Oversampling, Quasi Affine Frames and Wave Packets

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Abstract

In [11], three of the authors obtained a characterization of certain types of reproducing systems. In this work, we apply these results and methods to various affine–like, wave packets and Gabor systems to determine their frame properties. In particular, we study how oversampled systems inherit properties (like the frame bounds) of the original systems. Moreover, our approach allows us to study the phenomenon of oversampling in much greater generality than is found in the literature.

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1 Preliminaries

In order to describe the types of reproducing systems that we will consider in this study, we introduce the following concepts and notation.

A countable family $\{e_\alpha : \alpha \in \mathcal{A}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **frame** if there exist constants $0 < A \le B < \infty$ satisfying

$$A\,\|v\|^2 \;\le\; \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \;\le\; B\,\|v\|^2$$

for all $v \in \mathcal{H}$. If only the right hand side inequality holds, we say that $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a **Bessel system** with constant B. A frame is a **tight frame** if A and B can be chosen so that $A = B$, and is a **Parseval frame (PF)** if $A = B = 1$. Thus, if $\{e_\alpha : \alpha \in \mathcal{A}\}$ is a PF in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{\alpha\in\mathcal{A}} |\langle v, e_\alpha\rangle|^2 \tag{1.1}$$

for each $v \in \mathcal{H}$. This is equivalent to the reproducing formula

$$v = \sum_{\alpha\in\mathcal{A}} \langle v, e_\alpha\rangle\, e_\alpha \tag{1.2}$$

for all $v \in \mathcal{H}$, where the series in (1.2) converges in the norm of $\mathcal{H}$. We refer the reader to [12, Ch. 8] for the basic properties of frames that we shall use.

Let $\mathcal{P}$ be a countable collection of indices, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(\mathbb{R}^n)$ and $\{C_p : p \in \mathcal{P}\}$ a corresponding collection of matrices in $GL_n(\mathbb{R})$. For $y \in \mathbb{R}^n$, let T_y be the translation (by y) operator defined by $T_y\, f = f(\cdot - y)$. In [11] we study families of the form

$$\Phi^{\{g_p\}}_{\{C_p\}} \;=\; \big\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\big\}, \tag{1.3}$$

and we characterize those $\{g_p : p \in \mathcal{P}\}$ such that $\Phi^{\{g_p\}}_{\{C_p\}}$ is a PF (Parseval frame) for $L^2(\mathbb{R}^n)$. In order to state the main result of [11], we need to introduce the following notation:

$$\Lambda = \bigcup_{p\in\mathcal{P}} C_p^I\, \mathbb{Z}^n, \tag{1.4}$$

where $C_p^I = (C_p^T)^{-1}$ (= the inverse of the transpose of C_p), and for $\alpha \in \Lambda$,

$$\mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in C_p^I\, \mathbb{Z}^n\}. \tag{1.5}$$

If $\alpha = 0 \in \Lambda$, then $\mathcal{P}_0 = \mathcal{P}$; otherwise the best we can say is that $\mathcal{P}_\alpha \subset \mathcal{P}$. We note, in passing, that $L^* = C_p^I\, \mathbb{Z}^n$ is the dual of the translation lattice $L = C_p\, \mathbb{Z}^n$, in the sense that $\xi \in L^*$ iff $x \cdot \xi \in \mathbb{Z}$, for each $x \in L$. Let

$$\mathcal{D} = \mathcal{D}_E = \big\{f \in L^2(\mathbb{R}^n)\colon \hat f \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat f \text{ is compact in } \mathbb{R}^n \setminus E\big\}, \tag{1.6}$$

where E is a subspace of $\mathbb{R}^n$ of dimension smaller than n to be specified later in the various applications. We then have the following characterization result from [11, Thm.2.1]:

Theorem 1.1. *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Assume the local integrability condition (L.I.C.):*

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + C_p^I m)|^2 \, \frac{1}{|\det C_p|} \, |\hat{g}_p(\xi)|^2 \, d\xi < \infty \tag{1.7}$$

for all $f \in \mathcal{D}$. Then the system $\Phi^{\{g_p\}}_{\{C_p\}}$, given by (1.3), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|} \, \hat{g}_p(\xi) \, \overline{\hat{g}_p(\xi+\alpha)} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{1.8}$$

for each $\alpha \in \Lambda$, where δ is the Kronecker delta for $\mathbb{R}^n$.

The following result from the same paper will also be useful (cf. [11, Prop.4.1]).

Proposition 1.2. *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. If the system $\Phi^{\{g_p\}}_{\{C_p\}}$, given by (1.3), is Bessel with constant $\beta > 0$, then*

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} \, |\hat{g}_p(\xi)|^2 \le \beta \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{1.9}$$

We learned from personal communication that A. Ron and Z. Shen have developed, independently and by different methods, an approach to study families generated by countable unions of shift-invariant systems. Their results have many features that are similar to ours.

In many cases, we will consider applications of Theorem 1.1 to various variants of the affine systems. These systems involve the dilation operator D_A, $A \in GL_n(\mathbb{R})$, defined by

$$(D_A\, f)(x) = |\det A|^{1/2} f(Ax), \quad f \in L^2(\mathbb{R}^n).$$

Then the **affine systems** generated by a family $\Psi = \{\psi^1, \dots \psi^L\} \subset L^2(\mathbb{R}^n)$ and by the integral powers of the dilations D_A, $A \in GL_n(\mathbb{R})$, are the collections of the form

$$\mathcal{F}_A(\Psi) = \{D_A^j \, T_k \, \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1, \dots, L\}. \tag{1.10}$$

The collection $\Psi \subset L^2(\mathbb{R}^n)$ such that the affine system $\mathcal{F}_A(\Psi)$ is a PF for $L^2(\mathbb{R}^n)$ is called a **multi-wavelet** or a **wavelet** if $\Psi = \{\psi\} \in L^2(\mathbb{R}^n)$ is a single function. Observe that, in the literature, this terminology sometimes refers to a function which generates an affine orthonormal basis.

It is easy to see that the affine systems $\mathcal{F}_A(\Psi)$ are special cases of the systems given by (1.3). Indeed, by a simple calculation one obtains that $D_A^j\, T_k\, \psi^\ell = T_{A^{-j}k}\, D_A^j\, \psi^\ell$, which shows that the affine systems are obtained from (1.3), by choosing

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z},\, \ell = 1,2,\ldots,L\big\},$$

$$g_p = g_{(j,\ell)} = D_A^j\, \psi^\ell \quad \text{and} \quad C_p = C_{(j,\ell)} = A^{-j}, \qquad \text{for all } \ell = 1,\ldots,L \text{ and } j \in \mathbb{Z}.$$

The various variants of the affine systems $\mathcal{F}_A$ that will be discussed in this paper include the "quasi affine" and "oversampled" affine systems studied by a number of authors (cf. [4], [20], [17], [19], [15]). One of the novel feature of this paper is that all these systems can be represented in the form (1.3), which enables us to gain a better understanding of them. This approach allows us to include dilations that are more general than those found in the literature. In addition, several other systems (including Gabor systems, more general shift–invariant systems and wave packet systems) can be written in terms of collections of the form (1.3) and Theorem 1.1 can be applied to them. Moreover, we will discuss how the ideas used in the proof of this theorem can be applied to general frames (not just PF's). Section 2 will be devoted to the oversampling of the affine systems. The other applications, including oversampling of shift–invariant systems, dilation oversampling and wave packets, will be treated in the Sections 3, 4 and 5, respectively.

Before embarking in the applications of Theorem 1.1 to the oversampling of the affine systems, let us be more explicit about the dilation matrices we shall use. A matrix $M \in GL_n(\mathbb{R})$ is called **expanding** provided each of its eigenvalues λ satisfy $|\lambda| > 1$. As shown in [11, Sec.5], this is equivalent to the existence of constants k and γ, satisfying $0 < k \le 1 < \gamma < \infty$, such that

$$|M^j\, x| \;\ge\; k\, \gamma^j\, |x| \tag{1.11}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \ge 0$. The more general class of dilations that will be considered will be produced by those $M \in GL_n(\mathbb{R})$ that are **expanding on a subspace** $F \subset \mathbb{R}^n$ according to the following definition.

Definition 1.3. *Given $M \in GL_n(\mathbb{R})$ and a non-zero subspace F of $\mathbb{R}^n$, M* **is expanding on** *F if there exists a complementary (not necessarily orthogonal) subspace E of $\mathbb{R}^n$ with the following properties:*

(i) $\mathbb{R}^n = F + E$ *and* $F \cap E = \{0\}$;

(ii) $M(F) = F$ *and* $M(E) = E$, *that is, F and E are invariant under M;*

(iii) *condition (1.11) holds for all $x \in F$;*

(iv) given $r \in \mathbb{N}$, there exists $C = C(M, r)$ such that, for all $j \in \mathbb{Z}$, the set

$$\mathcal{Z}_r^j(E) = \{m \in E \cap \mathbb{Z}^n : |M^j\, m| < r\}$$

has less than C elements.

The characterization of those $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ for which $\mathcal{F}_A(\Psi)$ is a PF for $L^2(\mathbb{R}^n)$ when $B = A^T$ is expanding on a subspace $F \subset \mathbb{R}^n$ is the following:

Theorem 1.4 ([11]). *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. Then the system $\mathcal{F}_A(\Psi)$, given by (1.10), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_m} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi + m))} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{1.12}$$

and all $m \in \mathbb{Z}^n$, where $\mathcal{P}_m = \{j \in \mathbb{Z} : \; m \in B^j\, \mathbb{Z}^n\}$.

In order to illustrate the property of being expanding on a subspace, let us consider the case where $B \in GL_2(\mathbb{R})$. If both eigenvalues of B, say λ_1, λ_2, satisfy $|\lambda_1|, |\lambda_2| > 1$, then B is expanding, and, thus, is expanding on the subspace $F = \mathbb{R}^2$. In this case it is known that orthonormal wavelets (i.e., functions ψ such that $\mathcal{F}_A(\psi)$ is an orthonormal basis) always exist (as shown in [7]). If $|\lambda_1| = 1$ and $|\lambda_2| > 1$, then B is expanding on F, where F is the eigenspace corresponding to λ_2, and the complementary subspace E is the eigenspace corresponding to λ_1. For example, the matrix

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

is expanding on the eigenspace associated with the eigenvalue $\lambda = 2$. In [11, Example 5.15]) we explicitly construct a wavelet in $L^2(\mathbb{R}^2)$ with dilation matrix M. Furthermore, if $|\lambda_1| < 1 < |\lambda_2|$ and E, the eigenspace corresponding to λ_1, satisfies $\mathbb{Z}^2 \bigcap E = \varnothing$, then B is expanding on F, where F is the eigenspace corresponding to λ_2 (notice that item (iv) in Definition 1.3 is satisfied). In a very recent study, D. Speegle [22] has shown that there are examples of matrices in this class for which orthonormal wavelets exist, and others for which they do not exit. The following example illustrates this situation further.

