}$ for a sufficiently small ϵ. From the hypothesis, it follows that there exists u_0 near 0 such that $f'(u_0) \neq 0$, $f''(u_0) \neq 0$. By a translation to $(f(u_0), u_0)$ and possibly a rotation, we may assume that locally the boundary curve ∂S is given by $x_1 = Ax_2^2 + O(x_2^3)$, $|x_2| < \epsilon$, with $A \neq 0$. Further via a dilation $(x_1, x_2) \to (sx_1, sx_2)$ for $s \neq 0$, the equation for L becomes $sx_1 = A(sx_2)^2 + O((sx_2)^3)$, or $x_1 = Asx_2^2 + O(s^2x_2^3)$. We remark that this last transformation results in a dilation of ρ to $s\rho$ in the polar coordinates, and this will only affect our estimates below by a multiplication factor of $\frac{1}{s}$. Thus, if we choose $s = \frac{1}{2A}$, then we may write ∂S locally as $\{(\frac{1}{2}u^2 + O(u^3), u),\ |u| < \epsilon\}$ with $\alpha \in C_0^\infty(-\epsilon, \epsilon)$, $\alpha(0) \neq 0$.

Under the assumption above, we have:

$$
\begin{aligned}
I_{j,\ell,k} &= 2^{-\frac{3}{2}j} \int_0^\infty \int_0^{2\pi} \Gamma_{j,\ell}(\rho, \theta)\, e^{-2\pi i \rho \Theta(\theta)\cdot(2^{-2j}(k_1-\ell k_2), 2^{-j}k_2)} \\
&\quad \times \int_{-\epsilon}^{\epsilon} e^{2\pi i \rho\ \Theta(\theta)\cdot(\frac{1}{2}u^2+O(u^3),t)}\, \Theta(\theta)\cdot(-1, f'(u))\, \alpha(u)\, du\, d\theta\, d\rho \\
&= 2^{\frac{1}{2}j} \int_0^\infty \int_0^{2\pi} \Gamma_{j,\ell}(2^{2j}\rho, \theta)\, e^{-2\pi i 2^{2j}\rho \Theta(\theta)\cdot(2^{-2j}(k_1-\ell k_2), 2^{-j}k_2)} \\
&\quad \times \int_{-\epsilon}^{\epsilon} e^{2\pi i\, 2^{2j}\rho\ \Theta(\theta)\cdot(\frac{1}{2}u^2+O(u^3),u)}\, \Theta(\theta)\cdot(-1, f'(u))\, \alpha(u)\, du\, d\theta\, d\rho \\
&= 2^{\frac{1}{2}j} \int_0^\infty \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right) \Gamma_{j,\ell}(2^{2j}\rho, \theta)\, e^{-2\pi i 2^{2j}\rho \Theta(\theta)\cdot(2^{-2j}(k_1-\ell k_2), 2^{-j}k_2)} \\
&\quad \times \int_{-\epsilon}^{\epsilon} e^{2\pi i\, 2^{2j}\rho\ \Theta(\theta)\cdot(\frac{1}{2}u^2+O(u^3),u)}\, \Theta(\theta)\cdot(-1, f'(u))\, \alpha(u)\, du\, d\theta\, d\rho \\
&= P_1 + P_2,
\end{aligned}
$$

where

$$P_1 = 2^{\frac{1}{2}j} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W(\rho\cos\theta)\, V(2^j \tan\theta - \ell)\, e^{-2\pi i 2^{2j}\rho\Theta(\theta)\cdot(k_1-\ell k_2), 2^{-j}k_2)}$$
$$\times \int_{-\epsilon}^{\epsilon} e^{2\pi i\, 2^{2j}\rho\ \Theta(\theta)\cdot(\frac{1}{2}u^2+O(u^3),u)} \Theta(\theta)\cdot(-1, f'(u))\alpha(u) du\, d\theta\, d\rho$$
$$P_2 = 2^{\frac{1}{2}j} \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} W(\rho\cos\theta)\, V(2^j \tan\theta - \ell)\, e^{-2\pi i 2^{2j}\rho\Theta(\theta)\cdot(k_1-\ell k_2), 2^{-j}k_2)}$$
$$\times \int_{-\epsilon}^{\epsilon} e^{2\pi i\, 2^{2j}\rho\ \Theta(\theta)\cdot(\frac{1}{2}u^2+O(u^3),u)} \Theta(\theta)\cdot(-1, f'(u))\alpha(u) du\, d\theta\, d\rho.$$

Using the hypothesis that W is odd, we have that $P_2 = \overline{P}_1$ so that $I_{j,\ell,k} = 2\Re[P_1]$, where the symbol $\Re[\cdot]$ denotes the real part.

For $f(u) = \frac{1}{2}u^2 + O(u^3)$, valid for u near 0, and for $|2^{-j}k_2| \le \frac{\epsilon}{2}$, we can write

$$f(u) = f(2^{-j}k_2) + f'(2^{-j}k_2)(u - 2^{-j}k_2) + \tfrac{1}{2}f''(2^{-j}k_2)(u - 2^{-j}k_2)^2$$
$$+ O((u - 2^{-j}k_2)^3), \qquad u \text{ near } 2^{-j}k_2.$$

Let $v = u - 2^{-j}k_2$, so that $\alpha(u) = \alpha(v+2^{-j}k_2)$. As a function of v, $\alpha(v+2^{-j}k_2)$ is supported on $\{v : |v + 2^{-j}k_2| < \epsilon\} \subset (-2\epsilon, 2\epsilon)$. By Lemma 2, we may replace $f(u)$ (still for u near $2^{-j}k_2$) by $g(v) = Av^2 + Bv + C$ on $|v| < \frac{\epsilon}{2}$, where $A = \frac{1}{2}f''(2^{-j}k_2) = 1 + O(2^{-j}k_2)$, $B = f'(2^{-j}k_2) = 2^{-j}k_2 + O((2^{-j}k_2)^2)$, $C = f(2^{-j}k_2) = \frac{1}{2}(2^{-j}k_2)^2 + O((2^{-j}k_2)^3)$, when ϵ is sufficiently small. Using this change of variable from u to v, we can now rewrite P_1 as

$$P_1 = 2^{\frac{1}{2}j} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W(\rho\cos\theta)\, V(2^j \tan\theta - \ell)\, e^{-2\pi i\, 2^{2j}\ \rho\, \Theta(\theta)\cdot(k_1-\ell k_2), 2^{-j}k_2)}$$
$$\times \int_{-2\epsilon}^{2\epsilon} e^{i\,\lambda\, g(v)} \psi(v)\, dv\, d\theta\, d\rho,$$

where $\lambda = 2\pi\,\rho\, 2^{2j}$, $\psi(v) = \Theta(\theta)\cdot(-1, f'(v + 2^{-j}k_2))\,\alpha(v + 2^{-j}k_2)$ and

$$g(v) = \Theta(\theta)\cdot(Av^2 + Bv + C,\, v)$$
$$= \cos\theta \left(A(v + \frac{B+\tan\theta}{2A})^2 + C - \frac{(B+\tan\theta)^2}{4A} \right).$$

