Edge analysis and identification using the continuous shearlet transform

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Abstract

It is well known that the continuous wavelet transform has the ability to identify the set of singularities of a function or distribution f. It was recently shown that certain multidimensional generalizations of the wavelet transform are useful to capture additional information about the geometry of the singularities of f. In this paper, we consider the continuous shearlet transform, which is the mapping $f \in L^2(\mathbb{R}^2) \to \mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{ast} \rangle$, where the analyzing elements ψ_{ast} form an affine system of well localized functions at continuous scales $a > 0$, locations $t \in \mathbb{R}^2$, and oriented along lines of slope $s \in \mathbb{R}$ in the frequency domain. We show that the continuous shearlet transform allows one to exactly identify the location and orientation of the edges of planar objects. In particular, if $f = \sum_{n=1}^{N} f_n \chi_{\Omega_n}$ where the functions f_n are smooth and the sets Ω_n have smooth boundaries, then one can use the asymptotic decay of $\mathcal{SH}_\psi f(a,s,t)$, as $a \to 0$ (fine scales), to exactly characterize the location and orientation of the boundaries $\partial\Omega_n$. This improves similar results recently obtained in the literature and provides the theoretical background for the development of improved algorithms for edge detection and analysis.

Key words: Analysis of singularities; continuous wavelets; curvelets; directional wavelets; edge detection; shearlets; wavelets
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1 Introduction

Let $\mathcal{A}$ be the *affine group* on $\mathbb{R}^2$, consisting of the pairs $(M,t) \in GL_2(\mathbb{R}) \times \mathbb{R}^2$, with group operation $(M,t)\cdot(M',t') = (MM', t+Mt')$. The *affine systems* generated by $\psi \in L^2(\mathbb{R}^2)$ are obtained from the action of the quasi regular representation of $\mathcal{A}$ on ψ, and are the collections of functions of the form

$$\Big\{\psi_{M,t}(x) = |\det M|^{-\frac{1}{2}}\psi(M^{-1}(x-t)) :\ (M,t) \in \mathcal{A}\Big\}.$$

Let $\Lambda \subset \mathcal{A}$ be defined by $\{(M,t) : M \in G, t \in \mathbb{R}^2\}$ where G is a subset of $GL_2(\mathbb{R})$. If there is a function $\psi \in L^2(\mathbb{R}^2)$ such that any $f \in L^2(\mathbb{R}^2)$ can be recovered via the reproducing formula

$$f = \int_{\mathbb{R}^2}\int_G \langle f, \psi_{M,t}\rangle\, \psi_{M,t}\, d\lambda(M)\, dt,$$

where λ is a measure on G, then ψ is a *continuous wavelet* with respect to Λ. In this case, the *continuous wavelet transform* (with respect to Λ) is the mapping

$$f \to \mathcal{W}_\psi f(M,t) = \langle f, \psi_{M,t}\rangle, \quad (M,t) \in \Lambda.$$

There is a variety of examples of wavelet transforms [18,24]. The simplest case is when the matrices M in the dilations group G have the form $M(a) = a\, I$, where $a > 0$ and I is the identity matrix. In this situation, the continuous wavelet transform of f,

$$\mathcal{W}_\psi f(a,t) = a^{-1}\int_{\mathbb{R}^2} f(x)\, a^{-1}\, \overline{\psi(a^{-1}(x-t))}\, dx, \tag{1}$$

is *isotropic* since the dilation factor a is the same for all coordinate directions. Notice that this is the “standard” continuous wavelet transform used in a large part of the wavelet literature.

A remarkable property of the continuous wavelet transform $\mathcal{W}_\psi(a,t)$ is its ability to identify the singularities of a signal f. In fact, using an appropriate, well-localized continuous wavelet ψ, the continuous wavelet transform will signal the location of the singularities of f through its asymptotic decay at fine scales. Namely, if f is smooth apart from a discontinuity at a point x_0, then the isotropic wavelet transform $\mathcal{W}_\psi f(a,t)$ will decay rapidly as a approaches 0, unless t is near x_0 [15,22]. Thus, the locations for the slow decay of $\mathcal{W}_\psi f(a,t)$, as $a \to 0$, can be used to resolve the *singular support* of f, that is, the set of points where f is not regular.

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However, the isotropic wavelet transform is unable to provide additional information about the *geometry* of the set of singularities of f. In many situations, such as in the study of the propagation of singularities associated with PDEs or in image processing applications concerning the detection and analysis of edges, it is useful to not only identify the location of singularities, but also their geometrical properties, such as, for example, the orientation and curvature of discontinuity curves. For that, one needs to consider alternative transforms which are able to capture such features. In particular, this can be achieved by considering continuous wavelet transforms which take full advantage of the affine group structure associated with the affine systems.

In some previous work [8,16,17], the authors and their collaborators have shown that the continuous wavelet transforms associated with the *shearlet groups* have exactly such desirable properties. For each $0 < \alpha < 1$, a shearlet group is a subgroup of the affine group $\mathcal{A}$ consisting of the elements (M_{as}, t), where $M_{as} = \begin{pmatrix} a & -a^\alpha s \\ 0 & a^\alpha \end{pmatrix}$, $a > 0$, $s \in \mathbb{R}$, and $t \in \mathbb{R}^2$. Notice that M_{as} is the product of the matrices $B_s\, A_a$, where $A_a = \begin{pmatrix} a & 0 \\ 0 & a^\alpha \end{pmatrix}$ is an anisotropic dilation matrix, and $B_s = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$ is a non-expanding matrix called *shear matrix*. The continuous wavelet transform associated with the shearlet group is called the *continuous shearlet transform*. For each $0 < \alpha < 1$, this is the mapping

$$f \to \mathcal{SH}_\psi^\alpha f(a, s, t) = \langle f, \psi_{ast} \rangle,$$

on the transform domain $\{(a, s, t) :\ a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2\}$, where the analyzing elements ψ_{ast}, called *shearlets*, are the affine functions

$$\psi_{ast}(x) = |\det M_{as}|^{-\frac{1}{2}} \psi(M_{as}^{-1}(x - t)),$$

and form a collection of well localized functions at various scales, orientations and locations, controlled by the variables a, s and t, respectively. Hence, unlike the isotropic wavelet transform $\mathcal{W}_\psi f(a, t)$, given by (1), the shearlet transform depends on three variables: the *scale* a, the *shear* s and the *translation* t. This transform has several similarities to the continuous curvelet transform, previously introduced by Candès and Donoho [3]. Notice however that analyzing elements of the curvelet transform do not form an affine system.

By combining the localization properties of the traditional (isotropic) wavelet transform with the ability to capture the geometry of two-dimensional functions, the continuous shearlet transform, as well as the curvelet transform, have been shown to be particularly effective in describing both the location and the orientation of distributed singularities. Indeed these transforms decay rapidly at points where a function f is regular. This is not necessarily true for the points where f is discontinuous.

Recall that these properties play a basic role in the fundamental sparsity result of curvelets [2], where it is proved that curvelet coefficients of a function f have rapid decay for locations where f is regular. For t on a discontinuity curve, and normal orientation θ, the curvelet coefficients decay as

$$|\langle f, \psi_{j,\theta,t} \rangle| \leq C\, 2^{-\frac{3}{4}j}, \quad j \to \infty.$$

This upper bound estimate, however, does not guarantee that the curvelet coefficients have necessarily slow decay rate (as $2^{-\frac{3}{4}j}$) at t. To effectively "detect" the discontinuity, one has to show that an appropriate lower bound estimates also holds (the study of such lower estimates will be a significant part of our results below).

The goal of this paper is to use the continuous shearlet transform to provide a very precise description of "edge" discontinuities. Indeed we will show that, if a 2-dimensional function f consists of several smooth regions Ω_n, $n = 1, \ldots, N$, separated by smooth boundaries γ_n, at which jumps occur, then the continuous shearlet transform $\mathcal{SH}_\psi^\alpha f(a, s, t)$ will signal the location and orientation of the boundaries through its asymptotic decay at fine scales. For example, when $\alpha = 1/2$, the locations and orientations associated with the slow decay at fine scale,

$$\mathcal{SH}_\psi^{\frac{1}{2}} f(a, s, t) \sim a^{\frac{3}{4}} \quad a \to 0,$$

are exactly those of the boundary curves γ_n.

This is a refinement of the results from [16], where it is proved that the continuous shearlet transform exactly characterizes the wavefront set of a distribution. [2] More precisely, it is proved that the wavefront set of a distribution f is the closure of the set of (s, t) near which $\mathcal{SH}_\psi^{\frac{1}{2}} f(a, s, t)$ is not of rapid decay as $a \to 0$.

Our study is motivated by image applications, where f is used to model an image, and the curves γ_n are the edges of the image f. The results presented in this paper, combined with the mathematical structure of the shearlet transform which is amenable to efficient numerical implementations, makes the shearlet based approach very effective for the design of improved edge detection and analysis algorithms. This is confirmed by preliminary numerical tests conducted by one of the authors and his collaborators [25]. A more detailed study of the numerical applications of the shearlet approach to edge detection will appear in a separate paper.

The paper is organized as follows. In Section 2 we recall the definition and basic properties of the shearlet transform. In Section 3 we examine in detail the shearlet transform of the characteristic function of a disc (Section 3.1) and of more general convex bodies with nonvanishing curvature (Section 3.2).

[2] A similar result is obtained by using the continuous curvelet transform [3].

In Section 4, we extend these results to the case of more general compactly supported functions.

2 The shearlet transform

We recall the basic properties of the shearlet transform, which was introduced in [16]. Consider the subspace of $L^2(\mathbb{R}^2)$ given by $L^2(C)^\vee = \{f \in L^2(\mathbb{R}^2) : \text{supp}\, \hat{f} \subset C\}$, where C is the "horizontal cone" in the frequency plane:

$$C = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geq 2 \text{ and } |\tfrac{\xi_2}{\xi_1}| \leq 1\}.$$

The following proposition, which is a simple generalization of a result from [16], provides sufficient conditions on the function ψ for obtaining a reproducing system of continuous shearlets on $L^2(C)^\vee$.

Proposition 2.1 *Let $0 < \alpha < 1$, and consider the shearlet group $\Lambda_\alpha = \{(M_{as}, t) : 0 < a \leq \frac{1}{4}, -\frac{3}{2} \leq s \leq \frac{3}{2}, t \in \mathbb{R}^2\}$, where $M_{as} = \begin{pmatrix} a & -a^\alpha s \\ 0 & a^\alpha \end{pmatrix}$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_2 \neq 0$, let ψ be given by*

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(\tfrac{\xi_2}{\xi_1}),$$

where:

(i) $\psi_1 \in L^2(\mathbb{R})$ satisfies the Calderòn condition

$$\int_0^\infty |\hat{\psi}(a\xi)|^2 \,\frac{da}{a^{2\alpha}} = 1 \qquad \textit{for a.e. } \xi \in \mathbb{R},$$

and supp $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$;

(ii) $\|\psi_2\|_{L^2} = 1$ and supp $\hat{\psi}_2 \subset [-1, 1]$.

Then, for all $f \in L^2(C)^\vee$,

$$f(x) = \int_{\mathbb{R}^2} \int_{-\frac{3}{2}}^{\frac{3}{2}} \int_0^{\frac{1}{4}} \langle f, \psi_{ast}\rangle\, \psi_{ast}(x)\, \frac{da}{a^{2+2\alpha}}\, ds\, dt,$$

with convergence in the L^2 sense, where $\psi_{ast}(x) = |\det M_{as}|^{-\frac{1}{2}} \psi(M_{as}^{-1}(x - t))$.

If the assumptions of Proposition 2.1 are satisfied, we say that the functions

$$\Psi = \{\psi_{ast} : 0 < a \leq \tfrac{1}{4}, -\tfrac{3}{2} \leq s \leq \tfrac{3}{2}, t \in \mathbb{R}^2\}$$

are *continuous shearlets* for $L^2(C)^\vee$ and that the corresponding mapping from $f \in L^2(C)^\vee$ into $\mathcal{SH}_\psi^\alpha f(a, s, t) = \langle f, \psi_{ast}\rangle$ is the *continuous shearlet transform* on $L^2(C)^\vee$ with respect to Λ_α.

Observe that, in the frequency domain, a shearlet ψ_{ast} has the form:

$$\hat{\psi}_{ast}(\xi_1, \xi_2) = a^{\frac{1+\alpha}{2}} \hat{\psi}_1(a\,\xi_1)\, \hat{\psi}_2(a^{\alpha-1}(\tfrac{\xi_2}{\xi_1} - s))\, e^{-2\pi i \xi t}.$$

This shows each function $\hat{\psi}_{ast}$ has support:

$$\operatorname{supp} \hat{\psi}_{ast} \subset \{(\xi_1, \xi_2) : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}],\, |\tfrac{\xi_2}{\xi_1} - s| \le a^{1-\alpha}\}.$$

That is, its frequency support is a pair of trapezoids, symmetric with respect to the origin, oriented along a line of slope s. The support becomes increasingly thin as $a \to 0$. This is illustrated in Figure 1.

The case $\alpha = \frac{1}{2}$ corresponds to the so-called *parabolic scaling*, and plays a special role in the sparsity result of shearlets and curvelets (see [2,11]). Unlike the sparsity result, however, we will show that the shearlet transform $\mathcal{SH}_\psi^\alpha$ for all $\alpha \in (0, 1)$ is effective to detect edges. Indeed we will see that the case $\alpha \neq \frac{1}{2}$ provides some interesting observations (see remarks after Theorem 3.3).