Example 1.5. Consider

$$M_1 = \begin{pmatrix} 2 & 0 \\ a & 2/3 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 2 & a \\ 0 & 2/3 \end{pmatrix},$$

where $a \in \mathbb{R}$ is irrational. In either case, the only invariant proper subspaces are F, the eigenspace corresponding to $\lambda = 2$, and E, the eigenspace corresponding to $\lambda = 2/3$. For M_1, condition (iv) in Definition 1.3 is not satisfied, and thus the matrix cannot be expanding on F (the only expanding invariant subspace). On the other hand, M_2 is expanding on a subspace F: in fact, since $E = \{u(3a/4, 1) : u \in \mathbb{R}\}$ with a irrational, then $E \cap \mathbb{Z}^2 = \{0\}$ and, thus, condition (iv) in Definition 1.3 is satisfied. However,

$$M_1^{-1} = \begin{pmatrix} 1/2 & 0 \\ -3a/4 & 3/2 \end{pmatrix},$$

turns out to be expanding on a subspace F (F is the eigenspace associated with the eigenvalue $\lambda = 3/2$). In fact $E = \{u(1, 3a/4) : u \in \mathbb{R}\}$ (the eigenspace associated with the eigenvalue $\lambda = 1/2$) satisfies $E \cap \mathbb{Z}^n = \{0\}$ and, thus, condition (iv) in Definition 1.3 is satisfied.

It is clear that if $B = M_2$, then Theorem 1.4 applies to this case. Furthermore, in view of the observation that we made after announcing Theorem 1.4, when $B = M_1$ then Theorem 1.4 also applies, since M_1^{-1} is expanding on a subspace.

In dimensions larger than 2 the situation is more complicated. For example, there are matrices B with eigenvalues $|\lambda_i| \geq 1$, for all i, and $|\det B| > 1$ that are not expanding on a subspace (personal communication by A. Jaikin).

Another comment involves the local integrability condition (L.I.C.), given by (1.7). Observe that this condition is not mentioned in Theorem 1.4. In the proof of this theorem in [11], it is shown that the property that $B = A^T$ is expanding on $F \subset \mathbb{R}^n$ implies that if $\mathcal{F}_A(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$, then the L.I.C. is true for all $f \in \mathcal{D}_E$, where $E \in \mathbb{R}^n$ is the subspace complementary to F. Furthermore, it is shown that if the functions Ψ satisfy the condition (1.12), then the L.I.C. also holds. Thus, we do not need to state the L.I.C. for $\mathcal{F}_A(\Psi)$ in Theorem 1.4. These examples also illustrate why $\mathcal{D}_E$, defined by (1.6), is chosen to be dependent on E.

We now examine Theorem 1.4 in the case that $B = A^T$ is an integral matrix. Let $I(B) = \bigcap_{i \in \mathbb{Z}} B^i(\mathbb{Z}^n)$. We consider the three cases: $m = 0$, $m \in I(B) \setminus \{0\}$ and $m \in \mathbb{Z}^n \setminus I(B)$. Since $B\,\mathbb{Z}^n \subseteq \mathbb{Z}^n$, then $\{B^i\,\mathbb{Z}^n : i \in \mathbb{Z}\}$ is a decreasing sequence of sets and, obviously, $\{0\} \subseteq I(B)$. One may have some $m \neq 0$ in this set. For example, let $B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, with $\lambda \in \mathbb{Z}, \lambda > 1$; then $B^k = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^k \end{pmatrix}$, and $\begin{pmatrix} m_1 \\ 0 \end{pmatrix} \in I(B)$ for each $m_1 \in \mathbb{Z}$. If B is expanding, however, then only $m = 0$ is in $I(B)$.

When $m \in I(B)$, then $\mathcal{P}_m = \mathbb{Z}$, since $\mathcal{P}_m = \{j \in \mathbb{Z} : m \in B^j\,\mathbb{Z}^n\}$. On the other hand, if $m \in \mathbb{Z}^n \setminus I(B)$, then there are $j_0 \in \mathbb{Z}$ and $r \in \mathbb{Z}^n \setminus B\,\mathbb{Z}$ such that $m = B^{j_0}\,r$. In this case, after an appropriate change of variables, similar to the one we made above, equation (1.12) can be rewritten in the form

$$\sum_{\ell=1}^{L}\sum_{j\ge 0}\hat{\psi}^\ell(B^j\eta)\,\overline{\hat{\psi}^\ell(B^j(\eta+r))}=0 \quad \text{for a.e. } \eta\in\mathbb{R}^n.$$

We thus obtain the following refinement of Theorem 1.4 when $B\,\mathbb{Z}^n\subseteq\mathbb{Z}^n$.

Theorem 1.6. *Let $\Psi=\{\psi^1,\cdots,\psi^L\}\subset L^2(\mathbb{R}^n)$ and let $A\in GL_n(\mathbb{R})$ be an integral matrix such that $B=A^T$ is expanding on a subspace of $\mathbb{R}^n$. Then the affine system $\mathcal{F}_A(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if the following conditions hold:*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}^\ell(B^{-j}\xi)|^2=1 \quad \textit{for a. e.} \quad \xi\in\mathbb{R}^n, \tag{1.13}$$

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\hat{\psi}^\ell(B^{-j}\xi)\,\overline{\hat{\psi}^\ell(B^{-j}(\xi+m))}=0 \quad \textit{for a. e.} \quad \xi\in\mathbb{R}^n, \tag{1.14}$$

for all $m\in I(B)=\bigcap_{i\in\mathbb{Z}}B^i(\mathbb{Z}^n)$, $m\neq 0$, and

$$\sum_{\ell=1}^{L}\sum_{j\ge 0}\hat{\psi}^\ell(B^j\xi)\,\overline{\hat{\psi}^\ell(B^j(\xi+r))}=0 \quad \textit{for a.e. } \xi\in\mathbb{R}^n, \tag{1.15}$$

and all $r\in\mathbb{Z}^n\setminus B(\mathbb{Z}^n)$ (observe that $r\notin I(B)$).

As is usually the case, almost all the results that we will discuss remain valid for **dual reproducing systems**, where one system is used for analyzing functions and another system for reconstructing functions. Since essentially no new ideas are involved in this extension, and, also, to limit the length of this paper, we will not present this material here.

2 Oversampling of the affine systems

The notion of oversampling in the context of affine systems was introduced by Chui and Shi in [4] in the following manner. Given the dyadic affine system in $L^2(\mathbb{R})$,

$$\mathcal{F}_2(\psi)=\{D_2^j\,T_k\,\psi : j,k\in\mathbb{Z}\},$$

the corresponding **oversampled affine system** is obtained by using a larger collection of translations. More precisely, it is defined as

$$\mathcal{F}_2^m(\psi)=\{m^{-1/2}\,D_2^j\,T_{\frac{k}{m}}\,\psi : j,k\in\mathbb{Z}\},$$

where m is an odd number. It is shown in [4] that if the original affine system $\mathcal{F}_2(\psi)$ is a frame for $L^2(\mathbb{R})$, then the oversampled affine system $\mathcal{F}_2^m(\psi)$ is also a frame for $L^2(\mathbb{R})$ with the same frame bounds. This result is known as the Second Oversampling Theorem.

This notion of oversampling has been extended to higher dimensions and investigated by a number of authors (cf. [20], [3], [19], [15]). We will show that our methods, involving the use of Theorem 1.1 and other results from [11], can be applied to obtain all these results, as well as others. Not only will we consider higher dimensions, but we shall also consider an arbitrary change in the lattice of translations at each "scale" (or "resolution") associated with the dilations A^j.

The quasi affine systems, introduced by Ron and Shen [21], provide an important example of "scale-dependent" oversampling. Recall that the **quasi affine** $\widetilde{\mathcal{F}}_2(\psi)$, associated with $\mathcal{F}_2(\psi)$, is defined by $\widetilde{\mathcal{F}}_2(\psi) = \{\widetilde{\psi}_{j,k} : j,k \in \mathbb{Z}\}$, where

$$\tilde{\psi}_{j,k} = \begin{cases} 2^{j/2}\, D_2^j\, T_{2^j k}\, \psi, & j\ <\ 0 \\ D_2^j\, T_k\, \psi, & j\ \geq\ 0. \end{cases}$$

The same definition applies if the dilation 2 is replaced by any integer $a > 1$. It has been observed by many authors that the quasi affine systems enjoy many features that make the study of their properties easier than the corresponding affine systems. For example, they form shift-invariant systems, which is not the case for the affine system $\mathcal{F}_a(\psi)$. It is also important to realize that these systems are equivalent to the affine systems, in the sense that exactly the same ψ generates a PF for both systems (cf. [21]). This fails to be the case if $a \notin \mathbb{N}$. If $a \in \mathbb{Q}$, however, M.Bownik [2] observed that one can extend the definition of quasi affine systems, so that the good properties still hold. This can be done using the notion of "scale-dependent" oversampling. We will show that our unified approach can be applied to obtain all these features. For simplicity, let us begin by showing an application of Theorem 1.1 to the quasi affine systems with dilation $a \in \mathbb{Q}$.

2.1 Example: one-dimensional rational quasi affine systems

Let $a = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $p > q \geq 2$, $(p,q) = 1$. Given the affine system $\mathcal{F}_a(\psi) = \{D_a^j\, T_k\, \psi : j,k \in \mathbb{Z}\}$, the corresponding quasi affine systems $\widetilde{\mathcal{F}}_a(\psi)$ is defined by $\widetilde{\mathcal{F}}_a(\psi) = \{\widetilde{\psi}_{j,k} : j,k \in \mathbb{Z}\}$, where

$$\tilde{\psi}_{j,k} = \begin{cases} p^{j/2}\, D_a^j\, T_{p^j k}\, \psi, & j\ <\ 0 \\ q^{-j/2}\, D_a^j\, T_{q^{-j} k}\, \psi, & j\ \geq\ 0. \end{cases} = \begin{cases} p^{j/2}\, T_{q^j k}\, D_a^j\, \psi, & j\ <\ 0 \\ q^{-j/2}\, T_{p^{-j} k}\, D_a^j\, \psi, & j\ \geq\ 0. \end{cases} \tag{2.1}$$

This definition also makes sense when $q = 1$, $p \geq 2$, in which case it gives us the "classical" quasi affine system $\widetilde{\mathcal{F}}_p(\psi)$. It is easy to show, in general, that the systems $\widetilde{\mathcal{F}}_a(\psi)$ are shift-invariant. In fact, given any $m \in \mathbb{Z}^n$, from (2.1), we have that, if $j < 0$,

$$T_m \widetilde{\psi}_{j,k} = p^{j/2} T_m T_{q^j k} D_a^j \psi = p^{j/2} T_{q^j k+m} D_a^j \psi = p^{j/2} T_{q^j (k+q^{-j}m)} D_a^j \psi = \widetilde{\psi}_{j,k+q^{-j}m},$$

and, similarly, if $j \geq 0$,

$$T_m \widetilde{\psi}^{\ell}_{j,k} = q^{-j/2} T_m T_{p^{-j}k} D_a^j \psi = q^{-j/2} T_{p^{-j}k+m} D_a^j \psi = q^{-j/2} T_{p^{-j}(k+p^j m)} D_a^j \psi = \widetilde{\psi}_{j,k+p^j m}.$$