The equation $g(v) = 0$ gives $v_\theta = -\frac{B+\tan\theta}{2A}$. Let $\phi(v) = g(v) - g(v_\theta)$. Then $\phi(v_\theta) = \phi'(v_\theta) = 0$ and $\phi''(v_\theta) = g''(v_\theta) = 2A \neq 0$. Hence we can rewrite P_1

as

$$P_1 = 2^{\frac{1}{2}j} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W(\rho\cos\theta)\, V(2^j \tan\theta - \ell)\, e^{-2\pi i\, 2^{2j}\, \rho\, \Theta(\theta)\cdot(k_1 - \ell k_2), 2^{-j} k_2)}$$
$$\times\, e^{i\,\lambda\, g(v_\theta)} \int_{-2\epsilon}^{2\epsilon} e^{i\,\lambda\,\phi(v)} \psi(v)\, dv\, d\theta\, d\rho.$$

Now we apply Lemma 3 for $x_0 = v_\theta$ to verify that there is a constant c corresponding to the coefficient a_0 and c_1 in the error term of the asymptotic expansion (11) such that

$$\begin{aligned}
&\int_{-2\epsilon}^{2\epsilon} e^{i\,\lambda\,\phi(v)} \psi(v)\, dv \\
&= a_0(v_\theta)\lambda^{-\frac{1}{2}} + O(\lambda^{-1}) \\
&= c\, 2^{-j}\, \sqrt{i}\, \Theta(\theta)\cdot(-1, f'(v_\theta + 2^{-j}k_2))\, \alpha(v_\theta + 2^{-j}k_2)\, (A\cos(v_\theta)\rho)^{-\frac{1}{2}} \\
&\quad + O(2^{-2j})(\theta,\rho),
\end{aligned}$$

where $|O(2^{-2j})(\theta,\rho)| \le c_1\, 2^{-2j}$.

Since we restrict k_2 according to $|2^{-j}k_2| \le \epsilon$, from the remark following the statement of Lemma 3, it is easy to see that the constant c_1 only depends on the upper bounds of finitely many derivatives of ϕ and ψ in the support of ψ, which is contained in the interval $(-2\epsilon, 2\epsilon)$. In particular, c_1 is independent of θ, ρ, j, ℓ, k_1 and k_2 as long as $|2^{-j}k_2| \le \epsilon$.

Using the last observation, we can write P_1 as $P_1 = P_{11} + P_{12}$, where

$$\begin{aligned}
P_{11} &= c\,\sqrt{i}\, 2^{-\frac{1}{2}j} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W(\rho\cos\theta)\, V(2^j\tan\theta - \ell)\, e^{-2\pi i\, 2^{2j}\, \rho\, \Theta(\theta)\cdot(k_1 - \ell k_2), 2^{-j}k_2)} \\
&\quad \times e^{i\,\lambda\, g(v_\theta)} \Theta(\theta)\cdot(-1, f'(v_\theta))\, \alpha(v_\theta + 2^{-j}k_2)\, (A\cos\theta\rho)^{-\frac{1}{2}}\, d\theta\, d\rho \\
&= c\,\sqrt{i}\, 2^{-\frac{1}{2}j} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W(\rho\cos\theta)\, V(2^j\tan\theta - \ell)\, (A\cos\theta\rho)^{-\frac{1}{2}} \alpha(v_\theta + 2^{-j}k_2) \\
&\quad \times e^{-2\pi i\rho\cos(\theta_t)\left(\frac{1}{4A}(t + 2^jB + \ell)^2 + k_1 - \ell k_2 - 2^{2j}C\right)} \Theta(\theta)\cdot(-1, f'(v_\theta))\, d\theta\, d\rho \\
P_{12} &= 2^{\frac{1}{2}j} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} W(\rho\cos\theta)\, V(2^j\tan\theta - \ell) \\
&\quad \times e^{-2\pi i\rho\cos(\theta_t)\left(\frac{1}{4A}(t + 2^jB + \ell)^2 + k_1 - \ell k_2 - 2^{2j}C\right)} O(2^{-2j})(\theta,\rho)\, d\theta\, d\rho
\end{aligned}$$

In the above expression of P_{11} and P_{12}, let $t = 2^j \tan\theta - \ell$ so that $|t| \le 1$ and $\tan\theta(t) = 2^{-j}(t+\ell)$ so that $\theta_t := \theta(t) = \tan^{-1}(2^{-j}(t+\ell))$. It follows that $2^{2j}\phi(v_{\theta_t}) = 2^{2j}C - k_1 + \ell k_2 - \frac{(2^jB+t+\ell)^2}{4A}$. Thus with this change of variable we have

$$
\begin{aligned}
P_{11} &= c\sqrt{i}\,2^{-\frac{3}{2}j}\int_0^\infty\int_{-1}^1 W(\rho\cos\theta_t)\,V(t)\,\Theta(\theta_t)\cdot(-1,f'(v_{\theta_t}))\,\alpha(v_{\theta_t})\\
&\quad\times (A\cos\theta_t\rho)^{-\frac{1}{2}}\,e^{-2\pi i\rho\cos(\theta_t)\left(\frac{1}{4A}(t+2^jB+\ell)^2+k_1-\ell k_2-2^{2j}C\right)}\cos^2\theta_t\,dt\,d\rho\\
P_{12} &= 2^{-\frac{1}{2}j}\int_0^\infty\int_{-1}^1 W(\rho\cos\theta_t)\,V(t)\,e^{-2\pi i\rho\cos(\theta_t)\left(\frac{1}{4A}(t+2^jB+\ell)^2+k_1-\ell k_2-2^{2j}C\right)}\\
&\quad\times O(2^{-2j})(t,\rho)\,dt\,d\rho
\end{aligned}
$$

From the fact that $|O(2^{-2j})(t,\rho)| \le c_1\,2^{-2j}$, it is straightforward to see that

$$
|P_{12}| \le c\,2^{-\frac{5}{2}j}, \tag{12}
$$

where c is independent of j,ℓ and k.

Since $B = 2^{-j}k_2 + O((2^{-j}k_2)^2)$, we have

$$
2^jB = k_2 + O(2^{-j}k_2)\,k_2 = k_2(1+O(2^{-j}k_2)).
$$

It follows that, for any given ℓ with $|\ell| \le \frac{1}{2}\epsilon 2^j$, we can find an index $k_2(\ell) = k_{2,\ell}$ with $|2^{-j}k_{2,\ell}| < \epsilon$ such that $|2^jB+\ell| < \frac{1}{2}+\epsilon$. In addition, for the same ℓ we choose an index $k_1(\ell) = k_{1,\ell}$ such that $|k_{1,\ell} - \ell k_{2,\ell} - 2^{2j}C| \le \frac{1}{2}$.

Since $|2^{-j}t| \le 2^{-j}$ for all $|t| \le 1$, we have $\tan\theta(t) = 2^{-j}\ell + O(2^{-j})$, $\cos(\theta_t) = (1+(2^{-j}\ell)^2)^{-\frac{1}{2}} + O(2^{-j})$, $\sin(\theta_t) = (2^{-j}\ell)(1+(2^{-j}\ell)^2)^{-\frac{1}{2}} + O(2^{-j})$, and there exists $q_{j,\ell}$ with $|q_{j,\ell}| < \frac{\epsilon}{2}$ such that $|\alpha(q_{j,\ell}) - \alpha(v_{\theta_t})| = O(2^{-j})$ and $\alpha(q_{j,\ell}) \ne 0$.