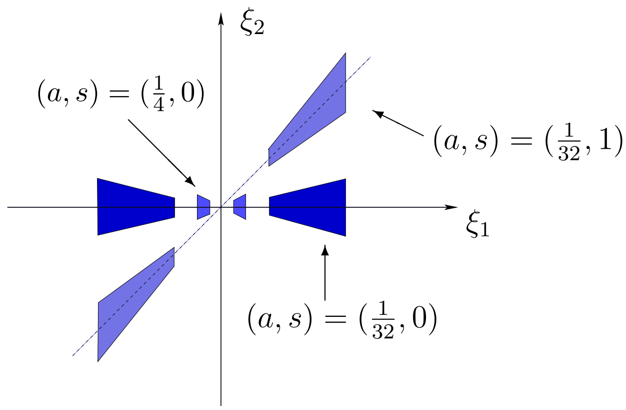

Fig. 1. Support of the shearlets $\hat{\psi}_{ast}$ (in the frequency domain) for different values of a and s.

There are a variety of examples of functions ψ_1 and ψ_2 satisfying the assumptions of Proposition 2.1. In particular, one can find a number of such examples with the additional property that $\hat{\psi}_1, \hat{\psi}_2 \in C_0^\infty$ [10,16]. For the kind of applications which will be described in this paper, some further additional properties are needed. In particular, we will require that $\hat{\psi}_1$ is a smooth odd function, and that $\hat{\psi}_2$ is an even smooth function which is decreasing on $[0, 1)$. We refer to Appendix A for details about their constructions. In the following, throughout the paper, we will assume that the functions ψ_1 and ψ_2 satisfy these assumptions.

As shown by Proposition 2.1, the continuous shearlet transform $\mathcal{SH}_\psi^\alpha$ provides a reproducing formula only for functions in a proper subspace of $L^2(\mathbb{R}^2)$. To extend the transform to all $f \in L^2(\mathbb{R}^2)$, we can introduce a similar transform to deal with the functions supported on the "vertical cone":

$$C^{(v)} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \geq 2 \text{ and } |\tfrac{\xi_2}{\xi_1}| > 1\}.$$

Specifically, let

$$\hat{\psi}^{(v)}(\xi) = \hat{\psi}^{(v)}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_2)\, \hat{\psi}_2(\tfrac{\xi_1}{\xi_2}),$$

where $\hat{\psi}_1$, $\hat{\psi}_2$ satisfy the same assumptions as in Proposition 2.1, and consider the shearlet group $\Lambda_\alpha^{(v)} = \{(N_{as}, t) : 0 < a \leq 1, -\frac{3}{2} \leq s \leq \frac{3}{2}, t \in \mathbb{R}^2\}$, where $N_{as} = M_{as}^T$. Then it is easy to verify that the functions

$$\Psi^{(v)} = \{\psi_{ast}^{(v)} = |\det N_{as}|^{-\frac{1}{2}} \psi^{(v)}(N_{as}^{-1}(x - t)) : 0 < a \leq \tfrac{1}{4}, -\tfrac{3}{2} \leq s \leq \tfrac{3}{2}, t \in \mathbb{R}^2\}$$

are continuous shearlets for $L^2(C^{(v)})^\vee$. The corresponding transform $\mathcal{SH}_\psi^{(v),\alpha} f(a, s, t) = \langle f, \psi_{ast}^{(v)} \rangle$ is the continuous shearlet transform on $L^2(C)^\vee$ with respect to $\Lambda_\alpha^{(v)}$. Finally, by introducing an appropriate window function W, we can represent the functions with frequency support on the set $[-2, 2]^2$ as

$$f = \int_{\mathbb{R}^2} \langle f, W_t \rangle W_t \, dt,$$

where $W_t(x) = W(x-t)$. As a result, we can represent any function $f \in L^2(\mathbb{R}^2)$ with respect of the full system of shearlets, as consisting of the horizontal shearlets ψ_{ast}, the vertical shearlets $\psi_{ast}^{(v)}$, and the coarse-scale isotropic functions W_t. We refer to [16] for more details about this representation. For our purposes, it is only the behavior of the fine-scale shearlets that matters. Indeed, in the following, we will apply the continuous shearlet transforms $\mathcal{SH}_\psi^\alpha$ and $\mathcal{SH}_\psi^{(v),\alpha}$, at fine scales ($a \to 0$), to resolve and precisely describe the boundaries of certain planar regions.

Finally, we recall that the *discrete* analogs of the continuous shearlet transform and continuous shearlets have been developed by the authors and their collaborators in [9–13,17].

3 Shearlet analysis of step edges

We will start by considering the behavior of the continuous shearlet transform of functions of the form $f = \chi_C$, where χ_C is the characteristic function of a planar region. This can be seen as a model of an image containing a *step* edge along the curve described by the boundary of C. More general functions will be examined in Section 4.

Since the properties of the "horizontal" continuous shearlet transform $\mathcal{SH}_\psi^\alpha$ are essentially identical to those of the "vertical" continuous shearlet transform $\mathcal{SH}_\psi^{(v),\alpha}$, in the following, it will be sufficient to examine the horizontal transform only.

3.1 Shearlet analysis of circular edges

We start with the case of circular edges. Let $D(R, x_0)$ be the ball in $\mathbb{R}^2$ of radius $R > 0$ and center x_0, and $B_{R,x_0}(x)$ be the characteristic function of $D(R, x_0)$. We will examine the asymptotic decay of the continuous shearlet transform $\mathcal{SH}_\psi^\alpha B_{R,x_0}(a, s, t) = \langle B_{R,x_0}, \psi_{ast}\rangle$, for $a \to 0$.

Observe first that, by a change of variables, we have that

$$\mathcal{SH}_\psi^\alpha B_{R,x_0}(a, s, t) = \mathcal{SH}_\psi^\alpha B_{R,0}(a, s, t - x_0).$$

Thus, there will be no loss of generality in assuming $x_0 = 0$. For simplicity of notation, in the following we will denote $B_R = B_{R,0}$.

The following results shows that the continuous shearlet transform of B_R exactly characterizes the set $\partial D(0, R)$, i.e., the boundary of the disc $D(R, 0)$. We start first with the case $\alpha = \frac{1}{2}$.

Theorem 3.1 *Let $t \in P = \{t = (t_1, t_2) \in \mathbb{R}^2 : |\frac{t_2}{t_1}| \leq 1\}$.*

If $t = t_0 = R\,(\cos\theta_0, \sin\theta_0)$, for some $|\theta_0| \leq \frac{\pi}{4}$, then

$$\lim_{a\to 0^+} a^{-\frac{3}{4}}\, \mathcal{SH}_\psi^{\frac{1}{2}} B_R(a, \tan\theta_0, t_0) \neq 0. \tag{2}$$

If $t = t_0$ and $s \neq \tan\theta_0$, or if $t \notin \partial D(0, R)$, then

$$\lim_{a\to 0^+} a^{-\gamma}\, \mathcal{SH}_\psi^{\frac{1}{2}} B_R(a, s, t) = 0, \qquad \text{for all } \gamma > 0. \tag{3}$$

For general $\alpha \in (0, 1)$, we have the following:

Theorem 3.2 *Let $t \in P = \{t = (t_1, t_2) \in \mathbb{R}^2 : |\frac{t_2}{t_1}| \leq 1\}$ and $\beta(\alpha)$ be*

$$\beta(\alpha) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } 0 < \alpha < \frac{1}{2} \\ \frac{\alpha+1}{2} & \text{if } \frac{1}{2} \leq \alpha < 1. \end{cases}$$

If $t = t_0 = R\,(\cos\theta_0, \sin\theta_0)$, for some $|\theta_0| \leq \frac{\pi}{4}$, then

$$\lim_{a\to 0^+} a^{-\beta(\alpha)}\, \mathcal{SH}_\psi^\alpha B_R(a, \tan\theta_0, t_0) \neq 0. \tag{4}$$

If $t = t_0$ *and* $s \neq \tan\theta_0$*, or if* $t \notin \partial D(0,R)$*, then*

$$\lim_{a \to 0^+} a^{-\gamma}\, \mathcal{SH}_\psi^\alpha B_R(a,s,t) = 0, \qquad \textit{for all } \gamma > 0. \tag{5}$$

Hence, the continuous shearlet transform $\mathcal{SH}_\psi^\alpha B_R(a,s,t)$ has "slow" decay only for $t = t_0$ on ∂D, the boundary of the disc $D(0,R)$, and for values of the shear variable $s = \tan\theta_0$ corresponding to the normal orientation to ∂D at t_0. For all other values of t and s the decay is fast. This behavior is illustrated in Figure 2.

Notice that the theorems only describes the behavior of the continuous shearlet transform at the points $t = (t_1,t_2) \in \mathbb{R}^2$ such that $|\frac{t_2}{t_1}| \leq 1$. For the points $t \in \mathbb{R}^2$ such that $|\frac{t_2}{t_1}| \geq 1$, it is sufficient to replace the "horizontal" shearlet transform $\mathcal{SH}_\psi^\alpha$ by the "vertical" shearlet transform $\mathcal{SH}_\psi^{(v),\alpha}$. Since the behavior of the transforms $\mathcal{SH}_\psi^\alpha$ and $\mathcal{SH}_\psi^{(v),\alpha}$ on each of those two regions is identical, in the following we will examine the situation for $\mathcal{SH}_\psi^\alpha$ only.

Finally, we recall that a result similar to Theorem 3.1 can be found in [3] and [16], for $R = 1$. Notice, however, that the proofs provided in those references do not completely justify the lower bound estimates:

$$|\mathcal{SH}_\psi^{\frac{1}{2}}(a,s,t)| \geq C\, a^{\frac{3}{4}},$$

with $C > 0$, for t on the boundary and s corresponding to normal orientation. As mentioned above, the lower bound estimate is very important if one wants to guarantee that the edge is detected. A careful argument is needed in our proof below to prove that such lower bound estimate (with $C > 0$) holds.

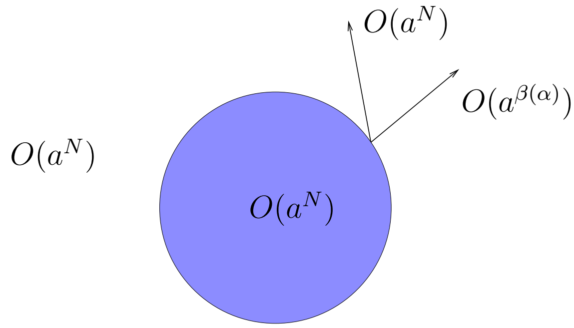

Fig. 2. Decay of the continuous shearlet transform of the disc $D(0,R)$. On the boundary, for normal orientation, the shearlet transform decays as $O(\beta(\alpha))$. For all other values of (t,s), the decay is as fast as $O(N)$, for any $N \in \mathbb{N}$.

Proof of Theorem 3.1.

For simplicity of notation, in the following we will write $\mathcal{SH}_\psi f = \mathcal{SH}_\psi^{\frac{1}{2}} f$.

A direct computation gives:

$$\begin{aligned}\mathcal{SH}_\psi B_R(a,s,t) &= \langle \widehat{B}_R, \widehat{\psi}_{ast}\rangle \\ &= a^{\frac{3}{4}} \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1}-s))\,e^{2\pi i \xi t}\,\widehat{B}_R(\xi_1,\xi_2)\,d\xi_1\,d\xi_2.\end{aligned}$$

For $R=1$, the Fourier transform $\widehat{B}_1(\xi_1,\xi_2)$ is the radial function:

$$\widehat{B}_1(\xi_1,\xi_2) = |\xi|^{-1}\,J_1(2\pi|\xi|),$$

where J_1 is the Bessel function of order 1, whose asymptotic behavior satisfies [23, Ch.8]:

$$J_1(2\pi|\xi|) = \frac{1}{\pi}\,|\xi|^{-\frac{1}{2}}\,\cos(2\pi|\xi| - \frac{3\pi}{4}) + O(|\xi|^{-3/2}) \quad \text{as } |\xi|\to\infty.$$

It is convenient to represent the integral above in polar coordinates. For $|\frac{t_2}{t_1}|\le 1$, $|s|\le\frac{3}{2}$, $R\ge R_0>0$ and $\frac{1}{2}R\le r\le 2R$, we write $t=(t_1,t_2)$ as $r(\cos\theta_0,\sin\theta_0)$, where $0\le|\theta_0|\le\frac{\pi}{4}$. Thus, we have:

$$\begin{aligned}&\mathcal{SH}_\psi B_R(a,s,r,\theta_0)\\ &= a^{\frac{3}{4}}\int_0^\infty\!\!\int_0^{2\pi} \hat{\psi}_1(a\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))e^{2\pi i\rho r\cos(\theta-\theta_0)}R^2\,\hat{B}_1(R\rho)d\theta\rho d\rho\\ &= R^2 a^{-\frac{5}{4}}\int_0^\infty\!\!\int_0^{2\pi} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))e^{2\pi i\frac{\rho r}{a}\cos(\theta-\theta_0)}\hat{B}_1(\frac{R\rho}{a})d\theta\rho d\rho\\ &= R^2 a^{-\frac{5}{4}}\int_0^\infty \eta(\rho,a)\,\hat{B}_1(\frac{R\rho}{a})\,\rho\,d\rho, \qquad (6)\end{aligned}$$

where

$$\begin{aligned}\eta(\rho,a) &= \int_0^{2\pi}\hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta-s))\,e^{2\pi i\frac{\rho r}{a}\cos(\theta-\theta_0)}\,d\theta,\\ &= \eta_1(\rho,a)+\eta_2(\rho,a),\end{aligned}$$

and

$$\begin{aligned}\eta_1(\rho,a) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta-s))\,e^{2\pi i\frac{\rho r}{a}\cos(\theta-\theta_0)}\,d\theta; \qquad (7)\\ \eta_2(\rho,a) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta-s))\,e^{2\pi i\frac{\rho r}{a}\cos(\theta-\theta_0)}\,d\theta\\ &= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta-s))\,e^{-2\pi i\frac{\rho r}{a}\cos(\theta-\theta_0)}\,d\theta. \qquad (8)\end{aligned}$$