We will now apply Theorem 1.1 to characterize those $\psi \in L^2(\mathbb{R})$ for which $\widetilde{\mathcal{F}}_a(\psi)$ is a PF. The reader can verify that the L.I.C., given by (1.7), is satisfied in this case (the proof for the higher dimensional case will be discussed in Theorem 2.4). Let $\mathcal{P} = \mathbb{Z}$, and

$$g_j = \begin{cases} p^{j/2} D_a^j \psi, & \text{if } j < 0, \\ q^{-j/2} D_a^j \psi, & \text{if } j \geq 0, \end{cases} \quad C_j = \begin{cases} q^j, & \text{if } j < 0, \\ p^{-j}, & \text{if } j \geq 0. \end{cases}$$

Under these assumptions, $\widetilde{\mathcal{F}}_a(\psi)$ is of the form (1.3) and Theorem 1.1 can be applied. We have:

$$\Lambda = (\bigcup_{j<0} (q^{-j}\mathbb{Z}) \cup (\bigcup_{j\geq 0} (p^j \mathbb{Z}) = \mathbb{Z}.$$

Therefore, for $\alpha = m \in \Lambda$, we obtain

$$\mathcal{P}_m = \{j < 0 : q^j m \in \mathbb{Z}\} \cup \{j \geq 0 : p^{-j} m \in \mathbb{Z}\}.$$

If $m = 0$, then $\mathcal{P}_0 = \mathbb{Z}$. On the other hand, for any $m \in \mathbb{Z} \setminus \{0\}$, we can write $m = p^{j_0} q^{j_1} r$, where $j_0, j_1 \geq 0$ and $r \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. Hence:

$$\mathcal{P}_m = \{j < 0 : p^{j_0} q^{j+j_1} r \in \mathbb{Z}\} \cup \{j \geq 0 : q^{j_1} p^{-j+j_0} r \in \mathbb{Z}\} = \{j \in \mathbb{Z} : -j_1 \leq j \leq j_0\}.$$

From (1.8), expressed in terms of the g_j we just defined, we obtain that $\widetilde{\mathcal{F}}_a(\psi)$ is a PF for $L^2(\mathbb{R})$ iff

$$\sum_{j \in \mathcal{P}_0} |\hat{\psi}(a^{-j}\xi)|^2 = \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^{-j}\xi)|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R} \tag{2.2}$$

and, for $r \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$,

$$\sum_{j \in \mathcal{P}_m} \hat{\psi}(a^{-j}\xi) \overline{\hat{\psi}(a^{-j}(\xi + m))} = \sum_{-j_1 \leq j \leq j_0} \hat{\psi}(a^{-j}\xi) \overline{\hat{\psi}(a^{-j}(\xi + p^{j_0} q^{j_1} r))} = 0 \tag{2.3}$$

for a.e. $\xi \in \mathbb{R}$. Let us compare now these characterization equations with the corresponding characterization equations for the affine system $\mathcal{F}_a(\psi)$. We apply Theorem 1.4. If $m = 0$, then $\mathcal{P}_0 = \mathbb{Z}$. On the other hand, for any $m \in \mathbb{Z} \setminus \{0\}$, we can write $m = p^{j_0} q^{j_1} r$, where $j_0, j_1 \geq 0$ and $r \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$. Hence:

$$\mathcal{P}_m = \{j \in \mathbb{Z} : (p/q)^{-j} m \in \mathbb{Z}\} = \{j \in \mathbb{Z} : p^{j_0-j} q^{j+j_1} r \in \mathbb{Z}\} = \{j \in \mathbb{Z} : -j_1 \leq j \leq j_0\}.$$

Thus, using equation (1.12), we have that $\mathcal{F}_a(\psi)$ is a PF for $L^2(\mathbb{R})$ iff

$$\sum_{j\in\mathcal{P}_0} |\hat{\psi}(a^{-j}\xi)|^2 = \sum_{j\in\mathbb{Z}} |\hat{\psi}(a^{-j}\xi)|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R} \tag{2.4}$$

and, for $r \in \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$,

$$\sum_{j\in\mathcal{P}_m} \hat{\psi}(a^{-j}\xi)\, \overline{\hat{\psi}(a^{-j}(\xi+m))} = \sum_{-j_1\le j\le j_0} \hat{\psi}(a^{-j}\xi)\, \overline{\hat{\psi}(a^{-j}(\xi+p^{j_0}\, q^{j_1}\, r))} = 0 \tag{2.5}$$

for a.e. $\xi \in \mathbb{R}$. The comparison of equations (2.2) and (2.3) with equations (2.4) and (2.5) shows that exactly the same ψ generates a PF for both $\widetilde{\mathcal{F}}_a(\psi)$ and $\mathcal{F}_a(\psi)$. Later on, in Section 2.3.3, we will consider the n-dimensional version of this example.

2.2 Characterization of oversampled affine systems

One of the features of the quasi affine systems described in the example above is that they are obtained from the affine system $\mathcal{F}_a(\psi)$ by changing the lattice of translation at each scale j. More generally, corresponding to the affine system $\mathcal{F}_A(\Psi)$, given by (1.10), we define the (scale-dependent) **oversampled affine systems** generated by Ψ relative to the sequence of non-singular matrices $\{R_j\}_{j\in\mathbb{Z}} \subset GL_n(\mathbb{R})$ as the collections of the form

$$\mathcal{F}_A^{\{R_j\}}(\Psi) = \{\psi^\ell_{j,k} = |\det R_j|^{-1/2}\, D_A^j\, T_{R_j^{-1}k}\, \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1,\dots,L\}, \tag{2.6}$$

where $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$. We will use the notation $B = A^T$, $S_j = R_j^T$, $j \in \mathbb{Z}$, and the matrices $\{R_j\}_{j\in\mathbb{Z}}$ will be called **oversampling matrices** for the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$. It is clear that when $R_j = R \in GL_n(\mathbb{R})$, for each $j \in \mathbb{Z}$, then one obtains the notion of oversampling that is usually found in the literature.

We want to find conditions on the oversampling matrices $\{R_j\}_{j=1}^n$ such that the oversampled affine system $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a PF whenever this is the case for the corresponding affine system $\mathcal{F}_A(\Psi)$. Later we will also consider the corresponding question about frames.

We start with the following simple observation, which shows that in order for the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6), to be a frame (or even a Bessel system), there are some restrictions on the choice of the oversampling matrices $\{R_j\}_{j=1}^n$.

Proposition 2.1. *If the oversampled system $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6), is a Bessel system with constant β, then, for each $\ell = 1, \cdots, L$,*

$$|\det R_j| \ge \frac{1}{\beta}\, \|\psi^\ell\|^2, \qquad \text{for each } j \in \mathbb{Z}.$$

Proof. Since $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a Bessel system with constant β, then

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}^{\ell}\rangle|^2 \le \beta \, \|f\|^2 \tag{2.7}$$

for all $f \in L^2(\mathbb{R}^n)$, where $\psi_{j,k}^{\ell} = |\det R_j|^{-1/2} \, D_A^j \, T_{R_j^{-1}k} \, \psi^{\ell}$. Equation (2.7) implies that, for any $j_0 \in \mathbb{Z}$, $k_0 \in \mathbb{Z}^n$, $1 \le \ell_0 \le L$:

$$|\langle \psi_{j_0,k_0}^{\ell_0}, \psi_{j_0,k_0}^{\ell_0}\rangle|^2 \le \beta \, \|\psi_{j_0,k_0}^{\ell_0}\|^2. \tag{2.8}$$

Since $\|\psi_{j_0,k_0}^{\ell_0}\|^2 = |\det R_{j_0}|^{-1} \, \|\psi^{\ell_0}\|^2$, from (2.8) we deduce:

$$|\det R_{j_0}|^{-2} \, \|\psi^{\ell_0}\|^4 \le \beta \, |\det R_{j_0}|^{-1} \, \|\psi^{\ell_0}\|^2,$$

and, thus, $|\det R_{j_0}| \ge \beta^{-1} \, \|\psi^{\ell_0}\|^2$, for all $j_0 \in \mathbb{Z}$, $1 \le \ell_0 \le L$. $\square$

The following proposition shows how Theorem 1.1 can be applied to obtain a general characterization of the oversampled systems $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6).

Proposition 2.2. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $A \in GL_n(\mathbb{R})$ and $\{R_j\}_{j\in\mathbb{Z}} \subset GL_n(\mathbb{R})$. Assume the local integrability condition (L.I.C.):*

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + B^j \, S_j \, m)|^2 \, |\hat{\psi}^{\ell}(B^{-j}\xi)|^2 \, d\xi < \infty \tag{2.9}$$

for all f in $\mathcal{D}$, where $\mathcal{D}$ is given by (1.6) and $S_j = R_j^T$, for each $j \in \mathbb{Z}$. Then the system $\mathcal{F}_A^{R_j}(\Psi)$, given by (2.6), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^{\ell}(B^{-j}\xi) \, \overline{\hat{\psi}^{\ell}(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.10}$$

and all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B^j \, S_j(\mathbb{Z}^n)$, where, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{j \in \mathbb{Z} : \, S_j^{-1} \, B^{-j}\alpha \in \mathbb{Z}^n\}$.

Remark. At first sight it is not clear what is the dependence on the oversampling matrices $\{R_j\}_{j\in\mathbb{Z}}$ in the characterization equation (2.10). We point out, however, that the dependence on the matrices $\{R_j\}_{j\in\mathbb{Z}}$ is actually "hidden" in the set $\mathcal{P}_\alpha$, which is defined in terms of the matrices $\{S_j\}_{j\in\mathbb{Z}}$.

Proof of Proposition 2.2. Let $\mathcal{P}$, $\{g_p\}_{p\in\mathcal{P}}$ and $\{C_p\}_{p\in\mathcal{P}}$ be defined by

$$\mathcal{P} = \big\{(j,\ell) : j \in \mathbb{Z}, \text{ and } \ell = 1, \dots, L\big\},$$

$$g_p(x) = g_{(j,\ell)}(x) = |\det R_j|^{-1/2} \, D_A^j \, \psi^{\ell}(x), \qquad C_p = C_{(j,\ell)} = A^{-j} \, R_j^{-1}. \tag{2.11}$$

With these assumptions, it follows that

$$T_{C_p k}\, g_p = |\det R_j|^{-1/2}\, T_{A^{-j} R_j^{-1} k}\, D_A^j\, \psi^\ell = |\det R_j|^{-1/2}\, D_A^j\, T_{R_j^{-1} k}\, \psi^\ell,$$

and so the collection $\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the scale-dependent oversampled affine system $\mathcal{F}_A^{R_j}(\Psi)$. We can now apply Theorem 1.1.