Similarly we can approximate $|f'(\tilde{q}_{j,\ell}) - f'(v_{\theta_t})| = O(2^{-j})$ and $f'(\tilde{q}_{j,\ell}) \ne 0$. To get the lower bound of P_{11}, after ignoring the higher order decay term we can then replace $\alpha(\theta_t)$ by a constant $\alpha(q_{j,\ell})$, $f'(v_{\theta_t})$ by a constant $f'(\tilde{q}_{j,\ell})$, and $\cos\theta$ by a constant $(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}$. Hence we have

$$
\begin{aligned}
P_{11} &= c\sqrt{i}\,2^{-\frac{3}{2}j}\delta_{j,\ell}A^{-\frac{1}{2}}(1+(2^{-j}\ell)^2)^{-\frac{3}{4}}\int_0^\infty e^{-2\pi i\rho(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}(k_{1,\ell}-\ell k_{2,\ell}-2^{2j}C)}\\
&\quad\times W((1+(2^{-j}\ell)^2)^{-\frac{1}{2}}\rho)\int_{-1}^1 V(t)\,e^{-2\pi i\rho(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}\frac{1}{4A}(t+2^jB+\ell)^2}dt\,\frac{d\rho}{\sqrt{\rho}},
\end{aligned}
$$

where

$$\delta_{j,\ell} = \alpha(q_{j,\ell}) \left(-(1+(2^{-j}\ell)^2)^{-1/2} + f'(\tilde{q}_{j,\ell})(2^{-j}\ell)(1+(2^{-j}\ell)^2)^{-1/2} \right).$$

With the assumptions we made, letting $D = k_{1,\ell} - \ell k_{2,\ell} - 2^{2j}C$ and $p = 2^j B + \ell$, we now write

$$P_{11} = c\sqrt{i}\, 2^{-\frac{3}{2}j}\, \delta_{j,\ell} A^{-\frac{1}{2}} (1+(2^{-j}\ell)^2)^{-\frac{3}{4}} \int_0^\infty e^{-2D\pi i \rho (1+(2^{-j}\ell)^2)^{-\frac{1}{2}}}$$
$$\times W((1+(2^{-j}\ell)^2)^{-\frac{1}{2}}\rho)\, h_{j,\ell,k}(\rho)\, \frac{d\rho}{\sqrt{\rho}},$$

where

$$h_{j,\ell,k}(\rho) = \int_{-1}^{1} \overline{V(t)}\, e^{-2\pi i \rho \left(1+(2^{-j}\ell)^2\right)^{-\frac{1}{2}})\frac{1}{4A}(t+p)^2} dt.$$

A direct computation shows that

$$\begin{aligned}
h_{j,\ell,k}(\rho) &= \int_{-\infty}^{\infty} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}(t+p)^2} V(t) dt \\
&= \int_{-\infty}^{\infty} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2} V(u-p) du \\
&= \int_{0}^{\infty} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2} V(u-p) du \\
&+ \int_{-\infty}^{0} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2} V(u-p) du \\
&= \int_{0}^{\infty} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2} V(u-p) du \\
&- \int_{\infty}^{0} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}v^2} V(-v-p) dv \\
&= \int_{0}^{\infty} e^{-2\pi i \rho\, \frac{1}{4A}(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2} \left(V(-u-p) + V(u-p)\right) du \\
&= a_{j,\ell,k}(\rho) - i\, b_{j,\ell,k}(\rho) \\
&= a_{1,j,\ell,k}(\rho) + a_{2,j,\ell,k}(\rho) - i\left(b_{1,j,\ell,k}(\rho) + b_{2,j,\ell,k}(\rho)\right),
\end{aligned}$$

where $a_{j,\ell,k} = a_{1,j,\ell,k} + a_{2,j,\ell,k}$, $b_{j,\ell,k} = b_{1,j,\ell,k} + b_{2,j,\ell,k}$ and

$$\begin{aligned}
a_{1,j,\ell,k}(\rho) &= \int_0^\infty \cos\left(\tfrac{1}{2A}\pi\rho\,(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\right) V(-u-p)\, du, \\
a_{2,j,\ell,k}(\rho) &= \int_0^\infty \cos\left(\tfrac{1}{2A}\pi\rho\,(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\right) V(u-p)\, du, \\
b_{1,j,\ell,k}(\rho) &= \int_0^\infty \sin\left(\tfrac{1}{2A}\pi\rho\,(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\right) V(-u-p)\, du, \\
b_{2,j,\ell,k}(\rho) &= \int_0^\infty \sin\left(\tfrac{1}{2A}\pi\rho\,(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\right) V(u-p)\, du.
\end{aligned}$$

Using the above notation we can write $P_{11} = \Re[P_{11}] + i\Im[P_{11}]$, where

$$\begin{aligned}
\Re[P_{11}] &= c\, 2^{-\frac{3}{2}j}\delta_{j,\ell}A^{-\frac{1}{2}}(1+(2^{-j}\ell)^2)^{-\frac{3}{4}}\int_0^\infty W((1+(2^{-j}\ell)^2)^{-\frac{1}{2}}\rho)\,\rho^{-\frac{1}{2}}\\
&\quad\times \Big((a_{j,\ell,k}(\rho)+b_{j,\ell,k}(\rho))\cos(2D\pi\rho(1+(2^{-j}\ell)^2)^{-\frac{1}{2}})\\
&\quad+\ (a_{j,\ell,k}(\rho)-b_{j,\ell,k}(\rho))\sin(2D\pi\rho(1+(2^{-j}\ell)^2)^{-\frac{1}{2}})\Big)\,d\rho;\\
\Im[P_{11}] &= c\, 2^{-\frac{3}{2}j}\delta_{j,\ell}A^{-\frac{1}{2}}(1+(2^{-j}\ell)^2)^{-\frac{3}{4}}\int_0^\infty W((1+(2^{-j}\ell)^2)^{-\frac{1}{2}}\rho)\,\rho^{-\frac{1}{2}}\\
&\quad\times \Big((a_{j,\ell,k}(\rho)-b_{j,\ell,k}(\rho))\cos(2D\pi\rho(1+(2^{-j}\ell)^2)^{-\frac{1}{2}})\\
&\quad-\ (a_{j,\ell,k}(\rho)+b_{j,\ell,k}(\rho))\sin(2D\pi\rho(1+(2^{-j}\ell)^2)^{-\frac{1}{2}})\Big)\,d\rho.
\end{aligned}$$

From $I_{j,\ell,k_\ell} = 2\Im[P_1] = 2\Im[P_{11}+P_{12}]$, we have $|I_{j,\ell,k_\ell}| \ge 2(|\Im[P_{11}]| - |\Im[P_{12}]|)$. by (12), we have $|\Im[P_{12}]| \le |P_{12}| \le C\,2^{-\frac{5}{2}j}$. Thus in order to show $|I_{j,\ell,k_\ell}| \ge c_0 2^{-\frac{3}{2}j}$ for some $c_0 > 0$ and all $j > J$, it is enough to show $|\Im[P_{11}]| \ge c\,2^{-\frac{3}{2}j}$ for some $c > 0$ and all $j > J$.