In the last equality, we have used the fact that $\hat{\psi}_1(\rho\cos(\theta+\pi)) = \hat{\psi}_1(-\rho\cos\theta) = -\hat{\psi}_1(\rho\cos\theta)$. Observe that the support condition of $\hat{\psi}_2$ implies that the integrals (7) and (8) are nonzero only on the set

$$\mathcal{I} = \{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] : |\tan\theta - s| \le a^{\frac{1}{2}}\}.$$

Using the asymptotic estimate for J_1, for small a we have:

$$\begin{aligned}
&\mathcal{SH}_\psi B_R(a,s,t)\\
&= \frac{R^2}{\pi} a^{\frac{-5}{4}} \int_0^\infty \eta(\rho,a)\, R^{-\frac{3}{2}}\, \rho^{-\frac{3}{2}} a^{\frac{3}{2}} \left(\cos\Big(\frac{2\pi R\rho}{a} - \frac{3\pi}{4}\Big) + O\big((\frac{R\rho}{a})^{-3/2}\big)\right) \rho\, d\rho\\
&= a^{\frac{1}{4}}\, \frac{R^{\frac{1}{2}}}{\pi}\, (I+E),
\end{aligned}$$

where

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \eta(\rho,a)\, \cos\Big(\frac{2\pi R\rho}{a} - \frac{3\pi}{4}\Big)\, \rho^{-\frac{1}{2}}\, d\rho\\
E &= \int_{-\infty}^{\infty} \eta(\rho,a)\, O\big((\frac{R\rho}{a})^{-3/2}\big)\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}$$

We will now show that the function $\eta(\rho,a)$ vanishes for ρ outside a compact region which does not contain the origin. To show that this is the case, recall that the support assumption on $\hat{\psi}_2$ implies that the integrals (7) and (8) are defined only for $\theta \in \mathcal{I}$, that is, for θ such that $|\tan\theta - s| \le a^{\frac{1}{2}}$. Since we assumed that $|s| \le \frac{3}{2}$, this inequality implies that $|\tan\theta| \le \frac{3}{2} + a^{\frac{1}{2}}$. For a small, this implies that the set $\mathcal{I}$ reduces to a small interval around the origin. More precisely, let us assume that $a^{\frac{1}{2}} \le \sqrt{3} - \frac{3}{2}$ (indeed we are only interested to values of a approaching 0). Hence, under this assumption, we have that $|\tan\theta| \le \sqrt{3}$ or $|\theta| \le \pi/3$. At the same time, the assumption on the support of $\hat{\psi}_1$ implies that the integrals (7) and (8) are non-vanishing only when $\frac{1}{2} \le |\rho\cos\theta| \le 2$. Thus, the support of the function $\eta(\rho,\cdot)$ must be contained in the set $\frac{1}{2} \le |\rho| \le 4$. It follows that the integral E is nonzero only for ρ defined on a compact region which does not contain the origin. Thus, as $a \to 0$, we have that $a^{-\beta} E = O(a^{\frac{3}{2}-\beta}) \to 0$ uniformly for s, r, R, for each $\beta \in (0,1)$.

It remains to examine the integral I to see under which conditions we have that $\lim_{a\to 0} a^{-\beta} |\mathcal{SH}_\psi B_R(a,s,t)| > 0$, for some $\beta \in (0,1)$. The proof will be divided in several cases depending on the values of s and r.

Case 1: $s \ne \tan\theta_0$.

We will examine again the behavior of the integrals (7) and (8). The support condition on $\hat{\psi}_2$ implies that these integrals are nonzero only for θ such that $|\tan\theta - s| = |\tan\theta - \tan\theta_0 + \tan\theta_0 - s| \le a^{\frac{1}{2}}$. Hence $|\tan\theta - \tan\theta_0| \ge |\tan\theta_0 - s| - a^{\frac{1}{2}}$, which implies that, for $\theta \in \mathcal{I}$, $|\cos(\theta-\theta_0)'| = |\sin(\theta-\theta_0)| \ge C_0 > 0$ uniformly for all small a. Thus, repeated integration by parts on (7) and (8) yields that, for each $N \in \mathbb{N}$ there is a C_N such that $|\eta(\rho,a)| \le C_N\, a^{\frac{1}{2}N}$. This implies that $a^{-\beta}\, I \to 0$, as $a \to 0$, for any $\beta > 0$.

Case 2: $s = \tan\theta_0$, but $r \ne R$

It will be convenient to use Euler's formula for the cosine function and write the integral I as $I = I_1 + I_2$, where

$$
\begin{aligned}
I_1 &= \frac{e^{-\frac{3\pi i}{4}}}{2} \int_0^\infty \eta(\rho,a)\, e^{2\pi i \frac{R\rho}{a}}\, \rho^{-\frac{1}{2}}\, d\rho;\\
I_2 &= \frac{e^{\frac{3\pi i}{4}}}{2} \int_0^\infty \eta(\rho,a)\, e^{-2\pi i \frac{R\rho}{a}}\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}
$$

Next, using again equations (7) and (8) to express η as $\eta = \eta_1 + \eta_2$, we can write $I_1 = I_{11} + I_{12}$ and $I_2 = I_{21} + I_{22}$, where, for $j = 1, 2$ we have

$$
I_{1j} = e^{-\frac{3\pi i}{4}} \int_0^\infty \eta_j(\rho,a)\, e^{2\pi i \frac{R\rho}{a}}\, \rho^{-\frac{1}{2}}\, d\rho, \quad I_{2j} = e^{\frac{3\pi i}{4}} \int_0^\infty \eta_j(\rho,a)\, e^{-2\pi i \frac{R\rho}{a}}\, \rho^{-\frac{1}{2}}\, d\rho.
$$

We will now examine the behavior of the integrals (7) and (8) for $s = \tan\theta_0$. The support condition of $\hat{\psi}_2$ imposes that $a^{-\frac{1}{2}}|(\tan\theta - \tan\theta_0)| \le 1$, or $|\tan\theta - \tan\theta_0| \le a^{\frac{1}{2}}$. It follows that there exists $\delta > 0$ such that, for all $a < \delta$, we have:

$$
1 + \tan\theta\, \tan\theta_0 \ge \frac{1}{2}. \tag{9}
$$

Let $u = a^{-\frac{1}{2}} \tan(\theta - \theta_0)$, which gives that $\theta = \theta_a(u) = \theta_0 + \tan^{-1}(a^{\frac{1}{2}} u)$. Thus, we have

$$
\begin{aligned}
\eta_1(\rho,a) &= a^{\frac{1}{2}}\, e^{2\pi i \frac{\rho r}{a}}\, T(a,\rho,r),\\
\eta_2(\rho,a) &= -a^{\frac{1}{2}}\, e^{-2\pi i \frac{\rho r}{a}}\, \overline{T(a,\rho,r)},
\end{aligned}
$$

where

$$
\begin{aligned}
T(a,\rho,r) = &\int_{-2}^{2} \hat{\psi}_1(\rho\cos\theta_a(u))\, \hat{\psi}_2(u(1+\tan\theta_a(u)\tan\theta_0))\\
&\times e^{-i2\pi\frac{\rho r}{a}\,(1-\frac{1}{\sqrt{1+au^2}})}\, \frac{1}{1+au^2}\, du.
\end{aligned}
$$

In the last expression we have used the identity $\tan\theta - \tan\theta_0 = \tan(\theta-\theta_0)(1+\tan\theta\,\tan\theta_0)$. Using (9) and the support condition of $\hat{\psi}_2$, it follows that the integral above is only nonzero for $|u| \le 2$. Observe that $\theta_a^{(n)}(u)$ is bounded for each $n \ge 0$, for all $0 < a < \delta$ and for all $|u| \le 2$. Also observe that $\lim_{a\to 0}\theta_a(u) = \theta_0$, uniformly for $|u| \le 2$.

Using the notation introduced above, we have:

$$
\begin{aligned}
I_{11} &= \frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{\frac{1}{2}} \int_0^\infty e^{\frac{2\pi i\rho}{a}(R+r)}\, T(a,\rho,r)\, \rho^{-\frac{1}{2}}\, d\rho,\\
I_{12} &= -\frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{\frac{1}{2}} \int_0^\infty e^{\frac{2\pi i\rho}{a}(R-r)}\, \overline{T(a,\rho,r)}\, \rho^{-\frac{1}{2}}\, d\rho,\\
I_{21} &= \frac{e^{\frac{3\pi i}{4}}}{2}\, a^{\frac{1}{2}} \int_0^\infty e^{-\frac{2\pi i\rho}{a}(R-r)}\, T(a,\rho,r)\, \rho^{-\frac{1}{2}}\, d\rho,\\
I_{22} &= -\frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{\frac{1}{2}} \int_0^\infty e^{-\frac{2\pi i\rho}{a}(R+r)}\, \overline{T(a,\rho,r)}\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}
$$

To estimate the integrals I_{kj}, $k,j = 1,2$, for $a \to 0$, we will integrate by parts over the variable ρ (notice that both R and r are positive and $r \ne R$). Observe that

$$\lim_{a\to 0} \frac{\frac{1}{a}\big(1 - \frac{1}{\sqrt{1+au^2}}\big)}{\frac{1}{2}u^2} = 1.$$

This shows that, for a sufficiently small, say $a < \delta$, for some $\delta > 0$, one can replace $\frac{1}{a}(1 - \frac{1}{\sqrt{1+au^2}})$ by $\frac{1}{2}u^2$. It follows that for each $n \ge 0$ there is a constant C_n such that $|\frac{d^nT}{d\rho^n}| \le C_n$, for all $0 < a < \delta$. Thus, repeated integrating by parts on the variable ρ for I_{kj}, $k,j = 1,2$, yields that, for each $n \ge 0$, there is a constant C_n such that $|I_{kj}| \le C_n\, a^n$, for $0 < a < \delta$. This implies that $a^{-\beta}\,|I_{kj}| \to 0$, as $a \to 0$, for any $\beta > 0$.

Case 3: $s = \tan\theta_0$, $r = R$

As discussed in the Case 2, we need to examine the integrals I_{kj}, $k,j = 1,2$. For the same reason as in the Case 2, we have that $a^{-\beta}|I_{kk}| \to 0$, for $k = 1,2$, as $a \to 0$. It remains to study I_{12} and I_{21}.

Notice that $\lim_{a\to 0}\frac{1}{a}(1 - \frac{1}{\sqrt{1+au^2}}) = \frac{u^2}{2}$. Hence, using the fact that $\sec^2\theta_0 \ge 1$ and $\mathrm{supp}\,(\hat{\psi}_2) \subset [-1,1]$, we have:

$$
\begin{aligned}
\lim_{a\to 0} a^{-\frac{3}{4}}\,|\mathcal{SH}_\psi B_R(a,s,t)| &= \lim_{a\to 0} a^{-\frac{3}{4}}\,|\mathcal{SH}_\psi B_R(a,\tan\theta_0,R,\theta_0)|\\
&= \frac{R^{\frac{1}{2}}}{\pi}\,\lim_{a\to 0} a^{-\frac{1}{2}}\,|I_{12} + I_{21}|
\end{aligned}
$$

$$= \left| -\frac{e^{-\frac{3\pi i}{4}}}{2} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\,\rho^{-\frac{1}{2}}\,\overline{h(\rho,R)}\,d\rho \right.$$
$$\left. + \frac{e^{\frac{3\pi i}{4}}}{2} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\,\rho^{-\frac{1}{2}}\,h(\rho,R)\,d\rho \right|, \qquad (10)$$

where
$$h(\rho,R) = \frac{R^{\frac{1}{2}}}{\pi} \int_{-1}^{1} \hat{\psi}_2(u\sec^2\theta_0)\,e^{-\pi i\rho R u^2}\,du.$$
Observe that

$$-\frac{e^{-\frac{3\pi i}{4}}}{2}\,\overline{h(\rho,R)} + \frac{e^{\frac{3\pi i}{4}}}{2}\,h(\rho,R)$$
$$= \frac{iR^{\frac{1}{2}}}{\sqrt{2}\,\pi} \int_{-1}^{1} \hat{\psi}_2(u\sec^2\theta_0)\,\Big(\sin(\pi\rho Ru^2)+\cos(\pi\rho Ru^2)\Big)\,du$$
$$= \frac{\sqrt{2}iR^{\frac{1}{2}}}{\pi} \int_0^1 \hat{\psi}_2(u\sec^2\theta_0)\,\Big(\sin(\pi\rho Ru^2)+\cos(\pi\rho Ru^2)\Big)\,du.$$

In the last step we have used the fact that $\hat{\psi}_2$ is even. Using the last equality in (10), we have

$$\lim_{a\to 0} a^{-\frac{3}{4}}\,|\mathcal{SH}_\psi B_R(a,\tan\theta_0,R,\theta_0)|$$
$$= \frac{\sqrt{2R}}{\pi}\,\left|\int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\,\rho^{-\frac{1}{2}} \int_0^1 \hat{\psi}_2(u\sec^2\theta_0)\,\Big(\sin(\pi\rho Ru^2)+\cos(\pi\rho Ru^2)\Big)\,du\,d\rho\right|.$$

We claim that there exists a $c_0 > 0$ such that the above limit is larger than c_0 for all $R \geq R_0 > 0$. To save a unnecessary notation, in the following we will only consider the case $\theta_0 = 0$. The argument for a general $|\theta_0| \leq \frac{\pi}{4}$ is identical.

Let $g(\rho,R) = 2R^{\frac{1}{2}} \int_0^1 \hat{\psi}_2(u\sec^2\theta_0)\,(\sin(\pi\rho Ru^2)+\cos(\pi\rho Ru^2))\,du$. By the support condition on $\hat{\psi}_1$, it is clear that our claim is satisfied provided we can show that:

$$(A)\;\; g(\rho,R) > 0 \;\text{ for each }\; \rho \geq \frac{1}{2},\; R > 0;$$
$$(B)\;\; g(\rho,R) \geq C > 0 \;\text{ for all }\; \rho \geq \frac{1}{2},\; R > R_0 > 0.$$

To show that (A) holds, observe first that

$$g(\rho,R) = \int_0^{\rho R} v^{-\frac{1}{2}}\,\hat{\psi}_2\Big(\sqrt{\tfrac{v}{\rho R}}\Big)\,\left(\sin(\pi v)+\cos(\pi v)\right)\,dv$$

$$
\begin{aligned}
&= \int_0^{\rho R} v^{-\frac{1}{2}}\, \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \sin(\pi v)\, dv + \int_0^{\rho R} v^{-\frac{1}{2}}\, \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv \\
&= g_1(\rho, R) + g_2(\rho, R).
\end{aligned}
$$

We will start by showing that $g_1(\rho, R) > 0$ for each $\rho \geq \frac{1}{2}$ and each $R > 0$.