Under these assumptions for $\mathcal{P}$, g_p and C_p, the *L.I.C.* (1.7) gives (2.9),

$$\Lambda = \bigcup_{p\in\mathcal{P}} C_p^I\, \mathbb{Z}^n = \bigcup_{j\in\mathbb{Z}} B^j\, S_j\, \mathbb{Z}^n,$$

and, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{p \in \mathcal{P} : C_p^T\, \alpha \in \mathbb{Z}^n\} = \{j \in \mathbb{Z} : S_j^{-1}\, B^{-j}\, \alpha \in \mathbb{Z}^n\}$. By direct computation, from (1.8), we obtain:

$$\begin{aligned}
&\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|}\, \hat{g}_p(\xi)\, \overline{\hat{g}_p(\xi+\alpha)} = \\
&\quad = \sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} |\det A|^j\, |\det R_j|\, |\det R_j|^{-1}\, |\det A|^{-j}\, \hat{\psi}(B^{-j}\xi)\, \overline{\hat{\psi}(B^{-j}(\xi+\alpha))} \\
&\quad = \sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} \hat{\psi}(B^{-j}\xi)\, \overline{\hat{\psi}(B^{-j}(\xi+\alpha))}.
\end{aligned}$$

which gives (2.10). □

While the "integrability" condition (2.9) is not guaranteed to hold in general, there are some important special choices of the oversampling matrices $\{R_j\}_{j\in\mathbb{Z}}$, which we will discuss in the following, for which we can show that (2.9) is satisfied. In all these cases, under the assumption that the dilation matrix A is such that $B = A^T$ is expanding on a subspace, we will be able to remove condition (2.9) from the hypothesis of Proposition 2.2. Before stating this result, we need to recall the following fact from [11, Prop. 5.6].

Proposition 2.3. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. If*

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \le \beta \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{2.12}$$

where $\beta > 0$, then

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + B^j\, m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty \tag{2.13}$$

for all $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is given by (1.6) and E is the complementary subspace to F.

Remark. Inequality (2.13) is exactly the L.I.C., given by (1.7), corresponding to the affine systems $\mathcal{F}_A(\Psi)$. Thus, Proposition 2.3 shows that (2.12) implies the L.I.C. for $\mathcal{F}_A(\Psi)$, when $B = A^T$ is expanding on a subspace.

We thus obtain the following:

Theorem 2.4. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. Let $\{R_j\}_{j\in\mathbb{Z}} \in GL_n(\mathbb{R})$ be in one of the following three classes:*

(I) $R_j = R \in GL_n(\mathbb{Z})$ for each $j \in \mathbb{Z}$ (observe: $GL_n(\mathbb{Z})$ denotes the subset of $GL_n(\mathbb{R})$ of matrices with integer entries).

(II) R_j satisfies $R_j\, A^j \in GL_n(\mathbb{Z})$ for each $j \in \mathbb{Z}$.

(III) $R_j = \begin{cases} R\, A^{-j+j_0} & j < j_0 \\ R, & j \geq j_0, \end{cases}$ *where $j_0 \in \mathbb{Z}$ is fixed and $R \in GL_n(\mathbb{Z})$.*

Then the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi)\, \overline{\hat{\psi}^\ell(B^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.14}$$

and all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B^j\, S_j\, \mathbb{Z}^n$, where $S_j = R_j^T$, and, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{j \in \mathbb{Z} : \alpha \in B^j\, S_j\, \mathbb{Z}^n\}$.

Proof. In order to prove the Theorem, we only have to show that condition (2.9) is satisfied under the assumption that the matrices $\{R_j\}_{j\in\mathbb{Z}}$ are in one of the three classes described above. Then the proof follows immediately from Proposition 2.2. In the following, let $\mathcal{D} = \mathcal{D}_E$, where $\mathcal{D}$ is given by (1.6) and $E \subset \mathbb{R}^n$ is the subspace complementary to F.

Class (I). Let $R_j = R$, for each $j \in \mathbb{Z}$. If $\mathcal{F}_A^R(\Psi)$ is a PF, then, in particular, $\mathcal{F}_A^R(\Psi)$ is a Bessel family with Bessel constant $\beta = 1$. By applying Proposition 1.2 to the system $\mathcal{F}_A^R(\Psi)$ (the elements $\mathcal{P}$, g_p and C_p are given by equation (2.11), with $R_j = R$ for each $j \in \mathbb{Z}$), we deduce inequality (2.12). This inequality also holds if we assume (2.14) (take $\alpha = 0$). Therefore, we can apply Proposition 2.3, which gives inequality (2.13). As a consequence, we have

$$\begin{aligned} L(f) &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j\, S\, m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi \\ &\leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j\, k)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty, \end{aligned}$$

for all $f \in \mathcal{D}$, since $S = R^T \in GL_n(\mathbb{Z})$. This shows that condition (2.9) is satisfied in this case.

Class (II). Since $R_j\, A^j\, \mathbb{Z}^n \subseteq \mathbb{Z}^n$, for each $j \in \mathbb{Z}$, then (by transposing) $B^j\, S_j\, \mathbb{Z}^n \subseteq \mathbb{Z}^n$ for each $j \in \mathbb{Z}$. Thus, we have

$$\begin{aligned} L(f) &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} |\hat f(\xi + B^j\, S_j\, m)|^2\, |\hat\psi^\ell(B^{-j}\xi)|^2\, d\xi \\ &\le \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} |\hat f(\xi + k)|^2\, |\hat\psi^\ell(B^{-j}\xi)|^2\, d\xi \end{aligned} \tag{2.15}$$

for all $f \in \mathcal{D}$. If $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a PF, then, applying Proposition 1.2 to the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$ as was done for class (I), we obtain inequality (2.12) with $\beta = 1$. This inequality also holds if we assume (2.14) (take $\alpha = 0$). Furthermore, since $\hat f$ is compactly supported, there are only finitely many $k \in \mathbb{Z}^n$ (say, M of then) such that $\hat f(\xi + k)$ is contained in $\operatorname{supp}\hat f$. Using this fact and (2.12), from (2.15) we obtain:

$$L(f) \le \sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} |\hat f(\xi + k)|^2\, d\xi \le M\, |\operatorname{supp}\hat f|\, \|\hat f\|_\infty^2 < \infty$$

for all $f \in \mathcal{D}$, which shows that condition (2.9) is satisfied also in this case.

Class (III). Let $S = R^T$. For every $f \in \mathcal{D}$, we have

$$\begin{aligned} L(f) &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} |\hat f(\xi + B^j\, S_j\, m)|^2\, |\hat\psi^\ell(B^{-j}\xi)|^2\, d\xi \\ &= \sum_{\ell=1}^{L} \sum_{j<j_0} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} |\hat f(\xi + B^{j_0}\, S\, m)|^2\, |\hat\psi^\ell(B^{-j}\xi)|^2\, d\xi + \\ &\quad + \sum_{\ell=1}^{L} \sum_{j\ge j_0} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat f} |\hat f(\xi + B^j\, S\, m)|^2\, |\hat\psi^\ell(B^{-j}\xi)|^2\, d\xi \\ &= L_1(f) + L_2(f), \end{aligned}$$

where $L_1(f)$ and $L_2(f)$ denote the sums over $j < j_0$ and over $j \ge j_0$, respectively. If $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a PF, then, applying Proposition 1.2 to the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$ as was done for class (I), we obtain inequality (2.12) with $\beta = 1$. This inequality also holds if we assume (2.14) (take $\alpha = 0$), and so Proposition 2.3 applies. Consider first $L_1(f)$. Since $f \in \mathcal{D}$, there exists an $R > 0$ such that $\operatorname{supp}\hat f$ is contained in $\{\xi \in \mathbb{R}^n : |\xi| < R\}$. In order to have $L_1(f) \ne 0$, we must have $|\xi| < R$ and $|\xi + B^{j_0}\, S\, m| < R$. Therefore we must have $|B^{j_0}\, S\, m| < 2\, R$, which implies $|m| < 2\, \|(B^{j_0} S)^{-1}\|\, R$. This shows that the sum with respect to m must be finite, where the number of $m \in \mathbb{Z}^n$ is at most $M = 2^n \|(B^{j_0} S)^{-1}\|^n R^n$. Thus, using (2.12) we have

$$L_1(f) \le \sum_{\ell=1}^{L} \sum_{j \ge \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^{j_0} S m)|^2 |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi \le M \, |\operatorname{supp} \hat{f}| \, \|\hat{f}\|_\infty^2.$$

Finally consider $L_2(f)$. Since $S \in GL_n(\mathbb{Z})$, then $S\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and so

$$L_2(f) \le \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + B^{j} m)|^2 |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi,$$

which is finite by Proposition 2.3. Thus, $L(f)$ is finite and condition (2.9) is satisfied. □

The following application of Theorem 2.4 shows that if the matrices $\{R_j\}_{j \in \mathbb{Z}}$ are in the class (I), then the characterization equations of the oversampled affine systems $\mathcal{F}_A^R(\Psi)$ can be written in a simpler form, involving only the lattice points $m \in \mathbb{Z}^n$, instead of all the elements $\alpha \in \Lambda$.

Theorem 2.5. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $R \in GL_n(\mathbb{Z})$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. Then the system*

$$\mathcal{F}_A^R(\Psi) = \{|\det R|^{-1/2} D_A^j T_{R^{-1}k} \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1, \ldots, L\}. \tag{2.16}$$

is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_{Sm}} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + Sm))} = \delta_{m,0} \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{2.17}$$

and all $m \in \mathbb{Z}^n$, where $\mathcal{P}_{Sm} = \{j \in \mathbb{Z} : \, S m \in B^j S \mathbb{Z}^n\}$.