We only consider the case $p \ge 0$ and $D \ge 0$ as the other cases can be discussed similarly. We remark that, when $L = \{(\frac{1}{2}u^2, u),\, |u| < \epsilon\}$, we have $p = 0$ and $D = 0$. In fact, since $f(u) = \frac{1}{2}u^2$, then $B = f'(2^{-j}k_2) = 2^{-j}k_2$, so that $2^jB = k_2$. In this case, one chooses $k_2 = -\ell$ so that $p = 2^jB + \ell = 0$. Further, since $C = f(2^{-j}k_2) = \frac{1}{2}(2^{-j}k_2)^2$, then $D = k_{1,\ell} - \ell k_{2,\ell} - 2^{2j}C = k_{1,\ell} + \frac{1}{2}\ell^2$. Thus we can pick $k_{1,\ell} = -1/2\ell^2$ when ℓ is even to make $D = 0$

Since $D \le \frac{1}{2}$ and $\operatorname{supp} W \subset (0, \frac{1}{2})$, we see that in order to show $|\Im[P_{11}]| \ge c2^{-\frac{3}{2}j}$, it is sufficient to show that, for all $\rho \in (0, \frac{1}{2})$, we have $a_{1,j,\ell,k}(\rho) > b_{1,j,\ell,k}(\rho) > 0$, and $a_{2,j,\ell,k}(\rho) > b_{2,j,\ell,k}(\rho) > 0$. Since $V(v)$ is decreasing for $v \ge 0$, we see that $V(u-p)$ is decreasing on $[p, \infty)$. Similarly, since $V(v)$ is increasing on $[-1, 0]$, we see that $V(-u-p)$ is decreasing on $[0, \infty)$. By Lemma 4, it follows that $b_{1,j,\ell,k}(\rho) > 0$ for $\rho \in (0, \frac{1}{2})$. For $b_{2,j,\ell,k}(\rho)$, we write

$$\begin{aligned}
&b_{2,j,\ell,k}(\rho)\\
&= \int_0^\infty \sin\Big(\tfrac{1}{2A}\pi\rho\,(1+(2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\Big)\,V(u-p)\,du\\
&= \sqrt{\tfrac{A}{2\pi\rho}}\,(1+(2^{-j}\ell)^2)^{\frac{1}{4}}\int_0^\infty \sin(v)\,V((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p)v^{-\frac{1}{2}}\,dv\\
&= \sqrt{\tfrac{A}{2\pi\rho}}\,(1+(2^{-j}\ell)^2)^{\frac{1}{4}}\Big(\int_0^\pi + \int_\pi^\infty\Big)\sin(v)\,V((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p)v^{-\frac{1}{2}}\,dv.
\end{aligned}$$

Since $p < \frac{1}{2} + \epsilon$ and $A = 1 + O(\epsilon)$, $\rho \in (0, \frac{1}{2})$, we see that, for $v \geq \pi$, we have

$$(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} > 2 - \epsilon > 1 + p.$$

It follows that $V(\frac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p) = 0$ when $v \geq \pi$. Combined with the observation that the integrand is positive on $[0, \pi]$, it follows that $b_{2,j,\ell,k}(\rho) > 0$.

To show that $a_{1,j,\ell,k}(\rho) - b_{1,j,\ell,k}(\rho) > 0$ for $\rho \in (0, \frac{1}{2})$, we observe that

$$\begin{aligned}
&a_{1,j,\ell,k}(\rho) - b_{1,j,\ell,k}(\rho) = \\
&= \int_0^\infty \Big(\cos\Big(\tfrac{\pi\rho}{2A}(1 + (2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\Big) - \sin\Big(\tfrac{\pi\rho}{2A}(1 + (2^{-j}\ell)^2)^{-\frac{1}{2}}u^2\Big)\Big)V(-u - p)du \\
&= \sqrt{\tfrac{A}{2\pi\rho}}(1 + (2^{-j}\ell)^2)^{\frac{1}{4}}\int_0^\infty (\cos v - \sin v)V(-(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p)\,v^{-\frac{1}{2}}dv.
\end{aligned}$$

We will now consider separately the integral $\int_0^\infty (\dots)\,dv$ for the cases where $v \in [0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \infty]$.

First we show that there is no contribution for $v > \frac{\pi}{2}$. In fact, since $|2^{-j}\ell| \leq \epsilon$, $\rho \in (0, \frac{1}{2})$ and $A = 1 + O(\epsilon)$, it follows that for sufficiently small $\epsilon > 0$ we have that for $v > \frac{\pi}{2}$

$$\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v > 1.$$

Since $\operatorname{supp} V \subset [-1, 1]$, it follows that, for all $v \geq \frac{\pi}{2}$,

$$V(-(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p) = 0.$$

Thus, for $\rho \in (0, \frac{1}{2})$ we have

$$\int_{\frac{\pi}{2}}^\infty (\cos v - \sin v)\,V(-(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p)\,v^{-\frac{1}{2}}dv = 0.$$

Next, by applying the change of variable $s = \frac{\pi}{2} - v$ to the integral defined over $(\frac{\pi}{4}, \frac{\pi}{2})$, we observe

$$\begin{aligned}
&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos v - \sin v)\,V(-(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p)\,v^{-\frac{1}{2}}dv \\
&= -\int_0^{\frac{\pi}{4}} (\cos s - \sin s)\,V(-(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}(\tfrac{\pi}{2} - s))^{\frac{1}{2}} - p)\,(\tfrac{\pi}{2} - s)^{-\frac{1}{2}}ds \\
&= -\int_0^{\frac{\pi}{4}} (\cos v - \sin v)\,V(-(\tfrac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac{1}{2}}(\tfrac{\pi}{2} - v))^{\frac{1}{2}} - p)\,(\tfrac{\pi}{2} - v)^{-\frac{1}{2}}dv.
\end{aligned}$$

Hence, from the above observations it follows that we can write

$$
\begin{aligned}
&a_{1,j,\ell,k}(\rho) - b_{1,j,\ell,k}(\rho) \\
&= \sqrt{\tfrac{A}{2\pi\rho}}(1+(2^{-j}\ell)^2)^{\frac14}\int_0^{\frac{\pi}{4}}(\cos v - \sin v)\Big(V(-(\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12}v)^{\frac12} - p)\,v^{-\frac12} \\
&\quad - V(-(\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12}(\tfrac{\pi}{2}-v))^{\frac12} - p)\,(\tfrac{\pi}{2}-v)^{-\frac12}\Big)\,dv.
\end{aligned}
$$

Since $V(-u-p)$ is decreasing for $u \ge 0$ and $(v)^{-\frac12} \ge (\frac{\pi}{2}-v)^{-\frac12}$ for $v \in [0, \frac{\pi}{4}]$, we have

$$
V(-(\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12}v)^{\frac12}-p)\,v^{-\frac12} \ge V(-(\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12}(\tfrac{\pi}{2}-v))^{\frac12}-p)\,(\tfrac{\pi}{2}-v)^{-\frac12}
$$

and the inequality is strict for some $v \in (0, \frac{\pi}{4})$. Thus we conclude that

$$
a_{1,j,\ell,k}(\rho) - b_{1,j,\ell,k}(\rho) > 0 \quad \text{for } \rho \in (0, \tfrac12).
$$