If $\rho R \leq 1$, it is trivial to see that $g_1(\rho, R) > 0$ since $\hat{\psi}_2(0) \geq \frac{1}{2}$, $\hat{\psi}_2(x) \geq 0$ on $[0, 1]$ and $\sin(\pi x) > 0$ on $(0, 1)$. Now assume that $1 < \rho R \leq 2$. Since $\hat{\psi}_2(x)$ is decreasing on $(0, 1)$, we have

$$
\begin{aligned}
g_1(\rho, R) &= \int_0^1 v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \sin(\pi v)\, dv + \int_1^{\rho R} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \sin(\pi v)\, dv \\
&= \int_0^1 v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \sin(\pi v)\, dv - \int_0^{\rho R - 1} (v+1)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+1}{\rho R}})\, \sin(\pi v)\, dv \\
&\geq \int_0^1 \Big(v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}}) - (v+1)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+1}{\rho R}}) \Big)\, \sin(\pi v)\, dv > 0.
\end{aligned}
$$

For $\rho R > 2$, one can find $k \geq 1$ and $0 < \zeta \leq 2$ such that $\rho R = 2k + \zeta$. In this case, we have:

$$
\begin{aligned}
g_1(\rho, R) &= \int_0^{2k} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \sin(\pi v)\, dv + \int_{2k}^{\rho R} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \sin(\pi v)\, dv \\
&= g_0(\rho, R) + g_\zeta(\rho, R),
\end{aligned}
$$

where

$$
\begin{aligned}
g_0(\rho, R) &= \sum_{j=0}^{k-1} \int_0^1 \Big((v+2j)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2j}{\rho R}}) + \\
&\quad - (v+2j+1)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2j+1}{\rho R}}) \Big)\, \sin(\pi v)\, dv; \\
g_\zeta(\rho, R) &= \int_0^\zeta (v+2k)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2k}{\rho R}})\, \sin(\pi v)\, dv.
\end{aligned}
$$

It is easy to verify that $g_0 > 0$ and $g_\zeta \geq 0$ and hence $g_1 > 0$.

Next we consider $g_2(\rho, R)$. If $\rho R \leq \frac{1}{2}$, it is trivial to see that $g_2(\rho, R) > 0$ since $\cos(\pi x) > 0$ on $(0, \frac{1}{2})$. Now assume that $\frac{1}{2} < \rho R \leq \frac{3}{2}$. Since $\cos(\pi v) < 0$ on $(\frac{1}{2}, \frac{3}{2})$, we have

$$
\begin{aligned}
g_2(\rho, R) &= \int_0^{\frac{1}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv + \int_{\frac{1}{2}}^{\rho R} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv \\
&\geq \int_0^{\frac{1}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv + \int_0^{\frac{3}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv
\end{aligned}
$$

$$\geq c_0\, \hat{\psi}_2(\sqrt{\tfrac{1}{2\rho R}}),$$

where $c_0 = \int_0^{\frac{3}{2}} v^{-\frac{1}{2}} \cos(\pi v)\, dv$. Since $\int_0^{\sqrt{\frac{3\pi}{2}}} \cos u^2 du \geq 0.4 > 0$, it follows that $c_0 > 0$.

For $\frac{3}{2} < \rho R \leq \frac{5}{2}$, since $\cos(\pi v) > 0$ on $(\frac{3}{2}, \frac{5}{2})$, we have:

$$\begin{aligned}
g_2(\rho, R) &= \int_0^{\frac{1}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv + \int_{\frac{1}{2}}^{\frac{3}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv \\
&\quad + \int_{\frac{3}{2}}^{\rho R} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv \\
&\geq \int_0^{\frac{1}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv + \int_{\frac{1}{2}}^{\frac{3}{2}} v^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v}{\rho R}})\, \cos(\pi v)\, dv \\
&\geq c_0\, \hat{\psi}_2(\sqrt{\tfrac{1}{2\rho R}}) > 0.
\end{aligned}$$

For $\rho R > \frac{5}{2}$, one can find $n \geq 1$ and $0 \leq \zeta < 1$ such that $\rho R = \frac{3}{2} + n + \zeta$. Let us examine the cases where n is even and odd separately. If $n = 2k$, for some $k \geq 1$, we have

$$\begin{aligned}
&g_2(\rho, R) \\
&\geq \hat{\psi}_2(\sqrt{\tfrac{1}{2\rho R}}) \int_0^{\frac{3}{2}} v^{-\frac{1}{2}}\, \cos(\pi v)\, dv \\
&+ \sum_{j=0}^{k-1} \int_{\frac{3}{2}}^{\frac{5}{2}} \Big((v+2j)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2j}{\rho R}}) - (v+2j+1)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2j+1}{\rho R}}) \Big) \cos(\pi v)\, dv \\
&+ \int_{\frac{3}{2}}^{\frac{3}{2}+\zeta} (v+2k)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2k}{\rho R}})\, \cos(\pi v)\, dv \\
&\geq c_0\, \hat{\psi}_2(\sqrt{\tfrac{1}{2\rho R}}) > 0.
\end{aligned}$$

If $n = 2k+1$, for some $k \geq 1$, we have

$$\begin{aligned}
&g_2(\rho, R) \\
&\geq \hat{\psi}_2(\sqrt{\tfrac{1}{2\rho R}}) \int_0^{\frac{3}{2}} v^{-\frac{1}{2}} \cos(\pi v)\, dv \\
&+ \sum_{j=0}^{k-1} \int_{\frac{3}{2}}^{\frac{5}{2}} \Big((v+2j)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2j}{\rho R}}) - (v+2j+1)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2j+1}{\rho R}}) \Big)\, \cos(\pi v)\, dv \\
&+ \int_{\frac{3}{2}}^{\frac{5}{2}} (v+2k)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2k}{\rho R}})\, \cos(\pi v)\, dv +
\end{aligned}$$

$$- \int_{\frac{3}{2}}^{\frac{3}{2}+\zeta} (v+2k+1)^{-\frac{1}{2}} \hat{\psi}_2(\sqrt{\tfrac{v+2k+1}{\rho R}}) \cos(\pi v)\, dv$$
$$\geq c_0\, \hat{\psi}_2(\sqrt{\tfrac{1}{2\rho R}}) > 0.$$

From the the estimates of $g_1(\rho, R)$ and $g_2(\rho, R)$, we see that (A) holds. Since $g_2(\rho, R) \geq c_0\, \hat{\psi}_2\,(\sqrt{\frac{1}{2\rho R}})$ for all $\rho > \frac{1}{2}$ and all $R > 0$, and since $\hat{\psi}_2(x) = \hat{\psi}_2(0)$ for $0 \leq x \leq \frac{1}{2}$, it follows that, for all $\rho > \frac{1}{2}$ and all $R > 4$, we have $g(\rho, R) \geq g_2(\rho, R) \geq c_0\, \hat{\psi}_2(\sqrt{\frac{1}{2\rho R}}) = c_0\, \hat{\psi}_2(0) > 0$. This shows that (B) also holds. □

To prove Theorem 3.2, we will need the following result, which is a special case of the method of stationary phase (Proposition 8.3 from [23]).

Proposition 3.1 *Let ϕ and ψ be smooth functions. Suppose $\phi(x_0) = \phi'(x_0) = 0$, while $\phi''(x_0) \neq 0$. If ψ is supported in a sufficiently small neighborhood of x_0, then*
$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\,\lambda\,\phi(x)}\, \psi(x)\, dx = a_0\, \lambda^{-1/2} + O(\lambda^{-1}),$$
as $\lambda \to \infty$, where
$$a_0 = \left(\frac{2\pi i}{\phi''(x_0)}\right)^{\frac{1}{2}} \psi(x_0).$$

Proof of Theorem 3.2.

Many steps in the proof follows by a simple adaptation of the arguments in Theorem 3.1. In the following, we will indicate where significant new arguments are needed.

Similarly to the proof of Theorem 3.1, we write

$$\mathcal{SH}_\psi^\alpha B_R(a,s,t) = a^{\frac{\alpha}{2}}\, \frac{R^{\frac{1}{2}}}{\pi}\, (I+E),$$

where E is negligible, as $a \to 0$,

$$I = \int_{-\infty}^{\infty} \eta(\rho, a)\, \cos\Big(\frac{2\pi R\rho}{a} - \frac{3\pi}{4}\Big)\, \rho^{-\frac{1}{2}}\, d\rho$$

and
$$\eta(\rho, a) = \int_0^{2\pi} \hat{\psi}_1(\rho\cos\theta)\, \hat{\psi}_2(a^{\alpha-1}(\tan\theta - s))\, e^{2\pi i \frac{\rho r}{a}\cos(\theta-\theta_0)}\, d\theta.$$

Now, Case 1 ($s \neq \tan\theta_0$) and Case 2 ($s = \tan\theta_0$ and $r \neq R$) can be discussed similarly to Theorem 3.1. The remaining Case 3 ($s = \tan\theta_0$, $r = R$) requires the following different argument.

We start by writing $I = I_{11} + I_{12} + I_{21} + I_{22}$, where

$$
\begin{aligned}
I_{11} &= \frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{1-\alpha} \int_0^\infty e^{\frac{2\pi i\rho}{a}(R+r)}\, T(a,\rho,r)\, \rho^{-\frac{1}{2}}\, d\rho,\\
I_{12} &= -\frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{1-\alpha} \int_0^\infty e^{\frac{2\pi i\rho}{a}(R-r)}\, \overline{T(a,\rho,r)}\, \rho^{-\frac{1}{2}}\, d\rho,\\
I_{21} &= \frac{e^{\frac{3\pi i}{4}}}{2}\, a^{1-\alpha} \int_0^\infty e^{-\frac{2\pi i\rho}{a}(R-r)}\, T(a,\rho,r)\, \rho^{-\frac{1}{2}}\, d\rho,\\
I_{22} &= -\frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{1-\alpha} \int_0^\infty e^{-\frac{2\pi i\rho}{a}(R+r)}\, \overline{T(a,\rho,r)}\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}
$$

and

$$
\begin{aligned}
T(a,\rho,r) = &\int_{-2}^{2} \hat{\psi}_1(\rho\cos\theta_a(u))\, \hat{\psi}_2(u(1+\tan\theta_a(u)\tan\theta_0))\\
&\times e^{-i2\pi\frac{\rho r}{a}\left(1-\frac{1}{\sqrt{1+a^{2-2\alpha}u^2}}\right)}\, \frac{1}{1+a^{2-2\alpha}u^2}\, du.
\end{aligned}
$$

Similarly to Theorem 3.2, we have that, for any $\beta > 0$, $a^{-\beta}|I_{kk}| \to 0$, for $k = 1, 2$, as $a \to 0$. It remains to study I_{12} and I_{21}. We consider two subcases: $\alpha < \frac{1}{2}$ and $\alpha > \frac{1}{2}$ (the case $\alpha = \frac{1}{2}$ was already discussed in Theorem 3.2).

Subcase 1: $\alpha < \frac{1}{2}$

Since $\alpha < 1/2$, we have $\lim_{a\to 0} \frac{1}{a}(1 - \frac{1}{\sqrt{1+a^{2-2\alpha}u^2}}) = 0$. Hence

$$
\lim_{a\to 0} T(a,\rho,\theta) = \hat{\psi}_1(\rho\,\cos\theta_0)\, \rho^{-\frac{1}{2}} \int_{-2}^{2} \hat{\psi}_2(u\,\sec^2\theta_0)\, du.
$$

Thus, by direct computation we have that:

$$
\begin{aligned}
&\lim_{a\to 0} a^{\frac{\alpha}{2}-1} |\mathcal{SH}_\psi^\alpha B_R(a,s,t)|\\
&= \lim_{a\to 0} a^{\frac{\alpha}{2}-1} |\mathcal{SH}_\psi^\alpha B_R(a,\tan\theta_0,R,\theta_0)|\\
&= \frac{R^{\frac{1}{2}}}{\pi} \lim_{a\to 0} a^{\alpha-1}\ |I_{12}+I_{21}|\\
&= \frac{R^{\frac{1}{2}}}{\pi} \left| -\frac{e^{-\frac{3\pi i}{4}}}{2} \int_0^\infty \hat{\psi}_1(\rho\,\cos\theta_0)\, \rho^{-\frac{1}{2}} \int_{-2}^{2} \hat{\psi}_2(u\,\sec^2\theta_0)\, du\, d\rho \right.\\
&\left. + \ \frac{e^{\frac{3\pi i}{4}}}{2} \int_0^\infty \hat{\psi}_1(\rho\,\cos\theta_0)\, \rho^{-\frac{1}{2}} \int_{-2}^{2} \hat{\psi}_2(u\,\sec^2\theta_0)\, du\, d\rho \right|\\
&= \frac{R^{\frac{1}{2}}}{\pi}\, \frac{\sqrt{2}}{2} \int_0^\infty \hat{\psi}_1(\rho\,\cos\theta_0)\, \rho^{-\frac{1}{2}} \int_{-1}^{1} \hat{\psi}_2(u\,\sec^2\theta_0)\, du\, d\rho > 0.
\end{aligned}
$$

In the last equation, we have used the fact that $\sec^2\theta_0 \geq 1$ and $\mathrm{supp}\,(\hat{\psi}_2) \subset [-1,1]$.