Proof. We apply Theorem 2.4 and adopt the same notation (observe that we need the assumption $R \in GL_n(\mathbb{Z})$ in order to apply this theorem). For any $\alpha \in \Lambda = \bigcup_{j \in \mathbb{Z}} B^j S(\mathbb{Z}^n)$, we can write $\alpha = B^{j_0} S m_0$ for some $j_0 \in \mathbb{Z}$ and some $m_0 \in \mathbb{Z}^n$. By making the change of variables $\xi = B^{j_0}\eta$ in the left hand side of (2.14), we obtain

$$\sum_{j \in \mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + \alpha))} = \sum_{j \in \mathcal{P}_{B^{j_0} S m_0}} \hat{\psi}^\ell(B^{-j+j_0}\eta) \, \overline{\hat{\psi}^\ell(B^{-j+j_0}(\eta + S m_0))} = \delta_{m,0}, \tag{2.18}$$

for a.e. $\xi \in \mathbb{R}^n$. Let $k = j - j_0$. Since $B^{-(k+j_0)}(B^{j_0} S m_0) = B^{-k} S m_0$, it follows that $j = k + j_0 \in \mathcal{P}_{B^{j_0} S m_0}$ if and only if $k \in \mathcal{P}_{S m_0}$. Using the change of indices $j = k + j_0$ in (2.18), we obtain:

$$\sum_{j \in \mathcal{P}_\alpha} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi + \alpha))} = \sum_{k \in \mathcal{P}_{S m_0}} \hat{\psi}^\ell(B^{-k}\eta) \, \overline{\hat{\psi}^\ell(B^{-k}(\eta + S m_0))}, \tag{2.19}$$

where $\mathcal{P}_{S m_0} = \{k \in \mathbb{Z} : S^{-1} B^{-k} S m_0 \in \mathbb{Z}^n\}$. We thus obtain equation (2.17). □

2.3 Oversampling theorems for frames

In the previous section, we have obtained the characterization equations of the oversampled affine systems $\mathcal{F}_A^{\{R_j\}}(\Psi)$ which form Parseval frames. By comparing these equations with the characterization equations of the corresponding affine systems $\mathcal{F}_A(\Psi)$, one can deduce conditions on the matrices $\{R_j\}_{j\in\mathbb{Z}}$ such that if $\mathcal{F}_A(\Psi)$ is a PF than also $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a PF. In this section we show that, using techniques from the unified characterization approach that we have described in the previous section, it is possible to consider not only Parseval frames but even more general frames. In order to illustrate the method that we shall use in dimension one, let $\psi_{j,k} = D_a^j\, T_k\, \psi$, and $\psi_{j,k}^{r_j} = r_j^{-1/2}\, D_a^j\, T_{r_j^{-1}k}\, \psi$, where $\psi \in L^2(\mathbb{R})$, $a, r_j \in \mathbb{R}$, $j, k \in \mathbb{Z}$, and define the functionals

$$N^2(f) = \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}\rangle|^2, \qquad N^2_{\{r_j\}}(f) = \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_{j,k}^{r_j}\rangle|^2.$$

Our method consists in expressing the functional $N^2_{\{r_j\}}(f)$ (corresponding to the oversampled affine system) as an average of $N^2(T_v\, f)$ (corresponding to the affine system) over a countable set of translates $v \in V$ (V depends on the oversampling sequence $\{r_j\}$). This idea extends and generalizes similar ideas that appeared in [21], [5] and [19].

We will consider oversampling matrices in the three classes defined in Theorem 2.4, and show that, under certain conditions on the oversampling matrices $\{R_j\}_{j\in\mathbb{Z}}$, if the affine system $\mathcal{F}_A(\Psi)$ is a frame, then the corresponding oversampled system $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is also a frame with the same frame bounds.

2.3.1 Class (I).

The first case we examine involves the matrices $\{R_j\}$ in the class (I), given by Theorem 2.4. This gives us the classical notion of oversampling which has been extensively studied in the literature (cf. [4], [20], [3], [19], [15]). The main result that we obtain is the following generalization of the so-called "Second Oversampling" theorem, which holds for dilation matrices that are not only expanding, but expanding on a subspace.

Theorem 2.6. *Let $S = R^T \in GL_n(\mathbb{Z})$ and $S^{-1}\, B\, S \in GL_n(\mathbb{Z})$, where the nonsingular matrix $B = A^T \in GL_n(\mathbb{Z})$ is expanding on a subspace F of $\mathbb{R}^n$. Assume that $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n = B\,S\,\mathbb{Z}^n$. If the affine system $\mathcal{F}_A(\Psi) = \{D_A^j\, T_k\, \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \dots, L\}$ is a frame, then the system $\mathcal{F}_A^R(\Psi)$, given by (2.16), is also a frame with the same frame bounds.*

Remark. (1) This theorem extends similar results in Chui-Shi [4], Ron-Shen [21], Chui-Czaja-Maggioni-Weiss [3], Laugesen [19] and Johnson [15], where only expanding matrices are considered. The proof that we will present uses several ideas from a theorem in [19].

(2) In dimension $n = 1$, with $A = B = a \in \mathbb{Z}$ and $R = S = r \in \mathbb{Z}$, the hypothesis $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n = B\,S\,\mathbb{Z}^n$ gives the condition $ma + nr = 1$ for $m, n \in \mathbb{Z}$; that is, a and r are relatively prime. Regarding this hypothesis, notice that we only need the assumption $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n \subseteq B\,S\,\mathbb{Z}^n$ in Theorem 2.6 since the converse inclusion is trivial. Also observe that, under the assumption that $S, S^{-1}\,B\,S \in GL_n(\mathbb{Z})$, this hypothesis can be replaced by $A^{-1}\,\mathbb{Z}^n \bigcap R^{-1}\mathbb{Z}^n \subseteq \mathbb{Z}^n$ (see [15, Sec. 5] for this and more comments about the notion of relative primality).

(3) In dimension $n = 1$, Theorem 2.6 requires $A = a \in \mathbb{Z}$. This assumption is not necessary in order to have oversampling that is preserving the frame bounds. We will later show (Theorem 2.12) a result similar to Theorem 2.6 for dilations $a \in \mathbb{Q}$ and more general matrices in $GL_n(\mathbb{Q})$.

In order to prove Theorem 2.6, some constructions are needed. Some of these ideas will also be used in the analysis of oversampling matrices in the classes (II) and (III) which will be discussed in Sections 2.3.2 and 2.3.3. We will use is the following result from [11, Prop. 2.4]:

Proposition 2.7. *Let $\mathcal{P}$ be a countable indexing set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$, $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$, and let $C_p^I = (C_p^T)^{-1}$. Assume that the L.I.C. given by (1.7) holds for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (1.6). Then, for each $f \in \mathcal{D}$, the function*

$$w(x) \;=\; \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{C_p k}\, g_p\rangle|^2$$

is a continuous function that coincides pointwise with the absolutely convergent series

$$\sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \hat{w}_p(m)\, e^{2\pi i C_p^I m\cdot x}\ ,$$

where,

$$\hat{w}_p(m) \;=\; \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C_p^I m)}\, \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi, \tag{2.20}$$

and the integral in (2.20) converges absolutely.

The application of Proposition 2.7 to the affine systems $\mathcal{F}_A(\Psi)$ gives the following result:

Proposition 2.8. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. If the system $\mathcal{F}_A(\Psi)$, given by (1.10), is a Bessel system for $L^2(\mathbb{R}^n)$ then, for each $f \in \mathcal{D} = \mathcal{D}_E$, where $\mathcal{D}_E$ is given by (1.6) and E is the complementary subspace to F, the function*

$$w(x) = N^2(T_x\, f) \;=\; \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, D_A^j\, T_k\, \psi^\ell\rangle|^2 \tag{2.21}$$

is a continuous function that coincides pointwise with the absolutely convergent series

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m\cdot x}\ ,$$

where the function $\hat{w}_{j,\ell}$ *is defined, for any* $\mu\in\mathbb{R}^n$*, by*

$$\hat{w}_{j,\ell}(\mu) \;=\; \int_{\mathbb{R}^n} \hat{f}(\xi)\,\overline{\hat{f}(\xi+B^j\mu)}\,\overline{\hat{\psi}^\ell(B^{-j}\xi)}\,\hat{\psi}^\ell(B^{-j}(\xi+B^j\mu))\,d\xi, \tag{2.22}$$

and the integral in (2.22) converges absolutely.

Proof. By choosing $\mathcal{P} = \{(j,\ell) : j\in\mathbb{Z},\, \ell = 1,2,\ldots,L\}$, $g_p = g_{(j,\ell)} = D_A^j\,\psi^\ell$, and $C_p = C_{(j,\ell)} = A^{-j}$, for all $\ell = 1,\ldots,L$, then the collection $\{T_{C_p\,k}\,g_p\}_{p\in\mathcal{P}}$ is the affine system $\mathcal{F}_A(\Psi)$. We will now apply Proposition 2.7. Under the assumptions that we made for $\mathcal{P}$, g_p and C_p, equation (2.20) gives (2.22), provided (1.7) is satisfied. Therefore, in order to complete the proof, we only have to show that the L.I.C. (1.7) holds. Arguing as in the proof of Theorem 2.4, we observe that, since $\mathcal{F}_A(\Psi)$ is a Bessel system, then, by Proposition 1.2, we have inequality (2.12). We can now apply Proposition 2.3 which gives (2.13). As observed in the Remark following Proposition 2.3, (2.13) is exactly inequality (1.7), for this choice of $\mathcal{P}$, g_p and C_p. □

Remark. Proposition 2.8 can be easily generalized to the case where the sum over $j\in\mathbb{Z}$, in (2.21), is replaced by a sum over a smaller set $j\in\mathcal{J}\subseteq\mathbb{Z}$. We will also use this generalization of Proposition 2.8 in the following.

The application of Proposition 2.7 to the oversampled affine system $\mathcal{F}_A^{R_j}(\Psi)$, with oversampling matrices in the classes given by Theorem 2.4, gives the following result.

Proposition 2.9. *Let* $\Psi = \{\psi^1,\cdots,\psi^L\}\subset L^2(\mathbb{R}^n)$, $A\in GL_n(\mathbb{R})$ *be such that the matrix* $B = A^T$ *is expanding on a subspace* F *of* $\mathbb{R}^n$*, and suppose that*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \le \beta \qquad \text{for a.e. } \xi\in\mathbb{R}^n, \tag{2.23}$$

where $\beta>0$*. If* $\{R_j\}_{j\in\mathbb{Z}}$ *is in one of the three classes given in Theorem 2.4, then, for each* $f\in\mathcal{D}=\mathcal{D}_E$*, where* $\mathcal{D}_E$ *is given by (1.6) and* E *is the complementary space to* F*, the function*

$$w(x) = N^2_{\{R_j\}}(T_x\, f) \;=\; \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} |\langle T_x f, |\det R_j|^{-1/2}\, D_A^j\, T_{R_j^{-1}k}\,\psi^\ell\rangle|^2 \tag{2.24}$$

is continuous and coincides pointwise with the absolutely convergent series

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(S_j m)\, e^{2\pi i B^j S_j m\cdot x}\ ,$$

where $S_j = R_j^T$ *and* $\hat{w}_{j,\ell}$ *is given by (2.22).*

Proof. If we choose $\mathcal{P}$, g_p and C_p as in (2.11), then the collection $\{T_{C_p k}\, g_p\}_{p\in\mathcal{P}}$ is the system $\mathcal{F}_A^{R_j}(\Psi)$, and, thus, we can apply Proposition 2.7. Under the assumptions that we made for $\mathcal{P}$, g_p and C_p, equation (2.20) gives the coefficients $\hat{w}_{j,\ell}(S_j m)$, where $\hat{w}_{j,\ell}$ is given by (2.22), provided (1.7) holds. Hence, in order to complete the proof, we only have to show that:

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j\, S_j\, m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi \tag{2.25}$$

is finite for each $f \in \mathcal{D}$ (in fact, this is exactly condition (1.7) in this particular case). Observe that, since (2.23) holds and B is expanding on a subspace, we can apply Proposition 2.3 which gives (2.13). We can now examine the expression (2.25) corresponding to the different classes of matrices $\{R_j\}_{j\in\mathbb{Z}}$.