Finally we need to show that $a_{2,j,\ell,k}(\rho) - b_{2,j,\ell,k}(\rho) > 0$, for $\rho \in (0, \frac12)$. Again, we have that $|2^{-j}\ell| < \epsilon$, $p < \frac12 + \epsilon$ and $A = 1 + O(\epsilon)$. We first fix a constant $\delta = O(\epsilon) > 0$, uniformly for all $p < \frac12 + \epsilon$, such that $(\frac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac12}v)^{\frac12} - p \ge 1$ when $v \ge \frac{9\pi}{16} + \delta$. Then we write

$$
\begin{aligned}
&a_{2,j,\ell,k}(\rho) - b_{2,j,\ell,k}(\rho) \\
&= \int_0^{\infty}\Big(\cos\Big(\tfrac{\pi\rho}{2A}(1+(2^{-j}\ell)^2)^{-\frac12}u^2\Big) - \sin\Big(\tfrac{\pi\rho}{2A}(1+(2^{-j}\ell)^2)^{-\frac12}u^2\Big)\Big)\,V(u-p)\,du \\
&= \sqrt{\tfrac{A}{2\pi\rho}}(1+(2^{-j}\ell)^2)^{\frac14}\int_0^{\infty}(\cos v - \sin v)\,V\Big((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12}v)^{\frac12} - p\Big)v^{-\frac12}\,dv.
\end{aligned}
$$

We will now consider separately the integral $\int_0^{\infty}(\dots)\,dv$ for the cases where $v \in [0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{9\pi}{16} + \delta]$ and $[\frac{9\pi}{16} + \delta, \infty)$.

As in the above argument, due to the fact that $\operatorname{supp} V \subset [-1, 1]$ one can show that the integral defined for $v > \frac{9\pi}{16} + \delta$ is equal to 0.

For $v \in [\frac{\pi}{2}, \frac{9\pi}{16} + \delta]$, we have that $(\frac{2A}{\pi\rho}\,(1 + (2^{-j}\ell)^2)^{\frac12}v)^{\frac12} - p > \frac34$ for all $\rho \in (0, \frac12)$. Thus, from the definition of V it follows that, for $v \in [\frac{\pi}{2} - \beta, \frac{9\pi}{16} + \delta]$,

$$
V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12}v)^{\frac12} - p\right) \le V(\tfrac34) = e^{-\frac{1}{2(-1/4)^2}} = e^{-8}.
$$

Hence a direct calculation gives

$$\left|\int_{\frac{\pi}{2}}^{\frac{9\pi}{16}+\delta} (\cos v - \sin v)\, V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right)\, v^{-\frac12} dv\right| \le \frac{\pi}{4}\, 2\, e^{-8}$$
$$\le 0.0006,$$

valid for all $\rho \in (0, \frac12)$.

For the integral over $v \in [0, \frac{\pi}{2}]$, we observe that

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} (\cos v - \sin v)\, V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right) v^{-\frac12}\, dv \\
&= \int_0^{\frac{\pi}{4}} (\cos v - \sin v) \Big(V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right) v^{-\frac12} \\
&- V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} (\tfrac{\pi}{2}-v)^{\frac12} - p\right) (\tfrac{\pi}{2}-v)^{-\frac12} \Big)\, dv.
\end{aligned}$$

By the monotonicity properties of V, we see that for $v \in [0, \frac{\pi}{4}]$,

$$\begin{aligned}
V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} (\tfrac{\pi}{2}-v))^{\frac12} - p\right) &\le V(1-\delta-p) \le V(1-\delta-(\tfrac12+\epsilon)) \\
&= V(\tfrac12-\epsilon-\delta))
\end{aligned}$$

and

$$V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right) \ge \min\{V(-(\tfrac12+\epsilon)), V(\tfrac12-\epsilon-\delta)\}.$$

Since by construction $V(-(\frac12+\epsilon)) > V(\frac12-\epsilon-\delta)$, for small ϵ and δ, it follows that for $v \in [0, \frac{\pi}{4}]$ we have

$$V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right) \ge V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} (\frac{\pi}{2}-v))^{\frac12} - p\right).$$

Using this observation and the fact that $v^{-\frac12} \ge (\frac{\pi}{2}-v)^{-\frac12}$ when $v \in [0, \frac{\pi}{4}]$, it follows that

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} (\cos v - \sin v)\, V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right) v^{-\frac12}\, dv \\
&\ge \int_0^{\frac{\pi}{16}} (\cos v - \sin v) \Big(V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} v)^{\frac12} - p\right) v^{-\frac12} \\
&-V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac12} (\tfrac{\pi}{2}-v)^{\frac12} - p\right) (\tfrac{\pi}{2}-v)^{-\frac12} \Big)\, dv.
\end{aligned}$$

We now observed that for $v \in [0, \pi/16]$ (corresponding to the domain of integration) we have that

$$V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac{1}{2}}(\tfrac{\pi}{2}-v))^{\frac{1}{2}} - p \right) \leq V(\tfrac{3}{4}) = e^{-8} \leq 0.0004$$

and

$$\begin{aligned} V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p \right) &\geq \min\{V(-p), V(\tfrac{1}{2}+\delta-p)\} \\ &\geq \min\{V(-(\tfrac{1}{2}+\epsilon)), V(\tfrac{1}{2}+\delta)\} \\ &\geq 0.1, \end{aligned}$$

so that

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} (\cos v - \sin v)\, V\left((\tfrac{2A}{\pi\rho}\,(1+(2^{-j}\ell)^2)^{\frac{1}{2}}v)^{\frac{1}{2}} - p \right) v^{-\frac{1}{2}}\, dv \\ &\geq \tfrac{\pi}{16}(\cos\tfrac{\pi}{16} - \sin\tfrac{\pi}{16})(0.1 - 0.0004) \geq 0.01. \end{aligned}$$

Combining this estimate with the one for the integral valid for $v \in [\frac{\pi}{2}, \frac{9\pi}{16}+\delta]$, we can now conclude that $a_{2,j,\ell,k}(\rho) - b_{2,j,\ell,k}(\rho) > 0$, for $\rho \in (0, \frac{1}{2})$.

This completes the proof. □

4 Extensions

The results presented above extend to other classes of singularities and to higher dimensions.

We start by considering distributions on $\mathbb{R}^2$ containing singularities supported on curves. Similar to [3], we can model such singularities by considering a distribution $\mathcal{P}$ supported along a curve $\lambda : [0,1] \to \mathbb{R}^2$ and defined as

$$\langle \mathcal{P}, \phi \rangle = \int_0^1 \phi(\lambda(s))\, \alpha(s)\, ds, \quad \text{for } \phi \in \mathcal{S}(\mathbb{R}^2), \tag{13}$$

where $\alpha \in C_0^\infty(0,1)$.

To apply our shearlet approach for the detection of the singularity curve, similar to Theorem 2 we need special assumptions on the shearlet generators. In addition to admissibility, we will assume (7) and replace (6) with the following condition:

$$W \text{ is even.} \tag{14}$$

We remark that, under this assumption, it is also possible to adopt the shearlet construction from [13], where W is a function in $C_0^\infty(\mathbb{R}^2)$, since in this case it is easy to satisfy the requirement that W is even.

Theorem 3 *Let $\mathcal{T}$ be given by* (13) *where $\lambda : [0,1] \to \mathbb{R}^2$ is a finite C^2 curve. Select admissible shearlets $\psi^{(\nu)}$, $\nu \in \{1,2\}$ satisfying the assumptions* (14) *and* (7)*. If λ is not linear, for a large j and each ℓ satisfying $|\ell| \le \epsilon\, 2^j$ with sufficiently small ϵ, one can find $k_\ell = (k_{1,\ell}, k_{2,\ell}) \in \mathbb{Z}^2$ such that the shearlet coefficients satisfy*

$$|\langle \mathcal{P}, \psi^{(\nu)}_{j,\ell,k_\ell} \rangle| \ge C\, 2^{\frac{1}{2}j},$$

where $C > 0$ is independent of ν, j, ℓ and k_ℓ.