Subcase 2: $\alpha > \frac{1}{2}$

Since $\theta_a^{(n)}(u)$ is uniformly bounded for each $n \geq 0$ for all small a and for $|u| \leq 2$, we can apply Proposition 3.1 to $T(a,\rho,r)$, by choosing $x_0 = 0$, $\phi(u) = -\pi\rho R u^2$ and $\lambda = a^{1-2\alpha}$. Hence, recalling that $\hat{\psi}_2(0) = 1$, and that $\frac{1}{a}(1 - \frac{1}{\sqrt{1+a^{2-2\alpha}u^2}})$ approaches $\frac{1}{2}a^{1-2\alpha}u^2$ as $a \to 0$, we have that

$$
\begin{aligned}
T(a,\rho,r) &\sim \sqrt{-\frac{i}{\rho\, R}}\, \hat{\psi}_1(\rho \cos\theta_0)\, a^{-\frac{1-2\alpha}{2}} \\
&= e^{\frac{3\pi i}{4}}\, \rho^{-1/2}\, R^{-1/2}\, \hat{\psi}_1(\rho\cos\theta_0)\, a^{-\frac{1-2\alpha}{2}}, \text{ as } a \to 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lim_{a\to 0} a^{-\frac{1}{2}}\, I_{12} &= \lim_{a\to 0} -\frac{e^{-\frac{3\pi i}{4}}}{2}\, a^{\frac{1-2\alpha}{2}} \int_0^\infty \overline{T(a,\rho,r)}\, \rho^{-\frac{1}{2}}\, d\rho, \\
&= -\frac{i}{2}\, R^{-1/2} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, \rho^{-1}\, d\rho.
\end{aligned}
$$

Similarly:

$$
\lim_{a\to 0} a^{-\frac{1}{2}}\, I_{21} = -\frac{i}{2}\, R^{-1/2} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, \rho^{-1}\, d\rho.
$$

It follows that

$$
\begin{aligned}
\lim_{a\to 0} a^{-\frac{1+\alpha}{2}} |\mathcal{SH}_\psi^\alpha B_R(a,s,t)| &= \lim_{a\to 0} a^{-\frac{1+\alpha}{2}}\, |\mathcal{SH}_\psi^\alpha B_R(a,\tan\theta_0, R, \theta_0)| \\
&= \frac{R^{\frac{1}{2}}}{\pi} \lim_{a\to 0} a^{-\frac{1}{2}}\, |I_{12} + I_{21}| \\
&= \frac{1}{\pi} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, \rho^{-1}\, d\rho > 0. \qquad \Box
\end{aligned}
$$

Theorem 3.2 implies the following corollary.

Corollary 3.1 *Let $t \in P$, and P, $\beta(\alpha)$ be defined as in Theorem 3.2. There is a constant $C \neq 0$, independent of $R > R_0 > 0$ such that*

$$
\begin{aligned}
&\sup_{|s|\leq\frac{3}{2},\, t\in P}\ \lim_{a\to 0^+} a^{-\beta(\alpha)}\, \mathcal{SH}_\psi^\alpha B_R(a,s,t) \\
&= \sup_{|\theta_0|\leq\frac{\pi}{4}} \left(\lim_{a\to 0^+} a^{-\beta(\alpha)}\, \mathcal{SH}_\psi^\alpha B_R(a,\tan\theta_0, t_0) \right) = C \neq 0,
\end{aligned}
$$

where $t_0 = R\,(\cos\theta_0, \sin\theta_0)$.

If the order of the limit and sup is reversed with respect to the results in Corollary 3.1, we obtain the following estimates for the continuous shearlet transform of B_R.

Theorem 3.3 *(i) For $\frac{1}{2} \le \alpha < 1$, there exist constants $C_1, C_2 > 0$, independent of $R \ge R_0$, such that*

$$\begin{aligned} C_1 &\le \liminf_{a\to 0^+} a^{-\frac{1}{2}(1+\alpha)} \sup_{|s|\le \frac{3}{2},\ \frac{1}{2}R\le |t|\le 2R} \mathcal{SH}_\psi^\alpha B_R(a,s,t) \\ &\le \limsup_{a\to 0^+} a^{-\frac{1}{2}(1+\alpha)} \sup_{|s|\le \frac{3}{2},\ \frac{1}{2}R\le |t|\le 2R} \mathcal{SH}_\psi^\alpha B_R(a,s,t) \le C_2. \end{aligned}$$

(ii) For $0 < \alpha < \frac{1}{2}$, there exist constants $C_1, C_2 > 0$, independent of $R \ge R_0$, such that

$$\begin{aligned} C_1 &\le \liminf_{a\to 0^+} a^{-(1-\frac{\alpha}{2})} \sup_{|s|\le \frac{3}{2},\ \frac{1}{2}R\le |t|\le 2R} \mathcal{SH}_\psi^\alpha B_R(a,s,t) \\ &\le \limsup_{a\to 0^+} a^{-(1-\frac{\alpha}{2})} \sup_{|s|\le \frac{3}{2},\ \frac{1}{2}R\le |t|\le 2R} \mathcal{SH}_\psi^\alpha B_R(a,s,t) \le C_2\, R^{\frac{1}{2}}. \end{aligned}$$

Remark: To more precisely characterize the edges of B_R, one would like to have $C_1 = C_2$, or, at least, $|C_1 - C_2|$ as small as possible. It turns out that this can be achieved by using values of $\alpha > \frac{1}{2}$. The details of this argument are quite technical and will be omitted.

For the proof of Theorem 3.3, we need the following fact, which can be found in [23, Ch.8] (Corollary to Proposition 8.2).

Proposition 3.2 *Let ϕ and ψ be smooth functions on $[a,b]$. For $k = 1, 2$, if $|\phi^{(k)}(x)| \ge 1$ on $[a,b]$, then*

$$\left| \int_a^b e^{i\lambda\phi(x)}\, \psi(x)\, dx \right| \le C_k\, \lambda^{-\frac{1}{k}} \left(|\psi(b)| + \int_a^b |\psi'|\, dx \right)$$

holds when (i) $k = 1$ or (ii) $k = 2$ and $\phi'(x)$ is monotonic on $[a,b]$. Here one can choose $C_1 = 3$ and $C_2 = 8$.

Proof of the Theorem 3.3. In the following, we will use the same C to indicate different uniform constants. We will refer to the notation introduced after equation (6) for the expressions of $\mathcal{SH}_\psi^\alpha B_R$, η_1, η_2 and $\mathcal{I}$.

We will start by obtaining some estimates on the function η_1.

Recall that the integral defining η_1, given by (7), is nonzero only on the set $\mathcal{I}$. Let $\mathcal{I}_1 = \{\theta : |\sin(\theta-\theta_0)| \geq \frac{1}{2}\} \bigcap \mathcal{I}$ and $\mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1$. Since $\cos^2(\theta-\theta_0) + \sin^2(\theta-\theta_0) = 1$, it follows that if $\mathcal{I}_2 \neq \emptyset$, then $|\cos(\theta-\theta_0)| \geq \frac{1}{2}$ on $\mathcal{I}_2$. We may assume that, on $\mathcal{I}_1$, $\sin(\theta-\theta_0)$ is monotonic, otherwise one can cut $\mathcal{I}_1$ into two intervals, on which $\sin(\theta-\theta_0)$ is monotonic. We will write $\eta_1 = \eta_{11}+\eta_{12}$, where η_{11} is the integral (7) restricted to $\mathcal{I}_1$ and η_{12} is the integral (7) restricted to $\mathcal{I}_2$.

Since $|s| \leq \frac{3}{2}$ and we are only interested to the small values of a, as in the proof of Theorem 3.2, we have that $|\tan\theta - s| \leq a^{1-\alpha}$; this implies that $|\tan\theta| < \sqrt{3}$ and hence $|\theta| < \frac{\pi}{3}$. It follows that, on the set $\mathcal{I}$, $(\tan\theta)' = \sec^2\theta < 4$. As a result, we have that $|\frac{d}{d\theta}(\hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2(a^{\alpha-1}(\tan\theta-s))| \leq C\,(\rho + a^{\alpha-1})$. Now we apply the Proposition 3.2 with $k=1$ to η_{11} to obtain that

$$|\eta_{11}(\rho, a, r, \theta_0, s)| \leq C\,\frac{a}{\rho r}\,(1 + \rho\, a^{1-\alpha}).$$

Here we used the fact that $|\mathcal{I}_1| \leq 2\,a^{1-\alpha}$. Similarly, by applying Proposition 3.2 with $k=2$ to η_{12}, we obtain

$$|\eta_{12}(\rho, a, r, \theta_0, s)| \leq C\,\sqrt{\frac{a}{\rho r}}\,(1 + \rho\, a^{\frac{1}{2}}).$$

The case for η_2 is proved very similarly.

Finally, notice that, as in the proof of Theorem 3.2, the function $\eta(\rho,\cdot)$ is compactly supported away from the origin. Hence, since $\frac{1}{2}R \leq r \leq 2R$, from the above estimates we conclude that

$$\begin{aligned}|\mathcal{SH}_\psi^\alpha B_R(a,s,t)| &\leq C\,R^2\,a^{\frac{1}{2}(\alpha-3)}\,\left((\frac{a}{R})^{\frac{1}{2}} + \frac{a}{R}\right)\,R^{-\frac{3}{2}}\,a^{\frac{3}{2}}\\ &= C\,\left(1 + (\frac{a}{R})^{\frac{1}{2}}\right)\,a^{\frac{1}{2}(\alpha+1)}.\end{aligned} \tag{11}$$

This proves the upper bound in (i). The lower bound of (i) follows from the calculations for the Case 3 (subcases 2 and 3) in the proof of Theorem 3.2.

For (ii), observe that

$$\begin{aligned}|\eta_1(\rho, a, r, \theta_0, s)| &= \left|\int_{\mathcal{I}} \hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2(a^{\alpha-1}(\tan\theta-s))\,e^{2\pi i\frac{\rho r}{a}\cos(\theta-\theta_0)}\,d\theta\right|\\ &\leq C\,|\mathcal{I}| \leq C\,a^{1-\alpha},\end{aligned}$$

uniformly for ρ, a, r, θ_0, s. Similarly for η_2. Thus, since $\eta(\rho,\cdot)$ is compactly supported away from the origin we have that

$$
\begin{aligned}
|\mathcal{SH}_\psi^\alpha B_R(a,s,t)| &= a^{\frac{\alpha}{2}} R^{\frac{1}{2}} \,|\int_0^\infty \eta(\rho,a)\,\cos\Big(\frac{2\pi R\rho}{a}-\frac{3\pi}{4}\Big)\,\rho^{-\frac{1}{2}}\,d\rho \\
&\quad + \int_{-\infty}^\infty \eta(\rho,a)\,O((\frac{R\rho}{a})^{-3/2})\,\rho^{-\frac{1}{2}}\,d\rho| \\
&\le C\,a^{(1-\frac{\alpha}{2})}\,R^{\frac{1}{2}}.
\end{aligned}
$$

This gives the upper bound in (ii). The lower bound for (ii) follows again from the calculations for the Case 3 (subcase 1) in the proof of Theorem 3.2. □

Notice that, as the proof of Theorem 3.3 shows, from the inequality (11) we deduce an estimate for the upper bound of $a^{-\frac{1}{2}(1+\alpha)}\,\mathcal{SH}_\psi^\alpha B_R(a,s,t)$ which is independent of R. Only for $\alpha < \frac{1}{2}$ our argument provides an upper bound for $a^{-(1-\frac{\alpha}{2})}\,\mathcal{SH}_\psi^\alpha B_R(a,s,t)$ which depends on R.

3.2 Step edges along general curves

In this section, we show that the results obtained in Section 3.1 for the characteristic function of the disc can be extended to a larger class of planar regions.

Let C be a convex body in $\mathbb{R}^2$ having a smooth boundary with everywhere positive curvature. Without of loss of generality, we may assume that the origin is an interior point of C. For any $\theta \in [0, 2\pi)$, we define

$$\Theta(\theta) = (\cos\theta, \sin\theta) \in S^1$$

such that there exists exactly one point $\sigma(\theta) \in \partial C$ at which the unit outer normal vector is $\Theta(\theta)$. Also let $K(\theta)$ be the curvature of ∂C at the point $\sigma(\theta)$. It is well known that both $\sigma(\theta)$ and $K(\theta)$ are smooth functions of θ. The following lemma lists some basic properties of $\sigma(\theta)$. Its proof follows easily from the assumptions on ∂C.

Lemma 3.1 *Let $C \subset \mathbb{R}^2$ be a convex body having a smooth boundary with everywhere positive curvature. Then*

(i) $\sigma(\theta)\cdot\Theta(\theta) = \sup_{x\in\partial C} \Theta(\theta)\cdot x$;

(ii) $\sigma'(\theta)\cdot\Theta(\theta) = 0$;

(iii) $\sigma(\theta_1) \neq \sigma(\theta_2)$ *for all* $\theta_1, \theta_2 \in [0, 2\pi)$.

Notice that $\Theta(\theta+\pi) = -\Theta(\theta)$. Hence Lemma 3.1(ii) implies that $\sigma'(\theta+\pi)\cdot\Theta(\theta) = 0$. Next, for any $\xi \in \mathbb{R}^2$, let $\xi = \rho\,\Theta(\theta)$ and define:

$$g(\xi) = g(\rho,\theta) = e^{\frac{\pi}{4}i}\,\rho^{-\frac{1}{2}} K^{-\frac{1}{2}}(\theta)\,e^{-2\pi i\sigma(\theta)\cdot\Theta(\theta)\rho}. \tag{12}$$

Notice that, by Lemma 3.1(i), we have

$$\overline{g(-\xi)} = \overline{g(\rho, \theta+\pi)} = e^{-\frac{\pi}{4}i}\rho^{-\frac{1}{2}}K^{-\frac{1}{2}}(\theta+\pi)\, e^{2\pi i\sigma(\theta+\pi)\cdot\Theta(\theta+\pi)\rho}.$$

The following classical result is due to C.S.Herz [14, Thm.3] (notice that in [14] the Fourier transform is defined with the positive sign in the exponent; this explains the minus sign in the expression of $\widehat{\chi_C}(\xi)$ which appears in the following statement of the theorem).