Class (I). Since $S_j = S \in GL_n(\mathbb{Z})$, for each $j \in \mathbb{Z}$, arguing as in the proof of Theorem 2.4, for all $f \in \mathcal{D}$ we have:

$$L(f) \le \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j\, k)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty.$$

Class (II). Since $R_j\, A^j\, \mathbb{Z}^n \subseteq \mathbb{Z}^n$, then $B^j S_j\, \mathbb{Z}^n \subseteq \mathbb{Z}^n$ for each $j \in \mathbb{Z}$. Using this observation and the fact that $\hat{f}$ is compactly supported, then arguing as in proof of Theorem 2.4, we have:

$$\begin{aligned} L(f) &\le \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + k)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi \\ &\le \beta \sum_{k\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + k)|^2\, d\xi \le \beta\, M\, |\operatorname{supp}\hat{f}|\, \|\hat{f}\|_\infty^2 < \infty, \end{aligned}$$

for some $K > 0$ and for all $f \in \mathcal{D}$.

Class (III). In this case we have

$$\begin{aligned} L(f) = &\sum_{\ell=1}^{L} \sum_{j<j_0} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^{j_0}\, S\, m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi + \\ &+ \sum_{\ell=1}^{L} \sum_{j\ge j_0} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^{j}\, S\, m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi. \end{aligned}$$

Also in this case, using the argument in the proof of Theorem 2.4, we have that the two sums are finite for all $f \in \mathcal{D}$. $\square$

The following proposition extends a result of R. Laugesen [19] to the case of dilation matrices $B = A^T$ that are expanding on a subspace.

Proposition 2.10. *Let $S = R^T \in GL_n(\mathbb{Z})$ and $S^{-1}\, B\, S \in GL_n(\mathbb{Z})$, where $B = A^T \in GL_n(\mathbb{Z})$ is expanding on a subspace F of $\mathbb{R}^n$. Assume that $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n = B\,S\,\mathbb{Z}^n$. Let V be a complete set of distinct representatives of $R^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. If $\mathcal{F}_A(\Psi)$ is a Bessel system, then, for each $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is given by (1.6) and E is the complementary space to F, we have*

$$N_R^2(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle f, |\det R|^{-1/2}\, D_A^j\, T_{R^{-1}k}\, \psi^\ell\rangle|^2 = \lim_{J\to\infty} \frac{1}{|\det R|} \sum_{v\in V} N^2(T_{A^J v}\, f),$$

where $J \in \mathbb{Z}$ and $N^2(T_{A^J v}\, f)$ is given by (2.21), with $x = A^J v$.

Proof. Since $\mathcal{F}_A(\Psi)$ is a Bessel system, we can apply Proposition 2.8. Thus, for each $f \in \mathcal{D}_E$ and any $v \in V$ we have

$$N^2(T_{A^J v}\, f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^{j+J} m\cdot v},$$

where $\hat{w}_{j,\ell}(m)$ is given by (2.22), with absolute convergence of the sum.

Recall the following property of finite groups (cf. [13, Lemma 23.19]):

Lemma 2.11. *Let $M \in GL_n(\mathbb{Z})$ and $q = |\det M|$. Choose a complete set $\{d_r\}_{r=0}^{q-1}$ of distinct representatives of the quotient group $M^{-1}\,\mathbb{Z}^n/\mathbb{Z}^n$, that is, $M^{-1}\mathbb{Z}^n = \bigcup_{r=0}^{q-1}(d_r + \mathbb{Z}^n)$. Then*

$$\frac{1}{q}\sum_{r=0}^{q-1} e^{2\pi i k\cdot d_r} = \begin{cases} 1 & \text{if } k \in M^T\mathbb{Z}^n \\ 0 & \text{if } k \in \mathbb{Z}^n \setminus M^T\mathbb{Z}^n. \end{cases}$$

Using Lemma 2.11 (with $M = R$) we are now going to show that if $j + J \geq 0$ then:

$$\frac{1}{|\det R|} \sum_{v\in V} e^{2\pi i B^{j+J} m\cdot v} = \begin{cases} 1 & \text{if } m \in S\,\mathbb{Z}^n \\ 0 & \text{if } m \in \mathbb{Z}^n \setminus S\,\mathbb{Z}^n. \end{cases} \tag{2.26}$$

In fact, if $m \in S\mathbb{Z}^n$, then $k = B^{j+J}m = B^{j+J}Sl$, for some $l \in \mathbb{Z}^n$. Thus $S^{-1}k = (S^{-1}BS)^{j+J}l \in \mathbb{Z}^n$ (since $S^{-1}BS \in GL_n(\mathbb{Z})$ and $j + J \geq 0$). On the other hand, if $m \notin S\mathbb{Z}^n$, then $Bm \notin S\mathbb{Z}^n$ (since $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n \subset B\,S\,\mathbb{Z}^n$) and, thus, by induction, $k = B^{j+J}m \notin S\mathbb{Z}^n$. This proves (2.26). Using (2.26), for each $f \in \mathcal{D}_E$ we have:

$$\frac{1}{|\det R|} \sum_{v\in V} N^2(T_{A^J v}\, f) =$$

$$= \frac{1}{|\det R|} \sum_{v\in V} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^{j+J} m\cdot v}$$

$$= \sum_{\ell=1}^{L} \sum_{j\geq -J} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m) \frac{1}{|\det R|} \sum_{v\in V} e^{2\pi i B^{j+J} m\cdot v} + \frac{1}{|\det R|} \sum_{v\in V} \sum_{\ell=1}^{L} \sum_{j<-J} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^{j+J} m\cdot v}$$

$$= \sum_{\ell=1}^{L} \sum_{j \geq -J} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(Sm) + \frac{1}{|\det R|} \sum_{v \in V} \sum_{\ell=1}^{L} \sum_{j < -J} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^{j+J} m \cdot v}. \tag{2.27}$$

Observe that, by Proposition 1.2, equation (2.23) is satisfied and, thus, we can apply Proposition 2.9, with $\{R_j\}$ in class (I), which gives:

$$N_R^2(f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(Sm), \tag{2.28}$$

with absolute convergence of the series. Since $\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} |\hat{w}_{j,\ell}(m)| < \infty$, then the second sum in (2.27) goes to zero as $J \to \infty$ and thus, using (2.28), for each $f \in \mathcal{D}_E$, we have

$$\lim_{J \to \infty} \frac{1}{|\det R|} \sum_{v \in V} N^2(T_{A^J v}\, f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(Sm) = N_R^2(f). \quad \square$$

Proof of Theorem 2.6. It suffices to prove the theorem for $f \in \mathcal{D}_E$, where E is the complementary space to F, since $\mathcal{D}_E$ is dense in $L^2(\mathbb{R}^n)$.

Since $\mathcal{F}_A(\Psi)$ is a frame, there are $0 < \alpha \leq \beta < \infty$ such that

$$\alpha\, \|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle f, D_A^j\, T_k\, \psi^\ell \rangle|^2 = N^2(f) \leq \beta\, \|f\|^2,$$

for all $f \in L^2(\mathbb{R}^n)$, and thus, since $\|T_x f\| = \|f\|$ for each $x \in \mathbb{R}^n$, this implies that

$$\alpha\, \|f\|^2 \leq \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle T_{A^J v}\, f, D_A^j\, T_k\, \psi^\ell \rangle|^2 = N^2(T_{A^J v}\, f) \leq \beta\, \|f\|^2, \tag{2.29}$$

for all $J \in \mathbb{Z}$, $v \in \mathbb{R}^n$. Let $v \in V$, where V is a complete set of distinct representative of the quotient group $R^{-1}\mathbb{Z}^n/\mathbb{Z}^n$, and apply the averaging operator $\lim_{J \to \infty} \frac{1}{|\det R|} \sum_{v \in V}$ to (2.29). Thus, using Proposition 2.10, for each $f \in \mathcal{D}_E$ we obtain:

$$\alpha\, \|f\|^2 \leq N_R^2(f) \leq \beta\, \|f\|^2.$$

These inequalities extend to all $f \in L^2(\mathbb{R}^n)$ by a standard density argument. $\square$

As we mentioned in the Remarks following Theorem 2.6, we can deduce a result similar to Theorem 2.6 for some matrices that do not satisfy the condition $S^{-1}BS \in GL_n(\mathbb{Z})$. The following result, which is *not* a consequence of Theorem 2.6, allows us to use dilation matrices $A \in GL_n(\mathbb{Q})$. For example, in the one-dimensional case, we can consider dilations $a \in \mathbb{Q}$ (this case was not allowed in Theorem 2.6, where a had to be integer-valued). As in Theorem 2.6, also in this case the idea of the proof consists in expressing $N^2_{\{R_j\}}(f)$ as an appropriate average over $N^2(T_v\, f)$, where v ranges over a finite set.

Theorem 2.12. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $R = S^T \in GL_n(\mathbb{Z})$ and assume that $A = P\,Q^{-1} \in GL_n(\mathbb{Q})$, where P and Q are commuting matrices in $GL_n(\mathbb{Z})$, and $B = A^T$ is expanding on a subspace $F \subseteq \mathbb{R}^n$. For $M = P$ or $M = Q$, assume that $R\,M\,R^{-1} \in GL_n(\mathbb{Z})$ and $M^T\,\mathbb{Z}^n \bigcap S\,\mathbb{Z}^n = M^T\,S\,\mathbb{Z}^n$. If the affine system $\mathcal{F}_A(\Psi)$, given by (1.10), is a frame for $L^2(\mathbb{R}^n)$, then the system $\mathcal{F}_A^R(\Psi)$, given by (2.16), is also a frame for $L^2(\mathbb{R}^n)$, and the frame bounds are the same.*

Proof. It suffices to prove the theorem for f in a dense subspace of $L^2(\mathbb{R})$. By Proposition 2.8, for each $f \in \mathcal{D}$, where $\mathcal{D}$ is given by by (1.6) with $E = \{0\}$, and for any $x \in \mathbb{R}$ we have

$$N^2(T_x\,f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle T_x\,f, D_A^j\,T_k\,\psi^\ell\rangle = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \hat{w}_{j,\ell}(m)\,e^{2\pi i B^j m\cdot x}, \tag{2.30}$$

where $\hat{w}_{j,\ell}(m)$ is given by (2.22), and the sum converges absolutely.