The proof of Theorem 3 follows the general idea of the proof of Theorem 2, with some important differences. Below, we present a sketch of the proof where we highlight the main differences with respect to the proof of Theorem 2.

4.1 Proof of Theorem 3 (sketch)

By a smooth partition, we can decompose λ into multiple sections that are parametrized either as a vertical or a horizontal curve. We will consider the case of a vertical curve $\lambda_v = \{(f(u), u),\, u \in (a,b)\}$. Hence, for $\phi \in \mathcal{S}(\mathbb{R}^2)$, we need to analyze the distribution $\mathcal{P}$ where

$$\langle \mathcal{P}, \phi \rangle = \int_a^b \phi(f(u), u)\, \alpha(u)\, du,$$

$\alpha \in C_0^\infty(a,b)$. Similar to the proof of Theorem 2, by applying the divergence theorem, we can express the Fourier transform of $\mathcal{P}$ as

$$\widehat{\mathcal{P}}(\rho, \theta) = \int_a^b e^{-2\pi i \rho \Theta(\theta)\cdot(f(u),u)} \alpha(u)\, du.$$

Considering only the analysis using horizontal shearlets, we will examine the terms

$$
\begin{aligned}
&\gamma_{j,\ell,k} = \langle \widehat{\mathcal{P}}, \hat{\psi}^{(1)}_{j,l,k} \rangle \\
&= 2^{-\frac{3}{2}j} \int_{\mathbb{R}^2} W(2^{-2j}\xi_1)\, V(2^j \tfrac{\xi_2}{\xi_1} - \ell)\, e^{-2\pi i \xi \cdot 2^{-2j}(k_1 - \ell k_2, 2^j k_2)} \int_a^b e^{-2\pi i \xi \cdot (f(u),u)} \alpha(u) du\, d\xi \\
&= 2^{-\frac{3}{2}j} \int_0^\infty \int_0^{2\pi} \Gamma_{j,\ell}(\rho,\theta) \int_a^b e^{-2\pi i \rho \Theta(\theta) \cdot \left(2^{-2j}(k_1 - \ell k_2) - f(u), 2^{-j}k_2 - u\right)} \alpha(u)\, du\, d\theta\, \rho\, d\rho,
\end{aligned}
$$

where $\Gamma_{j,\ell}(\rho,\theta) = W(2^{-2j}\rho\cos\theta)\, V(2^j \tan\theta - \ell)$. Similar to the proof of Theorem 2, we can now decompose $\gamma_{j,\ell,k}$ as the sum of two integrals $I_{j,\ell,k} + J_{j,\ell,k}$, where $I_{j,\ell,k}$ is defined for u near $2^{-j}k_2$ and $J_{j,\ell,k}$ for u in the complementary set. Then one can prove a version of Lemma 1 showing that $J_{j,\ell,k}$ has rapid decay, as a function of j, so that we only need to examine the integral $I_{j,\ell,k}$.

Next we introduce a local quadratic approximation of the curve λ_v by letting $u_0 = 2^{-j}k_2$ and $g(u) = f(u_0) + f'(u_0)(u - u_0) + \frac{f''(u_0)}{2}(u-u_0)^2$. We denote by $\widetilde{\mathcal{P}}$ the modified version of $\mathcal{P}$ obtained by replacing the curve λ_v with the curve $\widetilde{\lambda} = \{(g(u), u) :\ u \in (a,b)\}$ and let $\widetilde{\gamma}_{j,\ell,k} = \langle \psi_{j,l,k}, \widetilde{\mathcal{P}} \rangle$. The following statement is similar to Lemma 2, but its proof requires several modifications.

Lemma 5 *There exists a constant C, independent of j, ℓ, k and u_0 such that*

$$
|\gamma_{j,\ell,k} - \widetilde{\gamma}_{j,\ell,k}| \le C 2^{-\frac{1}{2}j}
$$

Proof. By direct computation, we have that

$$
\begin{aligned}
\gamma_{j,\ell,k} - \widetilde{\gamma}_{j,\ell,k} &= \int_a^b \left(\psi^{(1)}_{j,\ell,k}(f(u),u) - \psi^{(1)}_{j,\ell,k}(g(u),u) \right) \alpha(u)\, du \\
&= D_{j,\ell,k} + E_{j,\ell,k},
\end{aligned}
$$

where

$$
D_{j,\ell,k} = \int_{U_\epsilon} \left(\psi^{(1)}_{j,\ell,k}(f(u),u) - \psi^{(1)}_{j,\ell,k}(g(u),u) \right) \alpha(u)\, du
$$

and

$$
E_{j,\ell,k} = \int_{V_\epsilon} \left(\psi^{(1)}_{j,\ell,k}(f(u),u) - \psi^{(1)}_{j,\ell,k}(g(u),u) \right) \alpha(u)\, du.
$$

By Lemma 1 (modified version), for any $N > 0$ there is a constant $C_N > 0$ such that $|E_{j,\ell,k}| \le C_N\, 2^{-Nj}$ and, thus, it only remains to estimate the integral $D_{j,\ell,k}$. We can break up this integral as

$$D_{j,\ell,k} = D^{(1)}_{j,\ell,k} + D^{(2)}_{j,\ell,k},$$

where,

$$\begin{aligned} D^{(1)}_{j,\ell,k} &= \int_{|u-u_0|<2^{-\frac{7}{8}j}} \Big(\psi^{(1)}_{j,\ell,k}(f(u),u) - \psi^{(1)}_{j,\ell,k}(g(u),u)\Big)\, \alpha(u)\, du \\ D^{(2)}_{j,\ell,k} &= \int_{|u-u_0|\ge 2^{-\frac{7}{8}j}} \Big(\psi^{(1)}_{j,\ell,k}(f(u),u) - \psi^{(1)}_{j,\ell,k}(g(u),u)\Big)\, \alpha(u)\, du. \end{aligned}$$

We first estimate $D^{(1)}_{j,\ell,k}$. Recalling that $\psi^{(1)}_{j,\ell,k}(x) = 2^{\frac{3j}{2}} \psi^{(1)}(B^{\ell}_{(1)} A^{j}_{(1)} x - k)$ and using the Mean Value Theorem, we can write:

$$\begin{aligned} D^{(1)}_{j,\ell,k} &= 2^{\frac{3}{2}j} \int_{|u-u_0|<2^{-\frac{7}{8}j}} \Big(\psi^{(1)}\left(B^{\ell} A^{j}(f(u),u) - (k_1,k_2)\right) \\ &\quad - \psi^{(1)}\left(B^{\ell} A^{j}(g(u),u) - (k_1,k_2)\right)\Big)\, \alpha(u)\, du \\ &= C\, 2^{\frac{3}{2}j} \int_{|u-u_0|<2^{-\frac{7}{8}j}} \Big(\psi^{(1)}\left(2^{2j}(f(u) + \ell 2^{-j}u - 2^{-2j}k_1), 2^{j}(u - 2^{-j}k_2)\right) \\ &\quad - \psi^{(1)}\left(2^{2j}(g(u) + \ell 2^{-j}u - 2^{-2j}k_1), 2^{j}(u - 2^{-j}k_2)\right)\Big)\, \alpha(u)\, du \\ &= C 2^{\frac{7}{2}j} \int_{|u-u_0|<2^{-\frac{7}{8}j}} \frac{\partial \psi^{(1)}}{\partial x_1}\left(n(u,j,\ell,k), 2^{j}(u-u_0)\right)(f(u)-g(u))\alpha(u)\, du, \end{aligned}$$

