Remarks about Guido Weiss' contributions to wavelets

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Guido Weiss contributed very significantly to the study of wavelets and his work was instrumental to develop the mathematical theory of wavelets and make it into an integral part of harmonic analysis.

Wavelets were pioneered in the 1980's by Grossmann and Morlet to overcome the limitations of the short-time Fourier transform in signal processing applications. Their idea, based on the construction of function decompositions consisting of dilates and translates of a single template, was very fruitful and, through the seminal contributions of Lemariè and Meyer [8], Mallat [9] and Daubechies [2], led to the formulation of Multiresolution Analysis (MRA) and the definition of an algorithmic procedure to build wavelet decompositions customizable to the needs of signal processing applications.

When I met Guido in 2000, he had already published his outstanding monograph titled "A First Course on Wavelets" [6] jointly with E. Hernández, where everything that was currently known about the mathematics of wavelets was rigorously and carefully presented. Yet, a frequent complaint of Guido was that, "we really know nothing about wavelets". I interpreted his statement as Socratic attitude, where the act of relinquishing any assumptions precedes a renewed effort to a thoroughgoing investigation.

The notion of wavelet was still evolving during the 1990's. While MRA had been established as a very effective approach to build wavelet bases of the form

$$\mathcal{A}(\psi) = \{\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j \cdot -k), \quad j,k \in \mathbb{Z}\} \subset L^2(\mathbb{R}), \tag{1}$$

such approach left many fundamental mathematical questions unanswered, including questions about their multivariate extensions and the relation between *MRA wavelets* - those arising from the MRA construction or one of its variants - and *wavelets* understood in a more general sense.

In the univariate case, a *wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the collection $\mathcal{A}(\psi)$, given by (1), is an orthonormal basis (ONB) of $L^2(\mathbb{R})$; in this case, $\mathcal{A}(\psi)$ is called an ON wavelet system. Hence, a natural question is: what conditions are necessary or sufficient for ψ to be a wavalet? Additionally, as it was shown that there are wavelets that are not MRA wavelets [1], a related question is: what conditions are needed for ψ to be not simply a wavelet but also an MRA wavelet?

Guido led a number of students and collaborators to the investigation of these questions and it resulted in the discovery of two simple equations that completely characterize all (uni-variate) wavelets. That is, $\psi \in L^2(\mathbb{R})$ is a wavelet if and only if

$\|\psi\|_2 = 1$ and it satisfies

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1, \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{2}$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi+2m\pi))} = 0, \qquad \text{for a.e. } \xi \in \mathbb{R}, m \in 2\mathbb{Z}+1, \tag{3}$$

where $\hat{\psi}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\psi(x)\,dx$ is the Fourier transform of ψ. In fact, these equations were previously known, even though under more restrictive assumptions. Equation (2), in particular, was a known necessary condition and is often referred to as the *Calderón condition*, as versions of this resolution of identity have appeared in the works of Calderón. MRA wavelets could also be characterized using a simple equation; namely, a wavelet $\psi \in L^2(\mathbb{R})$ is an MRA wavelet if and only if

$$D_\psi(\xi) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi+2k\pi))|^2 = 1, \qquad \text{for a.e. } \xi \in \mathbb{R},$$

a fact first established by Guido's student Wang [10] and by Gripenberg [3], independently. The interested reader can find a detailed derivation and insightful discussion of these results in [6, Ch.7].

The line of investigation initiated by Guido went significantly further, leading to the discovery of characterization equations extending equations (2)-(3) to a much more general context. In a joint work with Guido and Eugenio Hernández [5], we considered very general collections of functions of the form

$$\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\} \subset L^2(\mathbb{R}^n), \tag{4}$$

where $\{g_p : p \in \mathcal{P}\}$ is a countable family in $L^2(\mathbb{R}^n)$, T_y is the translation operator defined by $T_y f(x) = f(x-y)$ and $\{C_p : p \in \mathcal{P}\}$ is a collection of $n \times n$ invertible matrices. Under minor technical conditions, it turns out that the system (4) is a Parseval frame[1] of $L^2(\mathbb{R}^n)$ if and only if it satisfies

$$\sum_{p\in\mathcal{P}} \tfrac{1}{|\det C_p|} |\hat{g}_p(\xi)|^2 = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}^n \tag{5}$$

$$\sum_{j\in\mathcal{P}_\alpha} \tfrac{1}{|\det C_p|} \hat{g}_p(\xi)\,\overline{\hat{g}_p(\xi+\alpha)} = 0, \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \alpha \in \Lambda, \tag{6}$$

where $\Lambda = \bigcup_{p\in\mathcal{P}} C_p^{-t}(\mathbb{Z}^n)$ and $\mathcal{P}_\alpha = \{p \in \mathcal{P} : C_p^t\alpha \in \mathbb{Z}^n\}$. One can show that equations (5) and (6) not only imply equations (2) and (3) but can also be used to derive characterization equations of Gabor systems, wave packet systems and general multidimensional wavelets. For instance, by letting $\mathcal{P} = \{(j,\ell) : j \in \mathbb{Z}, \ell = 1, \ldots, L\}$, $C_p = C_{j,\ell} = A^{-j}$, $g_p = g_{j,\ell} = D_A^j \psi^\ell$, where $\psi^\ell \in L^2(\mathbb{R}^n)$ and D_A is the dilation

[1] A set $\{\phi_\nu\} \subset L^2(\mathbb{R}^n)$ is a Parseval frame of $L^2(\mathbb{R}^n)$ if $\sum_\nu |\langle f, \phi_\nu\rangle|^2 = \|f\|^2$ holds for any $f \in L^2(\mathbb{R}^n)$. If, in addition, $\|\phi_\nu\| = 1$ for any ν, it follows that $\{\phi_\nu\}$ is also an ONB.

operator associated with the $n \times n$ invertible matrix A, defined by $(D_A f)(x) = |\det A|^{1/2} f(Ax)$, then the set (4) becomes the following affine system

$$\mathcal{A}(\psi^1, \ldots, \psi^\ell) = \{D_A^j T_k \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}. \tag{7}$$

The application of the general theory then yields that the affine system $\mathcal{A}(\psi^1, \ldots, \psi^\ell)$ is a Parseval frame of $L^2(\mathbb{R}^n)$ if and only if it satisfies the equations

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^\ell(B^j \xi)|^2 = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}^n \tag{8}$$

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_\alpha} \hat{\psi}(B^j \xi)\, \overline{\hat{\psi}(B^j(\xi + \alpha))} = 0, \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \alpha \in \bigcup_{j \in \mathbb{Z}} B^j \mathbb{Z}^n, \tag{9}$$

where $\mathcal{P}_\alpha = \{j \in \mathbb{Z} : B^{-j}\alpha \in \mathbb{Z}^n\}$ and $B = A^t$ is a $n \times n$ invertible matrix that is *expanding on a subspace* . The above equations characterize not only multivariate wavelet systems with expanding dilation matrices (i.e., matrices with eigenvalues whose modulus is larger than one) that are direct generalizations of the univariate wavelet basis (1) but also wavelet systems with non-expanding dilation matrices. This last observation was somewhat surprising, as it was commonly assumed that dilation matrices had to be expanding to generate a reproducing system. It was also very inspirational, as it motivated the construction of new wavelet-like representations of multivariate functions that are much more flexible than separable dyadic wavelets, such as *wavelets with composite dilations* [4] and *shearlets* [7]. This discovery spurred an intense research activity on the properties and applications of this class of wavelet-like systems for multivariate functions, with hundreds of publications.

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[1]See Definition 5.3 in [5].

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