Let V be a complete set of distinct representatives of the quotient group $R^{-1}\mathbb{Z}/\mathbb{Z}$ (the cardinality of V is $|\det R|$). Using Lemma 2.11 with $M = R$, we have

$$\frac{1}{|\det R|} \sum_{v\in V} e^{2\pi i k\cdot v} = \begin{cases} 1 & \text{if } k \in S\,\mathbb{Z}^n \\ 0 & \text{if } k \in \mathbb{Z}^n \setminus S\,\mathbb{Z}^n. \end{cases} \tag{2.31}$$

Suppose $j_1, j_2 \in \mathbb{Z}$, $j_1, j_2 \ge 0$. We claim that (2.31) implies the following relation:

$$\frac{1}{|\det R|} \sum_{v\in V} e^{2\pi i (P^T)^{j_1} (Q^T)^{j_2} m\cdot v} = \begin{cases} 1 & \text{if } m \in S\,\mathbb{Z}^n \\ 0 & \text{if } m \in \mathbb{Z}^n \setminus S\,\mathbb{Z}^n. \end{cases} \tag{2.32}$$

In order to prove the claim, observe first that the hypothesis $M^T\,\mathbb{Z}^n \bigcap S\,\mathbb{Z}^n = M^T\,S\,\mathbb{Z}^n$ is equivalent to

$$\mathbb{Z}^n \bigcap (M^T)^{-1}\,S\,\mathbb{Z}^n = S\,\mathbb{Z}^n, \tag{2.33}$$

and, under the assumption that $S, M \in GL_n(\mathbb{Z})$, we will show that (2.33) implies (and, thus, is equivalent to)

$$\mathbb{Z}^n \bigcap (M^T)^{-j}\,S\,\mathbb{Z}^n = S\,\mathbb{Z}^n, \quad \text{for each } j \ge 0. \tag{2.34}$$

In fact, if (2.33) holds, then, for any $\mu \in \mathbb{Z}^n$, we have that $\mu \in S\,\mathbb{Z}^n$ iff $M^T\,\mu \in S\,\mathbb{Z}^n$. This is equivalent to saying that, for any $\mu \in \mathbb{Z}^n$, we have $\mu \in S\,\mathbb{Z}^n$ iff $(M^T)^2\,\mu \in M^T\,S\,\mathbb{Z}^n \in S\,\mathbb{Z}^n$. And, similarly, by induction, this is equivalent to saying that, for any $\mu \in \mathbb{Z}^n$ and any $j \ge 0$, we have $\mu \in S\,\mathbb{Z}^n$ iff $(M^T)^j\,\mu \in (M^T)^j\,S\,\mathbb{Z}^n \subset S\,\mathbb{Z}^n$. The last statement is equivalent to the relation $\mathbb{Z}^n \bigcap (M^T)^{-j}\,S\,\mathbb{Z}^n = S\,\mathbb{Z}^n$, for any $j \ge 0$, and, thus, (2.33) implies (2.34).

For $m \in \mathbb{Z}^n$, write $l = (Q^T)^{j_2} m$ and $k = (P^T)^{j_1} l$. It is clear that $k, l \in \mathbb{Z}^n$. We have that $k = (P^T)^{j_1} l \in S\,\mathbb{Z}^n$ iff $l = (P^T)^{-j_1} k \in (P^T)^{-j_1} S\,\mathbb{Z}^n$. Thus, $l \in \mathbb{Z}^n \bigcap (P^T)^{-j_1} S\,\mathbb{Z}^n$, and, using (2.34) with $M = P$, this is equivalent to $l \in S\,\mathbb{Z}^n$. Next, observe that $m = (Q^T)^{-j_2} l \in (Q^T)^{-j_2}\, S\,\mathbb{Z}^n \bigcap \mathbb{Z}^n$. Thus, using (2.34) with $M = Q$, this is equivalent to $m \in S\,\mathbb{Z}^n$. This completes the proof of the claim.

Fix $J \in \mathbb{Z}$, $J > 0$. For any $j \in \mathbb{Z}$ such that $|j| \le J$, let $j_1 = J + j$, $j_2 = J - j$ (observe that $j_1, j_2 \ge 0$). Since P^T and Q^T commute, then $(P^T Q^T)^J B^j = (P^T)^{j_1} (Q^T)^{j_2}$. Applying this observation and equation (2.32) into (2.30), we deduce that, for any $f \in \mathcal{D}$,

$$
\begin{aligned}
\frac{1}{|\det R|} \sum_{v \in V} N^2(T_{(P\,Q)^J v}\, f) &= \frac{1}{|\det R|} \sum_{v \in V} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m \cdot (P\,Q)^J v} \\
&= \frac{1}{|\det R|} \sum_{v \in V} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(m)\, e^{2\pi i (P^T)^{j_1} (Q^T)^{j_2} m \cdot v} \\
&= \frac{1}{|\det R|} \sum_{v \in V} \sum_{\ell=1}^{L} \sum_{|j| \le J} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(m)\, e^{2\pi i (P^T)^{j_1} (Q^T)^{j_2} m \cdot v} \\
&+ \frac{1}{|\det R|} \sum_{v \in V} \sum_{\ell=1}^{L} \sum_{|j| > J} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(m)\, e^{2\pi i (P^T)^{j_1} (Q^T)^{j_2} m \cdot v} \\
&= \sum_{\ell=1}^{L} \sum_{|j| \le J} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(S\,m) \\
&+ \frac{1}{|\det R|} \sum_{v \in V} \sum_{\ell=1}^{L} \sum_{|j| > J} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(m)\, e^{2\pi i (P^T)^{j_1} (Q^T)^{j_2} m \cdot v}. \qquad (2.35)
\end{aligned}
$$

Since the series (2.30) converges absolutely, then the sum in (2.35) corresponding to $|j| > J$ goes to zero when $J \to \infty$. Thus,

$$
\lim_{J \to \infty} \frac{1}{|\det R|} \sum_{v \in V} N^2(T_{(P\,Q)^J v}\, f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{w}_{j,\ell}(S\,m). \qquad (2.36)
$$

Finally, since $\mathcal{F}_A(\Psi)$ is a Bessel system, by Proposition 1.2, equation (2.23) is satisfied and, thus, we can apply Proposition 2.9, which gives:

$$
N_R^2(f) = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle f, |\det R|^{-1/2}\, D_A^j\, T_{R^{-1} k}\, \psi^\ell \rangle|^2 = \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(S\,m), \qquad (2.37)
$$

where the sum converges absolutely. Comparing (2.36) and (2.37), we obtain

$$
\lim_{J \to \infty} \frac{1}{|\det R|} \sum_{v \in V} N^2(T_{(P\,Q)^J v}\, f) = N_R^2(f). \qquad (2.38)
$$

The proof now follows from (2.38) as in the last step of the proof of Theorem 2.6. $\square$

2.3.2 Class (III).

We will now examine the case of oversampling matrices $\{R_j\}$ in the class (III), defined in Theorem 2.4. We obtain the following result.

Theorem 2.13. *Let A, S and $W = S^{-1}\,B\,S$ be in $GL_n(\mathbb{Z})$, where $S = R^T$ and the matrix $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. Assume that $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n = B\,S\,\mathbb{Z}^n$. If the affine systems $\mathcal{F}_A(\Psi)$, given by (1.10), is a frame, then the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6), is also a frame with the same frame bounds, where $j_0 \in \mathbb{Z}$ and*

$$R_j = \begin{cases} R\,A^{-j+j_0} & j < j_0 \\ R, & j \geq j_0. \end{cases} \tag{2.39}$$

Remark. In the special case where $R = I$ in (2.39), the oversampled affine systems $\mathcal{F}_A^{\{R_j\}}(\Psi)$ are the n-dimensional extensions of the quasi affine systems that we have described at the beginning of Section 2. In this case, the systems $\mathcal{F}_A(\Psi)$ and $\mathcal{F}_A^{\{R_j\}}(\Psi)$ are equivalent in the sense that one is a frame if and only if the other is a frame, and the frame bounds are the same. This situation will be examined later in Theorem 2.16.

The main tool to prove this theorem is the following result. As in Theorem 2.6, we will write the functional $N^2_{\{R_j\}}(f)$ as an appropriate average over $N^2(T_v\,f)$, where v ranges over a finite set.

Proposition 2.14. *Let $B = A^T \in GL_n(\mathbb{Z})$, V_K be a complete set of distinct representatives of the quotient group $\mathbb{Z}^n/A^K\mathbb{Z}^n$, $K \geq 0$, and the oversampling matrices $\{\widetilde{R}_j\}$ be given by*

$$\widetilde{R}_j = \begin{cases} A^{-j} & j < 0 \\ I, & j \geq 0. \end{cases} \tag{2.40}$$

If $\mathcal{F}_A(\Psi)$, given by (1.10), is a Bessel system and B is expanding on a subspace $F \subseteq \mathbb{R}^n$, then, for each $f \in \mathcal{D}_E$, we have:

$$N^2_{\{\widetilde{R}_j\}}(f) = \lim_{K\to\infty} \frac{1}{|\det A|^K} \sum_{v\in V_K} N^2(T_v\,f), \tag{2.41}$$

where $N^2_{\{\widetilde{R}_j\}}(f)$ is given by (2.24), $N^2(f)$ is given by (2.21), $\mathcal{D}_E$ is given by (1.6) and E is the complementary space to F.

Proof. Since $\mathcal{F}_A(\Psi)$ is a Bessel system, we can apply Proposition 2.8, which gives that, for each $f \in \mathcal{D}_E$ and any $x \in \mathbb{R}^n$:

$$N^2(T_x\,f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m\cdot x}, \tag{2.42}$$

where $\hat{w}_{j,\ell}(m)$ is given by (2.22) and the sum converges absolutely. For $K > 0$, write

$$\begin{aligned}\frac{1}{|\det A|^K}\sum_{v\in V_K} N^2(T_v\, f) &= \frac{1}{|\det A|^K}\sum_{v\in V_K}\sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{j<-K}\hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m\cdot v}\\ &+\frac{1}{|\det A|^K}\sum_{v\in V_K}\sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{-K\le j<0}\hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m\cdot v}\\ &+\frac{1}{|\det A|^K}\sum_{v\in V_K}\sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{j\ge 0}\hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m\cdot v}\\ &= I_1(f;K) + I_2(f;K) + I_3(f), \end{aligned} \tag{2.43}$$

where I_1 is the sum for $j < -K$, I_2 is the sum for $-K \le j < 0$, and I_3 is the sum for $j \ge 0$.