where $n(u,j,\ell,k)$ is a number between $2^{2j}(f(u)+\ell 2^{-j}u - 2^{-2j}k_1)$ and $2^{2j}(g(u)+\ell 2^{-j}u - 2^{-2j}k_1)$. Next, using the observation that $|f(u)-g(u)| \le C|u-u_0|^3$ for u near u_0 and $\psi^{(1)} \in C_0^{\infty}$, we have that for every $N \in \mathbb{N}$ there is a constant C_N such that

$$\begin{aligned} |D^{(1)}_{j,\ell,k}| &\le C_N\, 2^{\frac{7}{2}j} \int_{|x_2-u_0|<2^{-\frac{7}{8}j}} (1 + |n(u,j,\ell,k)| + |2^{j}(u-u_0)|)^{-N} |u-u_0|^3 du \\ &\le C_N\, 2^{\frac{7}{2}j} 2^{-4j} \int_{\mathbb{R}} (1+|v|)^{-N} |v|^3\, dv \\ &\le C_N\, 2^{-\frac{1}{2}j}. \end{aligned}$$

For $D^{(2)}_{j,\ell,k}$, we use again the fact that $\psi_{j,\ell}$ is a rapidly decaying function. Therefore, for every $N \in \mathbb{N}$ there is conatact C_N such that

$$\begin{aligned}
|D^{(2)}_{j,\ell,k}| &\le C_N\, 2^{\frac{3}{2}j} \int_{|u-u_0|\ge 2^{-\frac{7}{8}j}} (1+2^j|u-u_0|)^{-N}\, du \\
&\le C_N\, 2^{\frac{3}{2}j}\, 2^{-Nj} \int_{|u-u_0|\ge 2^{-\frac{7}{8}j}} |u-u_0|^{-N}\, du \\
&\le C_N\, 2^{\frac{3}{2}j}\, 2^{-Nj}\, 2^{\frac{7}{8}(N-1)j} \\
&= C_N\, 2^{-(\frac{1}{8}N-\frac{5}{8})j}.
\end{aligned}$$

The statement follows if one chooses a large $N \ge 9$ so that $\frac{1}{8}N - \frac{5}{8} \ge \frac{1}{2}$. □

The rest of the proof follows closely the proof in Sec. 3.2 with a few important differences. As indicated above, we only need to estimate the integral $I_{j,\ell,k}$ which, in this case, can be expressed as

$$\begin{aligned}
I_{j,\ell,k} = 2^{-\frac{3}{2}j} \int_0^\infty \int_0^{2\pi} &\Gamma_{j,\ell}(\rho,\theta)\, e^{-2\pi i\rho\Theta(\theta)\cdot(2^{-2j}(k_1-\ell k_2),2^{-j}k_2)} \\
&\times \int_{-\epsilon}^{\epsilon} e^{2\pi i\rho\ \Theta(\theta)\cdot(\frac{1}{2}u^2+O(u^3),t)}\, \alpha(u)\, du\, d\theta\, \rho\, d\rho.
\end{aligned}$$

Note the additional factor ρ and the absence of the factor $\Theta(\theta)$ with respect to the similar integral in the proof in Sec. 3.2. Due to this difference, to carry over the same type of argument as in Sec. 3.2, we need to use the assumption that W is an even function. Up to these differences, the rest of the proof is very similar and will not be repeated here.

4.2 3D setting

Similar estimates can be derived in the 3-dimensional setting.

We recall that the shearlet construction extends to functions on $\mathbb{R}^3$. Given $\psi^{(1)}, \psi^{(2)}, \psi^{(3)} \in L^2(\mathbb{R}^2)$ and matrices

$$A_{(1)} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_{(2)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_{(3)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$B_{(1)}^{[(\ell_1,\ell_2)]} = \begin{pmatrix} 1 & \ell_1 & \ell_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{(2)}^{[(\ell_1,\ell_2)]} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_1 & 1 & \ell_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{(3)}^{[(\ell_1,\ell_2)]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell_1 & \ell_2 & 1 \end{pmatrix},$$

the *pyramid-based shearlets* generated by $\psi^{(1)}, \psi^{(2)}$ and $\psi^{(3)}$, respectively, are the collections of functions

$$\Psi^{(\nu)} = \{\psi_{j,\ell,k}^{(\nu)} = \psi^{(\nu)}(B_{(\nu)}^{[(\ell_1,\ell_2)]} A_{(\nu)}^j(\cdot - k)) : j \geq 0, -2^j \leq \ell_1, \ell_2 \leq 2^j, k \in \mathbb{Z}^3\},$$

where $\nu = 1,2,3$. Similar to the two-dimensional case, for $\xi = (\xi_1, \xi_2, \xi_3) \in \widehat{\mathbb{R}}^3$, $\nu \in \{1,2,3\}$, we say that $\psi^{(\nu)}$ is an *admissible shearlet* if

$$\hat{\psi}^{(\nu)}(\xi_1, \xi_2, \xi_3) = W(\xi_\nu)\, G_{(\nu)}(\xi_1, \xi_2, \xi_3),$$

where we have $G_{(1)}(\xi_1, \xi_2, \xi_3) = V(\frac{\xi_2}{\xi_1})V(\frac{\xi_3}{\xi_1})$, $G_{(2)}(\xi_1, \xi_2, \xi_3) = V(\frac{\xi_1}{\xi_2})V(\frac{\xi_3}{\xi_2})$, $G_{(3)}(\xi_1, \xi_2, \xi_3) = V(\frac{\xi_2}{\xi_3})V(\frac{\xi_1}{\xi_3})$, and the functions W and V satisfy the same assumptions as in Sec. 2.

Similar to Sec. 2, we can consider functions of the form $\mathcal{T} = \chi_S$, where S is a compact solid region with a smooth boundary having non-vanishing Gaussian curvature. We have the following result whose proof is similar to Theorem 2 and is omitted.

Theorem 4 *Let $\mathcal{T} = \chi_S$ where $S \subset \mathbb{R}^3$ is a bounded region having a smooth boundary ∂S with non-vanishing Gaussian curvature. Select admissible shearlets $\psi^{(\nu)}$, $\nu \in \{1,2,3\}$ satisfying the assumptions* (6)-(7)*. For a large j and each ℓ_1, ℓ_2 satisfying $|\ell_1|, |\ell_2| \leq \epsilon\, 2^j$ with sufficiently small $\epsilon > 0$, one can find $k_\ell = (k_{1,\ell_1,\ell_2}, k_{2,\ell_1,\ell_2}, k_{3,\ell_1,\ell_2})$ in $\mathbb{Z}^3$ such that the shearlet coefficients satisfy*

$$|\langle \mathcal{T}, \psi_{j,\ell_1,\ell_2,k_\ell}^{(\nu)} \rangle| \geq C\, 2^{-2j},$$

where $C > 0$ is independent of ν, j, ℓ_1, ℓ_2 and k_ℓ.