Theorem 3.4 *Let $C \subset \mathbb{R}^2$ be a convex body having a smooth boundary with everywhere positive curvature. Then asymptotically, for $|\xi| \to \infty$,*

$$\widehat{\chi_C}(\xi) = -\frac{1}{2\pi i|\xi|}\,\left(g(\xi) - \overline{g(-\xi)}\right) + O(|\xi|^{-\frac{5}{2}}),$$

where g is given by (12).

Observe that, if $C = D(R,0)$, as in Section 3.1, then $K(\theta) = R^{-1}$, $\sigma(\theta) = R\,\Theta(\theta)$ and, thus,

$$\begin{aligned}
\widehat{\chi_D}(\xi) &= \frac{1}{2\pi i|\xi|}\,|\xi|^{-1/2}R^{1/2}\,\left(e^{2\pi i R|\xi|}e^{-\frac{\pi i}{4}} - e^{-2\pi i R|\xi|}e^{\frac{\pi i}{4}}\right) + O(|\xi|^{-\frac{5}{2}})\\
&= \frac{1}{\pi}\,|\xi|^{-3/2}R^{1/2}\,\sin\left(2\pi i R|\xi| - \frac{\pi}{4}\right) + O(|\xi|^{-\frac{5}{2}})\\
&= \frac{1}{\pi}\,|\xi|^{-3/2}R^{1/2}\,\cos\left(2\pi i R|\xi| - \frac{3\pi}{4}\right) + O(|\xi|^{-\frac{5}{2}}).
\end{aligned}$$

The last formula coincides with the expression describing the asymptotic behavior of the Fourier transform of the characteristic function of the disc $\hat{B}_R(\xi) = R^2|R\xi|^{-1}J_1(2\pi R|\xi|)$, which was used in Section 3.1.

In the following, we will also need the following simple observation.

Lemma 3.2 *Let $\sigma(\theta)$ and $\Theta(\theta)$ be defined as above. Then*

$$\lim_{\theta\to\theta_0}\frac{(\sigma(\theta_0)-\sigma(\theta))\cdot\Theta(\theta)}{\tan^2(\theta-\theta_0)} = -\frac{1}{2}|\sigma'(\theta_0)|^2.$$

Proof. By direct calculation, using De L'Hôpital rule we have:

$$\begin{aligned}
&\lim_{\theta\to\theta_0}\frac{(\sigma(\theta_0)-\sigma(\theta))\cdot\Theta(\theta)}{\tan^2(\theta-\theta_0)}\\
&= \lim_{\theta\to\theta_0}\frac{(\sigma(\theta_0)-\sigma(\theta))\cdot\Theta'(\theta)}{2\tan(\theta-\theta_0)\,\sec^2(\theta-\theta_0)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\theta\to\theta_0} \frac{(-\sigma'(\theta))\cdot\Theta'(\theta) + (\sigma(\theta_0)-\sigma(\theta))\cdot\Theta''(\theta)}{2[\sec^4(\theta-\theta_0) + 2\tan^2(\theta-\theta_0)\sec^2(\theta-\theta_0)]}\\
&= -\frac{1}{2}\sigma'(\theta_0)\cdot\Theta'(\theta_0)\\
&= -\frac{1}{2}|\sigma'(\theta_0)|^2. \qquad \square
\end{aligned}
$$

Observe that, for any θ, there exist a unique $\phi \in S^1$ such that $\frac{\sigma(\theta)}{\|\sigma(\theta)\|} = \Theta(\phi)$. Hence, we can write ϕ as $h(\theta)$, where the mapping $\theta \to h(\theta)$ is a diffeomorphism on S^1.

We can now state the following result which is a generalization of Theorem 3.2.

Theorem 3.5 *Let $C \subset \mathbb{R}^2$ be a convex body having a smooth boundary with everywhere positive curvature. Let $t \in P$, where $P = \{r\,\Theta(\phi) \in \mathbb{R}^2 : \phi \in H,\ 0 \le r < \infty\}$ and $H = \{\phi = h(\theta) : |\theta| \le \frac{\pi}{4}$ or $|\theta - \pi| \le \frac{\pi}{4}\}$, and $\beta(\alpha)$ be given by*

$$
\beta(\alpha) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } 0 < \alpha < \frac{1}{2} \\ \frac{\alpha+1}{2} & \text{if } \frac{1}{2} \le \alpha < 1. \end{cases}
$$

If $t = \sigma(\theta_0)$, for some θ_0, then

$$
\lim_{a\to 0^+} a^{-\beta(\alpha)}\, \mathcal{SH}_\psi^\alpha \chi_C(a, \tan\theta_0, \sigma(\theta_0)) \neq 0.
$$

If $t = \sigma(\theta_0)$ and $s \neq \tan\theta_0$, or if $t \notin \partial C$, then

$$
\lim_{a\to 0^+} a^{-\gamma}\, \mathcal{SH}_\psi^\alpha \chi_C(a, s, t) = 0, \qquad \text{for all } \gamma > 0.
$$

Proof. In the following, we will only consider the case $\alpha = \frac{1}{2}$. The case of a general $\alpha \in (0,1)$ follows by adapting the same ideas we use in Theorem 3.2 for the disc. For simplicity, in the following we will use the notation $\mathcal{SH}_\psi = \mathcal{SH}_\psi^{1/2}$.

We also recall that s is only defined for $|s| \le \frac{3}{2}$.

Using Theorem 3.4, we have:

$$
\begin{aligned}
\mathcal{SH}_\psi \chi_C(a,s,t) &= \langle \widehat{\chi}_C, \hat{\psi}_{ast} \rangle\\
&= \int_{\mathbb{R}^2} \overline{\hat{\psi}_{ast}(\xi)}\, \widehat{\chi}_C(\xi)\, d\xi\\
&= a^{\frac{3}{4}} \int_{\mathbb{R}^2} \hat{\psi}_1(a\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_1} - s))\, e^{2\pi i \xi t}\, \widehat{\chi}_C(\xi_1,\xi_2)\, d\xi\\
&= \frac{a^{\frac{3}{4}}}{2\pi i}\,(I + E),
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \int_{\mathbb{R}^2} \hat{\psi}_1(a\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_2} - s))\, \frac{e^{2\pi i \xi t}}{|\xi|} \left(\overline{g(-\xi)} - g(\xi)\right) d\xi; \\
E &= 2\pi i \int_{\mathbb{R}^2} \hat{\psi}_1(a\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_2} - s))\, e^{2\pi i \xi t}\, O(|\xi|^{-\frac{5}{2}})\, d\xi.
\end{aligned}
$$

It is easy to verify that $\lim_{a\to 0} E = 0$. Hence it only remains to examine $\lim_{a\to 0} I$. To do that, let $I = I_1 - I_2$, where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^2} \hat{\psi}_1(a\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_2} - s))\, \frac{e^{2\pi i \xi t}}{|\xi|}\, \overline{g(-\xi)}\, d\xi; \\
I_2 &= \int_{\mathbb{R}^2} \hat{\psi}_1(a\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_2} - s))\, \frac{e^{2\pi i \xi t}}{|\xi|}\, g(\xi)\, d\xi.
\end{aligned}
$$

Converting ξ to polar coordinates, we have

$$
I_1 = e^{-\frac{\pi}{4}i} a^{-\frac{1}{2}} \left(I_{11} + I_{12}\right),
$$

where

$$
\begin{aligned}
I_{11} &= \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta+\pi)e^{2\pi i \frac{\rho}{a}(t+\sigma(\theta+\pi))\cdot\Theta(\theta+\pi)}\rho^{-\frac{1}{2}}\, d\theta d\rho; \\
&= \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta+\pi)e^{2\pi i \frac{\rho}{a}(t-\sigma(\theta+\pi))\cdot\Theta(\theta)}\rho^{-\frac{1}{2}}\, d\theta d\rho; \\
I_{12} &= \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta+\pi)e^{2\pi i \frac{\rho}{a}(t+\sigma(\theta+\pi))\cdot\Theta(\theta+\pi)}\rho^{-\frac{1}{2}}\, d\theta d\rho; \\
&= -\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta)e^{2\pi i \frac{\rho}{a}(t+\sigma(\theta))\cdot\Theta(\theta)}\rho^{-\frac{1}{2}}\, d\theta d\rho \\
&= -\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta)e^{2\pi i \frac{\rho}{a}(t-\sigma(\theta))\cdot\Theta(\theta+\pi)}\rho^{-\frac{1}{2}}\, d\theta d\rho.
\end{aligned}
$$

In the expression of I_{12}, we have used the fact that $\hat{\psi}_1$ is an odd function: hence $\hat{\psi}_1(\rho\cos(\theta+\pi)) = \hat{\psi}_1(-\rho\cos\theta) = -\hat{\psi}_1(\rho\cos\theta)$.

Similarly, we write

$$
I_2 = e^{\frac{\pi}{4}i} a^{-\frac{1}{2}} \left(I_{21} + I_{22}\right),
$$

where

$$
I_{21} = \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta)e^{2\pi i \frac{\rho}{a}(t-\sigma(\theta))\cdot\Theta(\theta)}\rho^{-\frac{1}{2}}\, d\theta d\rho;
$$

$$I_{22} = -\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s))K^{-\frac{1}{2}}(\theta+\pi)e^{2\pi i\frac{\rho}{a}(t-\sigma(\theta+\pi))\cdot\Theta(\theta+\pi)}\rho^{-\frac{1}{2}}d\theta d\rho.$$

We will now examine the asymptotic decay of the integrals I, as $a \to 0$, for different values of t and s.

Case 1: $t \notin \partial C$.

Since $t \notin \partial C$, there exists a constant $c_t > 0$ such that $\inf_{\theta\in[0,2\pi)} |t-\sigma(\theta)| = c_t$. Let $J = \{\theta : |\tan\theta - s| \le a^{\frac{1}{2}}\} \bigcap (-\frac{\pi}{2}, \frac{\pi}{2})$, $J_1 = \{\theta : |(t-\sigma(\theta))\cdot\Theta(\theta)| \ge \frac{c_t}{\sqrt{2}}\} \bigcap J$, and $J_2 = J \setminus J_1$. Since the vectors $\Theta(\theta), \Theta'(\theta)$ form an orthonormal basis in $\mathbb{R}^2$, it follows that, on the set J_2, we have $|(t-\sigma(\theta))\cdot\Theta'(\theta)| \ge \frac{c_t}{\sqrt{2}}$. By Lemma 3.1(ii), we have that

$$\frac{d}{d\theta}[(t-\sigma(\theta))\cdot\Theta(\theta)] = (t-\sigma(\theta))\cdot\Theta'(\theta).$$

Hence we can express each one of the integrals I_{kj}, for $k, j = 1, 2$, as a sum of a term where $\theta \in J_1$ and another term where $\theta \in J_2$, and integrate by parts as follows. On J_1, we integrate by parts with respect to the variable ρ; on J_2 we integrate by parts with respect to the variable θ. Doing this repeatedly, it yields that, for any positive integer N, $|I_{kj}| \le C_N\, a^{\frac{N}{2}}$, for $k, j = 1, 2$. Thus $\lim_{a\to 0} I = 0$.

Case 2: $t_0 = \sigma(\theta_0) \in P \bigcap \partial C$, but $s \ne \tan(\theta_0)$.

We will only consider the case $|\theta_0| \le \frac{\pi}{4}$. The argument for θ_0 such thta $|\theta_0 - \pi| \le \frac{\pi}{4}$ is similar. Since $|s| \le \frac{3}{2}$, there exists a θ_1 such that $s = \tan\theta_1$ where $|\theta_1| \le \frac{\pi}{3}$ and $\theta_1 \ne \theta_0$. From the support assumption of $\hat{\psi}_2$, it follows that $\theta \to \theta_1$ as $a \to 0$. Also, by Lemma 3.1(iii) it follows that (for all a sufficiently small) one can assume that $\inf_{\theta\in J} |\sigma(\theta_0) - \sigma(\theta)| = c_{\theta_0} > 0$. Now one can employ the same argument as in Case 1 to conclude that $\lim_{a\to 0} I = 0$.

Case 3: $t_0 = \sigma(\theta_0) \in P \bigcap \partial C$ and $s = \tan(\theta_0)$.