Similar to the two-dimensional case, we have the following interesting connection to the sparse approximation problem of cartoon-like images.

Let $\beta_{j,\ell_1,\ell_2,k}$ be the 3-dimensional shearlet coefficients of a 3-dimensional cartoon-like image with a smooth boundary having non-vanishing Gaussian curvature [12]. It was proved in [12, Thm. 3.3] that, for j sufficiently large, there exists a constant $C>0$ independent of j such that

$$\#\{(j,\ell_1,\ell_2,k) : |\beta_{j,\ell_1,\ell_2,k}| > C 2^{-2j}\} \le C 2^{2j}.$$

In this case, one can show the existence of the upper bound $|\beta_{j,\ell_1,\ell_2,k}| \le C2^{-2j}$ for all j,ℓ_1,ℓ_2 and k. Therefore our new result here implies that, for j sufficiently large, there is a constant $C>0$ such that

$$\#\{(j,\ell_1,\ell_2,k) : |\beta_{j,\ell_1,\ell_2,k}| > C 2^{-\frac{3}{2}j}\} \simeq 2^{2j}.$$

Appendix

We prove the existence of a function V satisfying the requirements of Sec. 1.2.

Let α be a nonnegative smooth bump function supported on $[-\frac{3}{4},-\frac{5}{8}]$ with $\int_{-\frac{3}{4}}^{-\frac{5}{8}} \alpha(u)\,du = 1$ and let $f_1(x) = \int_{-\infty}^{x} \alpha(u)du$ for $x \in (-\infty,\infty)$. It follows that $f_1 \in C^\infty(\mathbb{R})$ such that $f_1(x)=0$ for $x \le -\frac{3}{4}$ and $f_1(x)=1$ for $x \ge -\frac{5}{8}$. Let $f_2(x) = \sqrt{1-e^{-\frac{1}{x^2}}}$. Hence, define $f(x) = f_1(x)\,f_2(x)$, and $g(x) = \sqrt{1-f^2(x-1)}$. Finally define $V(x)$ by $V(x)=f(x)$ on $[-1,0)$, $V(x)=g(x)$ on $[0,1]$ and $V(x)=0$ otherwise. It is easy to check that $V \in C_0^\infty([-1,1])$ and satisfies the identity (5). Furthermore, V satisfies the following additional properties:

(i) V is increasing on $[-1,0]$ with $V(x)=0$ for $x \le -\frac{3}{4}$ and $V(x) = \sqrt{1-e^{-\frac{1}{x^2}}}$ for $x \in [-\frac{5}{8},0)$.

(ii) V is decreasing on $[0,1]$ with $V(0)=1$ and $V(x) = e^{-\frac{1}{2(x-1)^2}}$ for $x \in [\frac{3}{8},1)$.

References

1. E. J. Candès and D. L. Donoho, *New tight frames of curvelets and optimal representations of objects with C^2 singularities*, Comm. Pure Appl. Math., 56(2) (2004), pp. 219–266.
2. E. J. Candès and D. L. Donoho, *Continuous curvelet transform: I. Resolution of the wavefront set*, Appl. Comput. Harmon. Anal., 19 (2005), pp. 162–197.

3. D.L. Donoho and Kutyniok, *Microlocal analysis of the geometric separation problem*, Comm. Pure Appl. Math., 66 (2013), pp. 1–47.
4. M.A. Duval-Poo, F. Odone, and E. De Vito, *Edges and corners with shearlets*, IEEE Trans. Image Process. 24(11) (2015), pp. 3768–3780.
5. L. Evans and R. Gariepy, *Measure theory and fine properties of functions*, CRC Press, 1992.
6. P. Grohs, *Continuous shearlet frames and resolution of the wavefront set*, Monatshefte för Mathematik, 164(4) (2011), pp. 393–426.
7. K. Guo, G. Kutyniok, and D. Labate, *Sparse multidimensional representations using anisotropic dilation and shear operators*, in: Wavelets and Splines: Athens 2005 (Proceedings of the International Conference on the Interactions between Wavelets and Splines. Athens, GA, May 16–19, 2005), G. Chen and M. Lai (eds.)
8. K. Guo and D. Labate, *Optimally sparse multidimensional representation using shearlets*, SIAM J Math. Anal., 39 (2007), pp. 298–318.
9. K. Guo, and D. Labate, *Characterization and analysis of edges using the continuous shearlet transform*, SIAM on Imaging Sciences, 2 (2009), pp. 959–986.
10. K. Guo, and D. Labate, *Analysis and detection of surface discontinuities using the 3D continuous shearlet transform*, Appl. Comput. Harmon. Anal., 30 (2011), pp. 231–242.
11. K. Guo, and D. Labate, *Characterization of piecewise-smooth surfaces using the 3D continuous shearlet transform*, J. Fourier Anal. Appl., 18 (2012), pp. 488–516.
12. K. Guo, and D. Labate, *Optimally sparse representations of 3D data with C^2 surface singularities using Parseval frames of shearlets*, SIAM J Math. Anal., 44 (2012), pp. 851–886.
13. K. Guo, and D. Labate, *The construction of smooth Parseval frames of shearlets*, Math. Model. Nat. Phenom., 8(1) (2013), pp. 82–105.
14. K. Guo, and D. Labate, *Geometric separation of singularities using combined multiscale dictionaries*, J. Fourier Anal. Appl., 21(4) (2015), pp. 667–693.
15. S. Jaffard, *Pointwise smoothness, two-microlocalization and wavelet coefficients*, Publications Matematiques, 35 (1991), pp. 155–168.
16. S. Jaffard, Y. Meyer, *Wavelet methods for pointwise regularity and local oscillations of functions*, Memoirs of the AMS 123 n.587 (1996).
17. G. Kutyniok and D. Labate, *Resolution of the wavefront set using continuous shearlets*, Trans. Amer. Math. Soc., 361 (2009), pp. 2719–2754.
18. G. Kutyniok and D. Labate, *Introduction to shearlets*, in: *Shearlets: Multiscale analysis for multivariate data*, Editors: G. Kutyniok and D. Labate, Birkhäuser, Boston, 2012
19. G. Kutyniok and W. Lim, *Compactly supported shearlets are optimally sparse*, Journal of Approximation Theory, 163 (2011), pp. 1564-1589.
20. G. Kutyniok, P. Petersen, *Classification of edges using compactly supported shearlets*, to appear in Appl. Comput. Harmon. Anal. (2016) doi:10.1016/j.acha.2015.08.006.

21. D. Labate, W.-Q Lim, G. Kutyniok, and G. Weiss, *Sparse multidimensional representation using shearlets*, in Wavelets XI, edited by M. Papadakis, A. F. Laine, and M. A. Unser, SPIE Proc. 5914 (2005), SPIE, Bellingham, WA, 2005, pp. 254-262.
22. R. Reisenhofer, J. Kiefer, E. J. King, *Shearlet-based detection of flame fronts*, Experiments in Fluids 57(3) (2016), pp. 1–14.
23. E. M. Stein, *Harmonic analysis*, Princeton University Press, 1993.
24. D. Vera, *Democracy of shearlet frames with applications*, J. Approx. Theory 213 (2017), pp. 23–49.
25. S. Yi, D. Labate, G. R. Easley, and H. Krim, *A shearlet approach to edge analysis and detection*, IEEE Trans. Image Process. 18 (2009), pp. 929–941.