Let us first examine the term I_{21}. We assume that $|\theta_0| \le \frac{\pi}{4}$. As in the proof of Theorem 3.2, we use the change of variable $u = a^{-\frac{1}{2}} \tan(\theta - \theta_0)$. Hence, by applying Lemma 3.2 we have that

$$\lim_{a\to 0} a^{-\frac{1}{2}} I_{21}$$
$$= \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, K^{-\frac{1}{2}}(\theta_0) \int_{-1}^{1} \hat{\psi}_2(u\sec^2\theta_0)\, e^{-i\pi\rho|\sigma'(\theta_0)|u^2}\, du\, \rho^{-\frac{1}{2}}\, d\rho.$$

Similarly, recalling that $\Theta(\theta+\pi) = -\Theta(\theta)$, we have

$$
\begin{aligned}
&\lim_{a\to 0} a^{-\frac{1}{2}} I_{12} \\
&= -\int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, K^{-\frac{1}{2}}(\theta_0) \int_{-1}^1 \hat{\psi}_2(u\sec^2\theta_0)\, e^{i\pi\rho|\sigma'(\theta_0)|u^2}\, du\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}
$$

Also, using the same argument as in Case 2, it is easy to see that $|I_{11}| \le C_N\, a^{\frac{N}{2}}$ and $|I_{22}| \le C_N\, a^{\frac{N}{2}}$. Thus we have

$$
\begin{aligned}
&\lim_{a\to 0} 2\pi i a^{-\frac{3}{4}}\, \mathcal{SH}_\psi \chi_C(a,s,t) \\
&= \lim_{a\to 0} 2\pi i a^{-\frac{3}{4}}\, \mathcal{SH}_\psi \chi_C(a,\tan(\theta_0),\sigma(\theta_0)) \\
&= \lim_{a\to 0} e^{-\frac{\pi i}{4}} a^{-1/2} I_{12} - \lim_{a\to 0} e^{\frac{\pi i}{4}} a^{-1/2} I_{21} \\
&= -e^{-\frac{\pi}{4} i} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, K^{-\frac{1}{2}}(\theta_0) \int_{-1}^1 \hat{\psi}_2(u\sec^2\theta_0)\, e^{i\pi\rho|\sigma'(\theta_0)|u^2}\, du\, \rho^{-\frac{1}{2}}\, d\rho \\
&- e^{\frac{\pi}{4} i} \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0)\, K^{-\frac{1}{2}}(\theta_0) \int_{-1}^1 \hat{\psi}_2(u\sec^2\theta_0)\, e^{-i\pi\rho|\sigma'(\theta_0)|u^2}\, du\, \rho^{-\frac{1}{2}}\, d\rho \\
&= -\sqrt{2}\, K^{-\frac{1}{2}}(\theta_0) \int_0^\infty \hat{\psi}_1(\rho\cos\theta_0) \int_{-1}^1 \hat{\psi}_2(u\sec^2\theta_0) \\
&\quad \times \Big(\cos(\pi\rho|\sigma'(\theta_0)|u^2) + \sin(\pi\rho|\sigma'(\theta_0)|u^2)\Big)\, du\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\lim_{a\to 0} a^{-\frac{3}{4}}\ |\mathcal{SH}_\psi \chi_C(a,\tan(\theta_0),\sigma(\theta_0))| \\
&= \frac{\sqrt{2}}{2\pi} K^{-\frac{1}{2}}(\theta_0) \int_0^\infty \hat{\psi}_1(\rho\ \cos\theta_0) \int_{-1}^1 \hat{\psi}_2(u\sec^2\theta_0) \\
&\quad \times \Big(\cos(\pi\rho|\sigma'(\theta_0)|u^2) + \sin(\pi\rho|\sigma'(\theta_0)|u^2)\Big)\, du\, \rho^{-\frac{1}{2}}\, d\rho.
\end{aligned}
$$

The proof that $\lim_{a\to 0} a^{-\frac{3}{4}} |\mathcal{SH}_\psi \chi_C(a,\tan(\theta_0),\sigma(\theta_0))| > 0$ now follows exactly as in Theorem 3.2, by replacing R with $|\sigma'(\theta_0)|$. $\square$

From Theorem 3.3 we have the corollary

Corollary 3.2 *Let $H, P, \beta(\alpha)$ be defined as in Theorem 3.3. Then there is a constant $c > 0$ such that*

$$
\begin{aligned}
&\sup_{|s|\le\frac{3}{2},\, t\in P} \lim_{a\to 0^+} a^{-\beta(\alpha)}\, \mathcal{SH}_\psi^\alpha \chi_C(a,s,t) \\
&= \sup_{|\theta_0|\le\frac{\pi}{4}} \Big(\lim_{a\to 0^+} a^{-\beta(\alpha)}\, \mathcal{SH}_\psi^\alpha \chi_C(a,\tan\theta_0,\sigma(\theta_0))\Big) = c > 0.
\end{aligned}
$$

An extension of Theorem 3.3 to the case of a convex body C with boundary having nonvanishing curvature also follows easily from the argument used above.

4 Extensions and Generalizations

So far, we have only considered the continuous shearlet transform of characteristic functions of sets. In order to provide a more realistic model for a certain class of images containing edges, in this section, we extend our analysis to a more general class of compactly supported functions, which are not necessarily constant or piecewise constant.

More precisely, let Ω be a bounded open subset of $\mathbb{R}^2$ and assume a smooth partition

$$\Omega = \bigcup_{n=1}^{L} \Omega_n \cup \Gamma,$$

where:

(1) for each $n = 1, \ldots, L$, Ω_n is a connected open domain;
(2) each boundary $\partial_\Omega \Omega_n$ is generated by a C^3 curve γ_n and each of the boundary curves γ_n can be parametrized as $(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)$ where $\rho(\theta) : [0, 2\pi) \to [0, 1]$ is a radius function;
(3) $\Gamma = \bigcup_{n=1}^{L} \partial_\Omega \Omega_n$, where $\partial_\Omega X$ denotes the relative topological boundary in Ω of $X \subset \Omega$.

Finally, we define the space $E^{1,3}(\Omega)$ as the collection of functions which are compactly supported in Ω and have the form

$$f(x) = \sum_{n=1}^{L} f_n(x)\, \chi_{\Omega_n}(x) \;\text{ for }\; x \in \Omega \backslash \Gamma$$

where, for each $n = 1, \ldots, L$, $f_n \in C_0^1(\Omega)$ with $\sum_{|\alpha| \le 1} \|D^\alpha f_n\|_\infty \le C$ for some $C > 0$, and the sets Ω_n are pairwise disjoint in measure. Notice that the definition does not specify the function value along the boundary set Γ. For each x in a C^3 component of Γ, we define the *jump of f at x*, denoted by $[f]_x$, to be

$$[f]_x = \lim_{\epsilon \to 0^+} f(x + \epsilon\, v_x) - f(x - \epsilon\, v_x)$$

where v_x is an unit normal vector along Γ at x.

The functions in $E^{1,3}(\Omega)$ provide a reasonable model for images where the set Γ describes the boundaries of different objects. Each $u_n(x) = f_n(x)\, \chi_{\Omega_n}(x)$ models the relatively homogeneous interior of a single object. Notice that similar image models are quite common, for example, in the variational approach to image processing [6, Ch.3].

For $x \in \mathbb{R}^2$, $L > 0$, we denote the cube of center x and side-length $2L$ by $Q(x, L)$; that is, $Q(x, L) = [-L, L]^2 + x$. For $k = (k_1, k_2) \in \mathbb{Z}^2$, let $M \in \mathbb{N}$ be sufficiently large so that each of the boundary curves γ_n may be parametrized as either $(E(t_2), t_2)$ or $(t_1, E(t_1))$ in $Q(\frac{k}{M}, \frac{1}{M})$ if $Q(\frac{k}{M}, \frac{1}{M}) \cap \Gamma \neq \emptyset$.

We will now examine the behavior of the continuous shearlet transform of functions f in $E^{1,3}(\Omega)$. For simplicity, we will restrict our analysis to the $\mathcal{SH}_\psi^\alpha f$, with $\alpha = \frac{1}{2}$. The extension to the general situation $0 < \alpha < 1$ can be easily obtained by adapting the ideas which will presented in the following for $\alpha = \frac{1}{2}$. Similarly to Theorems 3.2 and 3.5, we will only consider the "horizontal" transform; one needs to consider the "vertical" shearlet transform $\mathcal{SH}_\psi^{(v)}$ to deal with the shear variable s for $|s| > 1$. We have the following result.

Theorem 4.1 *Let $f \in E^{1,3}(\Omega)$ and suppose that the boundary curve γ_n, for some n, is parametrized as $(E(t_2), t_2)$ in $Q(\frac{k}{M}, \frac{1}{M})$ for some $k \in \mathbb{Z}^2$, and that $t = (E(t_2), t_2) \in Q(\frac{k}{M}, \frac{1}{2M})$ for some t_2. If $s = -E'(t_2)$ and $|s| \leq 1$ then there exist positive constants C_1 and C_2 such that*

$$C_1 \, |[f]_t| \leq \lim_{a \to 0^+} a^{-\frac{3}{4}} \, |\mathcal{SH}_\psi f(a, s, t)| \leq C_2 \, |[f]_t|. \tag{13}$$

If $s \neq -E'(t_2)$ and $|s| \leq 1$, then

$$\lim_{a \to 0^+} a^{-\frac{3}{4}} \, |\mathcal{SH}_\psi f(a, s, t)| = 0. \tag{14}$$

Theorem 4.1 generalizes the results from Section 3 to functions which are not necessarily characteristic functions of sets and to sets which are not necessarily convex. In fact, we allow the curvature to be zero. However, the estimate we obtain here are weaker than the results from Section 3. In particular, the estimate (3) for the characteristic function of a disc (and similarly for other convex bodies whose boundary has nonvanishing curvature) holds for all $\gamma > 0$. On the other hand, for the more general functions in $E^{1,3}(\Omega)$, where the boundaries are not necessarily curved, we can only prove the weaker estimate (14).

To prove Theorem 4.1, we will need the following result from [16].

Theorem 4.2 *For $p \in \mathbb{R}$, consider the half-plane*

$$V_p = \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 \geq p x_2\}.$$

Let $t = (t_1, t_2) \in \mathbb{R}^2$ be such that $t_1 = p t_2$. If $s = -p$ and $|s| \leq 1$ then

$$\lim_{a \to 0^+} a^{-\frac{3}{4}} \, |\mathcal{SH}_\psi \chi_{V_p}(a, s, t)| > 0. \tag{15}$$

If $s \neq -p$, and $|s| \leq 1$ then

$$\lim_{a\to 0^+} a^{-\frac{3}{4}} \left|\mathcal{SH}_{\psi}\chi_{V_p}(a,s,t)\right| = 0. \tag{16}$$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. In the following, we will use the same c to indicate different uniform constants. Let p and a be positive real numbers such that $\frac{3}{8} < p < \frac{1}{2}$, and a be sufficiently small so that $Q(0,2a^p) + t \subset Q(\frac{k}{M}, \frac{1}{M})$ for all $t \in Q(\frac{k}{M}, \frac{1}{2M})$.

Let $t = (t_1, t_2)$ be a point on the curve γ_n. Without loss of generality, one may assume that $t = (0,0)$ and the curve γ_n is parametrized as $(E(t_2), t_2)$ in the set $Q(0, 2a^p)$ Thus, $E(0) = 0$. For simplicity, we will assume that $E'(0) = 0$. The proof of the case $E'(0) \neq 0$ can be obtained from the case $E'(0) = 0$ by applying a suitable change of coordinates.

Since $f \in E^{1,3}(\Omega)$, we may write the function f on $Q(0, 2a^p)$ as:

$$f(t_1,t_2) = f_0(t_1,t_2)\chi_{\{t_1 > E(t_2)\}\cap Q(0,2a^p)} + f_1(t_1,t_2)\chi_{\{t_1 < E(t_2)\}\cap Q(0,2a^p)}$$

where f_0 and $f_1 \in C^1(\Omega)$. Let

$$R = \{(t_1,t_2) \in Q(0,2a^p) : t_1 > E(t_2)\}.$$

We will start by considering the case $E''(0) > 0$. In this case, let us consider the osculating disc $\tilde{D}$ to γ_n at the origin, which is given by

$$\tilde{D} = \{(t_1,t_2) \in \mathbb{R}^2 : \left(t_1 - \frac{1}{E''(0)}\right)^2 + {t_2}^2 \leq \frac{1}{E''(0)^2}\}.$$

For $t_2 \leq \frac{1}{E''(0)}$, the boundary of $\tilde{D}$ is the half circle

$$P(t_2) = \frac{1}{E''(0)} - \sqrt{\frac{1}{E''(0)^2} - {t_2}^2}.$$

Notice that $P(0) = E(0) = 0$, $P'(0) = E'(0) = 0$ and $P''(0) = E''(0)$. Hence, by computing the power series of $E(t_2)$ around $t_2 = 0$ we have

$$\sup_{t_2\in[-2a^p,2a^p]} |E(t_2) - P(t_2)| = O(a^{3p}). \tag{17}$$

Thus:

$$\int_{Q(0,2a^p)} |\chi_R - \chi_{\tilde{D}}| = \int_{R\cap\tilde{D}} |\chi_R - \chi_{\tilde{D}}| + \int_{(R\cup\tilde{D})^c\cap Q(0,2a^p)} |\chi_R - \chi_{\tilde{D}}| + \int_{Q(0,2a^p)\cap(R\Delta\tilde{D})} |\chi_R - \chi_{\tilde{D}}|,$$

where $R\Delta\tilde{D} = (R\cap\tilde{D}^c)\cup(R^c\cap\tilde{D})$. Notice that

$$\int_{R\cap\tilde{D}} |\chi_R - \chi_{\tilde{D}}| = 0 \quad \text{and} \quad \int_{(R\cup\tilde{D})^c\cap Q(0,2a^p)} |\chi_R - \chi_{\tilde{D}}| = 0,$$

and that, by (17), we have

$$\int_{Q(0,2a^p)\cap(R\Delta\tilde{D})} |\chi_R - \chi_{\tilde{D}}| \le \int_{-2a^p}^{2a^p} |E(t_2) - P(t_2)|\, dt_2 \le c\, a^{4p}.$$

Thus, it follows that

$$\int_{Q(0,2a^p)} |\chi_R - \chi_{\tilde{D}}| \le c\, a^{4p}.$$

From the last equality it follows that

$$\begin{aligned} |\langle f_0(0)\chi_R, \psi^0_{as0}\rangle - \langle f_0(0)\chi_{\tilde{D}\cap Q(0,2a^p)}, \psi_{as0}\rangle| &\le |f_0(0)|\, \|\psi_{as0}\|_\infty \int_{Q(0,2a^p)} |\chi_R - \chi_{\tilde{D}}| \\ &\le c\, a^{4p-\frac{3}{4}}. \end{aligned} \tag{18}$$

Also, by the Mean Value Theorem (recall that $f_0 \in C^1$) we have

$$\sup_{x\in Q(0,2a^p)} |f_0(x) - f_0(0)| \le c\, a^p.$$

Hence, from the last two inequalities, observing that $4p > \frac{3}{4}$ (since $p > \frac{3}{8}$), we have that

$$\begin{aligned} |\langle f_0\chi_R, \psi_{as0}\rangle - \langle f_0(0)\chi_R, \psi_{as0}\rangle| &\le \langle |f_0 - f_0(0)|\chi_{Q(0,2a^p)}, |\psi_{as0}|\rangle \\ &\le c\, \|\psi\|_1\, a^p\, a^{4p-\frac{3}{4}} \\ &\le c\, \|\psi\|_1\, a^{\frac{3}{4}+p}. \end{aligned} \tag{19}$$

Since $\hat{\psi} \in C_0^\infty$, for each $n \in \mathbb{N}$, there exists $c_n > 0$ such that $|\psi(x)| \le c_n\,(1+|x|)^{-n}$. Hence, for $0 < a < 1$ and $\frac{3}{8} < p < \frac{1}{2}$, we have:

$$\begin{aligned} \langle \chi_{(Q(0,a^p))^c}, |\psi_{a00}|\rangle &\le c_n\, a^{-\frac{3}{4}} \iint_{(Q(0,a^p))^c} \left(1 + \sqrt{a^{-2}x_1^2 + a^{-1}x_2^2}\right)^{-n} dx_1\, dx_2 \\ &\le c_n\, a^{-\frac{3}{4}} \iint_{(Q(0,a^p))^c} \left(\sqrt{a^{-1}x_1^2 + a^{-1}x_2^2}\right)^{-n} dx_1\, dx_2 \\ &\quad \left(\text{Converting to polar coordinates with } \; r = \sqrt{x_1^2 + x_2^2}\right) \\ &\le 2\pi\, c_n\, a^{\frac{n}{2}-\frac{3}{4}} \int_{a^p}^\infty r^{1-n}\, dr \\ &= \frac{2\pi\, c_n}{n-2}\, a^{n(\frac{1}{2}-p)}\, a^{2p-\frac{3}{4}} \end{aligned}$$

$$\le c_n\, a^{n(\frac{1}{2}-p)}. \tag{20}$$

Notice that $B_s\,(Q(0,2a^p))^c \subset (Q(0,a^p))^c$ for $|s|\le 1$. This implies that

$$\langle \chi_{(Q(0,2a^p))^c}, |\psi_{as0}|\rangle = \langle \chi_{B_s\,(Q(0,2a^p))^c}, |\psi^0_{a00}|\rangle \le \langle \chi_{(Q(0,a^p))^c}, |\psi_{a00}|\rangle.$$

Thus, by (20), for $|s|\le 1$ we have that

$$|\langle f\,\chi_{(Q(0,2a^p))^c}, \psi_{as0}\rangle| \le \|f\|_\infty\,\langle \chi_{(Q(0,2a^p))^c}, |\psi_{as0}|\rangle \le c_n\, a^{(\frac{1}{2}-p)n}. \tag{21}$$

Since $\tilde D \cap Q(0,2a^p) = \tilde D \setminus (\tilde D \cap (Q(0,2a^p))^c)$, we have that

$$|\langle f_0(0)\,\chi_{\tilde D\cap Q(0,2a^p)}, \psi^0_{a00}\rangle| = |f_0(0)|\,|\langle \chi_{\tilde D}, \psi_{a00}\rangle - \langle \chi_{\tilde D\cap (Q(0,2a^p))^c}, \psi^0_{a00}\rangle|.$$

This implies that

$$\begin{aligned}
|\langle \chi_{\tilde D}, \psi_{a00}\rangle| - \langle \chi_{(Q(0,2a^p))^c}, |\psi_{a00}|\rangle &\le \frac{|\langle f_0(0)\,\chi_{\tilde D\cap Q(0,2a^p)}, \psi_{a00}\rangle|}{|f_0(0)|}\\
&\le |\langle \chi_{\tilde D}, \psi_{a00}\rangle| + \langle \chi_{(Q(0,2a^p))^c}, |\psi_{a00}|\rangle.
\end{aligned}$$

Hence, by (21) we have

$$\begin{aligned}
|f_0(0)\,|\Big(|\langle \chi_{\tilde D}, \psi_{a00}\rangle| - c_n\, a^{(\frac{1}{2}-p)n}\Big) &\le |\langle f_0(0)\chi_{\tilde D\cap Q(0,2a^p)}, \psi_{a00}\rangle|\\
&\le |f_0(0)|\Big(|\langle \chi_{\tilde D}, \psi_{a00}\rangle| + c_n\, a^{(\frac{1}{2}-p)n}\Big).
\end{aligned}$$

Finally, using the estimate (2), we conclude that

$$|\langle f_0(0)\,\chi_{\tilde D\cap Q(0,2a^p)}, \psi_{a00}\rangle| \sim |f_0(0)\,|a^{\frac{3}{4}} \quad \text{as } a\to 0^+. \tag{22}$$

We now estimate $|\langle f_0\chi_R, \psi_{a00}\rangle|$. Notice that

$$\Big||\langle f_0\chi_R, \psi_{a00}\rangle| - |\langle f_0(0)\chi_R, \psi_{a00}\rangle|\Big| \le |\langle (f_0 - f_0(0))\chi_R, \psi_{a00}\rangle|.$$

Hence, by (19) we have that

$$\Big||\langle f_0\chi_R, \psi_{a00}\rangle| - |\langle f_0(0)\chi_R, \psi_{a00}\rangle|\Big| \le c\, a^{\frac{3}{4}+p}.$$

Now, using (18) and the last inequality, it follows that

$$\Big||\langle f_0\chi_R, \psi_{a00}\rangle| - |\langle f_0(0)\chi_{\tilde D\cap Q(0,2a^p)}, \psi_{a00}\rangle|\Big| \le c\,(a^{4p-\frac{3}{4}} + a^{p+\frac{3}{4}}).$$

Thus, from the last inequality and (22) it follows that (recall $4p > \frac{3}{4}$)

$$|\langle f_0 \chi_R, \psi_{a00}\rangle| \sim |f_0(0)|a^{\frac{3}{4}} \quad \text{as } a \to 0^+. \tag{23}$$

Finally, we estimate $|\langle f\chi_{Q(0,2a^p)}, \psi_{a00}\rangle|$. We first observe that

$$\begin{aligned}|\langle f\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| &= \left|\langle (f_0\,\chi_R, \psi_{a00}\rangle + \langle f_1\,\chi_{Q(0,2a^p)\cap R^c}, \psi_{a00}\rangle\right| \\ &= \left|\langle (f_0 - f_1)\,\chi_R, \psi_{a00}\rangle + \langle f_1\,\chi_{Q(0,2a^p)}, \psi_{a00}\rangle\right|.\end{aligned}$$

Thus:

$$\Big||\langle f\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| - |\langle (f_0 - f_1)\chi_R, \psi_{a00}\rangle|\Big| \le |\langle f_1\chi_{Q(0,2a^p)}, \psi_{a00}\rangle|. \tag{24}$$

By the Mean Value Theorem (since $f_1 \in C^1$) we have:

$$\sup_{x\in Q(0,2a^p)} |f_1(x) - f_1(0)| \le c\,a^p.$$

Hence, arguing again as for inequality (19), we obtain:

$$|\langle f_1\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| \le c\,a^{p+\frac{3}{4}} + |\langle f_1(0)\chi_{Q(0,2a^p)}, \psi_{a00}\rangle|.$$

Since $\int_{\mathbb{R}^2} \psi_{a00} = \int_{Q(0,2a^p)\cup(Q(0,2a^p))^c} \psi_{a00} = 0$, (20) implies that, for sufficiently small $a > 0$,

$$|\langle f_1(0)\,\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| \le |\langle f_1(0)\,\chi_{(Q(0,2a^p))^c}, \psi_{a00}\rangle| \le c\,a^{(\frac{1}{2}-p)n},$$

and, thus,

$$|\langle f_1\,\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| \le c\,a^{(\frac{1}{2}-p)n}.$$

Thus, by (24), for sufficiently small $a > 0$ we have that

$$\Big||\langle f\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| - |\langle (f_0 - f_1)\chi_R, \psi_{a00}\rangle|\Big| \le c\,a^{(\frac{1}{2}-p)n}. \tag{25}$$

Notice that, for n large enough, $a^{(\frac{1}{2}-p)n} = o(a^{\frac{3}{4}})$. Using the last inequality and replacing f_0 by $f_0 - f_1$ in (23) we obtain:

$$|\langle f\chi_{Q(0,2a^p)}, \psi_{a00}\rangle| \sim |[f]_0|a^{\frac{3}{4}} \quad \text{as } a \to 0^+.$$

Since

$$\langle f, \psi_{a00}\rangle = \langle f\chi_{Q(0,2a^p)}, \psi_{a00}\rangle + \langle f\chi_{(Q(0,2a^p))^c}, \psi_{a00}\rangle,$$

inequalities (20) and (25) gives the proof of (13) when $E''(0) > 0$. The proof of the case $E''(0) < 0$ is similar.

When $E''(0) = 0$, we cannot define an osculating disc $\tilde{D}$ to the curve $(E(t_2), t_2)$ at the origin as done above. Instead, in this case, we consider the half plane

$$\tilde{H} = \{(t_1, t_2) \in \mathbb{R}^2 : t_1 > 0\}.$$

Then we can proceed very similarly to what done above. The only significant difference is that, to obtain (22), we will use the estimate (15) for the half-plane rather than the estimate (2) for the disc we used above.

To prove (14), we can follow the same structure of the proof described above. However we will now use estimates (3) and (16) for the non-normal orientation, rather than estimates (2) and (15) used above for the normal orientation. □

5 Conclusion

The idea of using wavelet-like continuous transforms for the analysis of the set of singularities of distributions can be traced back to the notion of *wave packet transforms*, introduced independently by Bros and Iagolnitzer [1] and Córdoba and Fefferman [7]. More recently, Candès and Donoho [3,4] have introduced the continuous curvelet transform, which uses parabolic scaling and rotations in polar coordinates and whose decay properties, at fine scales, can be used to identify the wavefront set of distributions. As mentioned above, Kutyniok and Labate [16] have proved a similar result using the continuous shearlet transform. The shearlet approach has the advantage of a simplified mathematical structure deriving from the use of the affine framework.

In this paper, we focused on the analysis of a special class of functions, consisting of several smooth regions separate by smooth boundaries at which jumps occur. For these functions, we have refined the wavefront characterization obtained in [3,16] by showing that the continuous shearlet transform exactly describes the location and orientation of the boundary set. As mentioned above, the functions considered in this paper are useful to model a large class of images with edges and, thus, our results indicate the great potential of the shearlet transform for the development of improved algorithms for edge detection and analysis.

Indeed, traditional wavelet methods such as the Wavelet Transform Modulus Maxima algorithm [19–21] are frequently applied to edge detection. Despite their success, these methods have a limited ability to detect the orientations of singularities, and this affects the accuracy of edge detection algorithms especially in the presence of noise. In fact, the estimated edge orientation is typically used to "track" edges and distinguish true edges from noise [5,27]. On the contrary, we have shown that the shearlet approach is especially adapted to capture the geometry of edges. Preliminary numerical tests conducted by one of the authors and his collaborators [25] have shown that a shearlet-based edge detector significantly outperforms traditional methods in the estimation of the edge orientation. In addition, its numerical implementation is very efficient, thanks to the fast implementation of the shearlet transform obtained in [9]. A more detailed discussion of the applications of the shearlet transforms to edge detection and analysis will appear in a separate publication.

Finally, we notice that the results provided in our paper do not address the situation where the boundaries are not smooth. In the special case where the boundary is polygonal set, the estimates for the decay of the shearlet transform are given in [16] (see [4,3] for the curvelet case). The more general situation of piecewise smooth boundaries requires further investigation and will be presented elsewhere.

A Construction of $\hat{\psi}_1$, $\hat{\psi}_2$

We briefly describe how to construct functions $\hat{\psi}_1$, $\hat{\psi}_2$ such that: (i) they satisfy the assumptions of Proposition 2.1; (ii) $\hat{\psi}_1$ is a smooth odd function; (iii) $\hat{\psi}_2$ is an even smooth function which is decreasing for $\xi \geq 0$.

(i) Let $f \in C_0^\infty((\frac{1}{2}, 2))$ such that $0 \leq f \leq 1$ and $f(x_0) \neq 0$ for some $x_0 \in (\frac{1}{2}, 2)$. By multiplying a suitable constant, we may assume that $\int_0^\infty \frac{f^2(x)}{x}\, dx = 1$. Let $\hat{\psi}_1(\xi)$ be the odd extension of f so that $\hat{\psi}_1$ satisfies the admissibility condition $\int_0^\infty \frac{|\hat{\psi}_1(a\xi)|^2}{a} da = 1$ for almost $\xi \in \mathbb{R}$.

(ii) Let $f \in C_0^\infty((0,1))$ with $f \geq 0$, $\int_0^1 f(x)\, dx = 1$, $f \geq 1$ on $(\frac{3}{4}, \frac{7}{8})$ and $f = 0$ on $(0, \frac{1}{2}]$. This can be obtained from a standard bump function on (0,1) with value 1 on $(\frac{3}{4}, \frac{7}{8})$. For $x \in [0,1)$, let $g(x) = 1 - \int_0^x f(s)\, ds$. Let $\phi(x)$ be the even extension of $g(x)$ on $(-1,1)$. It follows that $\phi \in C_0^\infty((-1,,1))$ with the property that ϕ is nonnegative and decreasing on $[0,1)$ such that $\phi(x) = \phi(0) = 1$ for $0 \leq x \leq \frac{1}{2}$. Let $\hat{\psi}_2(x) = \frac{\phi(x)}{\int_{-1}^1 |\phi(x)|^2\, dx}$. Then it is easy to verify that $\hat{\psi}_2 \in C_0^\infty((-1,,1))$ with the properties that $\hat{\psi}_2$ is decreasing on $[0,1)$, $\int_{-1}^1 |\hat{\psi}_2(x)|^2\, dx = 1$ and $\hat{\psi}_2(x) = \hat{\psi}_2(0) \geq \frac{1}{2}$ for $0 \leq x \leq \frac{1}{2}$.

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