If $j \ge 0$, then $B^j m\cdot v \in \mathbb{Z}^n$ whenever $m \in \mathbb{Z}^n$ and $v \in V_K$. Thus, under these assumptions, $e^{2\pi i B^j m\cdot v} = 1$ and, consequently, since V_K has cardinality $|\det A|^K$, we have

$$I_3(f) = \sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{j\ge 0}\hat{w}_{j,\ell}(m). \tag{2.44}$$

For $j < 0$, let V_j be a complete set of distinct representatives of the group $\mathbb{Z}^n/A^{-j}\mathbb{Z}^n$ (V_j has cardinality $|\det A|^{-j}$). We will need the following variant of Lemma 2.11 (which is easily obtained by setting $\delta_r = M d_r$ in Lemma 2.11):

Lemma 2.15. *Let $M \in GL_n(\mathbb{Z})$ and $q = |\det M|$. Choose a complete set $\{\delta_r\}_{r=0}^{q-1}$ of distinct representatives of the quotient group $\mathbb{Z}^n/M\mathbb{Z}^n$, that is, $\mathbb{Z}^n = \bigcup_{r=0}^{q-1}(\delta_r + M\,\mathbb{Z}^n)$. Then*

$$\frac{1}{q}\sum_{r=0}^{q-1} e^{2\pi i u\cdot\delta_r} = \begin{cases} 1 & \text{if } u \in \mathbb{Z}^n\\ 0 & \text{if } u \in (M^T)^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n. \end{cases}$$

Using Lemma 2.15, with $M = A^{-j}$ and $u = B^j\, m$, we have

$$\frac{1}{|\det A|^{-j}}\sum_{v\in V_j} e^{2\pi i B^j m\cdot v} = \begin{cases} 1 & \text{if } m \in B^{-j}\,\mathbb{Z}^n\\ 0 & \text{if } m \in \mathbb{Z}^n \setminus B^{-j}\,\mathbb{Z}^n. \end{cases} \tag{2.45}$$

We claim that for each $-K \le j < 0$ we have:

$$\frac{1}{|\det A|^{K}}\sum_{v\in V_K} e^{2\pi i B^j m\cdot v} = \begin{cases} 1 & \text{if } m \in B^{-j}\mathbb{Z}^n\\ 0 & \text{if } m \in \mathbb{Z}^n \setminus B^{-j}\mathbb{Z}^n. \end{cases} \tag{2.46}$$

Indeed, by the Third Homomorphism Theorem (cf., for example, [10]), for any $-K \le j < 0$, the quotient group $(\mathbb{Z}^n/A^K\mathbb{Z}^n)/(A^{-j}\mathbb{Z}^n/A^K\mathbb{Z}^n)$ is isomorphic to $\mathbb{Z}^n/A^{-j}\mathbb{Z}^n$. This implies that $\mathbb{Z}^n/A^K\mathbb{Z}^n = \bigcup_{v(j)\in V_j}\big(v(j) + A^{-j}\mathbb{Z}^n/A^K\mathbb{Z}^n\big)$, and, thus, each $v(K) \in V_K$ is of the form $v(K) = v(j) + A^{-j}\, v(K+j)$, where $v(K) \in V_K$ and $v(K+j) \in V_{K+j}$ (notice that $A^{-j}\mathbb{Z}^n/A^K\mathbb{Z}^n \simeq A^{-j}\,\mathbb{Z}^n/A^{K+j}\mathbb{Z}^n$). Since V_{K+j} has cardinality $|\det A|^{K+j}$, then V_K is made up of as many copies of V_j, and, thus, (2.46) follows from (2.45).

Using (2.46) into the expression for I_2, we can write:

$$I_2(f;K) = \sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{-K\le j<0}\hat{w}_{j,\ell}(B^{-j}m). \tag{2.47}$$

Since the sum in (2.42) is absolutely convergent, then

$$\lim_{K\to\infty} I_1(f;K) = 0. \tag{2.48}$$

Thus, using (2.44), (2.47) and (2.48) into (2.43), we deduce

$$\lim_{K\to\infty}\frac{1}{|\det A|^K}\sum_{v\in V_K} N^2(T_v\, f) = \sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{j<0}\hat{w}_{j,\ell}(B^{-j}m) + \sum_{\ell=1}^{L}\sum_{m\in\mathbb{Z}^n}\sum_{j\ge 0}\hat{w}_{j,\ell}(m). \tag{2.49}$$

Finally, since $\mathcal{F}_A(\Psi)$ is a Bessel system, by Proposition 1.2, equation (2.23) is satisfied and we can apply Proposition 2.9, which gives us

$$N^2_{\{\widetilde{R}_j\}}(f) = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n}\hat{w}_{j,\ell}(\widetilde{S}_j\, m), \tag{2.50}$$

with absolute convergence of the sum, where

$$\widetilde{S}_j = \widetilde{R}_j^T = \begin{cases} B^{-j} & j<0 \\ I, & j\ge 0. \end{cases}$$

The proof is completed by combining (2.49) and (2.50). □

We can now prove the theorem.

Proof of Theorem 2.13. To prove the theorem, it suffices to consider the case $j_0 = 0$ in (2.39). In fact, consider the system $\mathcal{F}_A^{\{R_j\}}(\psi)$, where $\{R_j\}$ is given by

$$R_j = \begin{cases} R\,A^{-j} & j<0 \\ R, & j\ge 0. \end{cases} \tag{2.51}$$

Applying the dilation operator $D_A^{j_0}$ to each element of $\mathcal{F}_A^{\{R_j\}}(\psi)$ and making the change of variables $j' = j + j_0$, we obtain:

$$D_A^{j_0}\mathcal{F}_A^{\{R_j\}}(\psi) = \{|\det R|^{-1/2}\,|\det A|^{j/2}\, D_A^{j+j_0}\, T_{A^jR^{-1}k}\,\psi : j<0,\, k\in\mathbb{Z}^n\}$$

$$
\begin{aligned}
&\bigcup\{|\det R|^{-1/2}\, D_A^{j+j_0}\, T_{R^{-1}k}\,\psi : j \geq 0,\ k \in \mathbb{Z}^n\} \\
&= \{|\det R|^{-1/2}\, |\det A|^{(j'-j_0)/2}\, D_A^{j'}\, T_{A^{j'-j_0}R^{-1}k}\,\psi : j' < j_0,\ k \in \mathbb{Z}^n\} \\
&\quad \bigcup\{|\det R|^{-1/2}\, D_A^{j'}\, T_{R^{-1}k}\,\psi : j' \geq j_0,\ k \in \mathbb{Z}^n\} \\
&= \mathcal{F}_A^{\{R_j^0\}}(\psi),
\end{aligned}
\tag{2.52}
$$

where the oversampling matrices $\{R_j^0\}$ are given by

$$
R_j^0 = \begin{cases} R\, A^{-j+j_0} & j < j_0 \\ R, & j \geq j_0. \end{cases}
$$

Since the dilation $D_A^{j_0}$ is a unitary operator, $\mathcal{F}_A^{\{R_j\}}(\psi)$ is a frame if and only $\mathcal{F}_A^{\{R_j^0\}}(\psi)$ is a frame, and the frame bounds are preserved. Therefore, in the following, we will write R_j as in (2.51) and $\widetilde{R}_j$ as in (2.40), so that $R_j = R\,\widetilde{R}_j$ for each $j \in \mathbb{Z}$.

Since $\mathcal{F}_A(\Psi)$ is a Bessel system, by Proposition 1.2, equation (2.23) is satisfied and, thus, using Proposition 2.9 we obtain that, for each $f \in \mathcal{D}_E$,

$$
N^2_{\{\widetilde{R}_j\}}(T_x\, f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(\widetilde{S}_j\, m)\, e^{2\pi i B^j \widetilde{S}_j \cdot x}
\tag{2.53}
$$

and

$$
N^2_{\{R_j\}}(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(S_j\, m),
\tag{2.54}
$$

where $N^2_{\{R_j\}}(T_x\, f)$ is given by (2.24), $\hat{w}_{j,\ell}(m)$ is given by (2.22), $S_j = R_j^T = \widetilde{R}_j^T\, R^T = \widetilde{S}_j\, S$, $\mathcal{D}_E$ is given by (1.6), E is the space complementary to F, and the sums converge absolutely.

We will now use an argument similar to the one in the proof of Proposition 2.10. Let U be a complete collection of distinct representatives of the quotient group $R^{-1}\mathbb{Z}^n/\mathbb{Z}^n$; U has cardinality $|\det R|$. Given $J \geq 0$, for any $f \in \mathcal{D}_E$, we write:

$$
\begin{aligned}
\frac{1}{|\det R|} \sum_{u\in U} N^2_{\{\widetilde{R}_j\}}(T_{A^J u} f) &= \frac{1}{|\det R|} \sum_{u\in U} \sum_{\ell=1}^{L} \sum_{j<-J} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(\widetilde{S}_j\, m)\, e^{2\pi i B^{j+J} \widetilde{S}_j m\cdot u} \\
&\quad + \frac{1}{|\det R|} \sum_{u\in U} \sum_{\ell=1}^{L} \sum_{j\geq -J} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(\widetilde{S}_j\, m)\, e^{2\pi i B^{j+J} \widetilde{S}_j m\cdot u} \\
&= I_1(J) + I_2(J),
\end{aligned}
\tag{2.55}
$$

where $I_1(J)$ is the sum when $j < -J$, $I_2(J)$ is the sum when $j \geq -J$, and the sums converge absolutely. Since the sum (2.55) converges absolutely, then $\lim_{J\to\infty} I_1(J) = 0$.

Next, using (2.26) for the expression for $I_2(J)$ (notice that (2.26) holds due to the hypotheses that $S, S^{-1}BS \in GL_n(\mathbb{Z})$ and $B\,\mathbb{Z}^n \cap S\,\mathbb{Z}^n = B\,S\,\mathbb{Z}^n$), we obtain that, for any $J \geq 0$,

$$I_2(J) = \sum_{\ell=1}^{L} \sum_{j \geq -J} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(\widetilde{S}_j S\, m) = \sum_{\ell=1}^{L} \sum_{j \geq -J} \sum_{m \in \mathbb{Z}^n} \hat{w}_{j,\ell}(S_j\, m). \tag{2.56}$$

Taking the limit when $J \to \infty$ in (2.55) and using (2.54) and (2.56) we have:

$$\lim_{J\to\infty} \frac{1}{|\det R|} \sum_{u \in U} N^2_{\{\tilde{R}_j\}}(T_{A^J u}\, f) = N^2_{\{R_j\}}(f). \tag{2.57}$$

Using (2.41) from Proposition 2.14, we finally obtain

$$N^2_{\{R_j\}}(f) = \lim_{J\to\infty} \frac{1}{|\det R|} \sum_{u \in U} \lim_{K\to\infty} \frac{1}{|\det A|^K} \sum_{v \in V_K} N^2(T_{v+A^J u}\, f),$$

where V_K is a complete collection of distinct representatives of the quotient group $\mathbb{Z}^n / A^K\, \mathbb{Z}^n$.

The proof now follows as in the (last step of the) proof of Theorem 2.6. □

As we mentioned before, if $R = I$ in (2.39), then $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is the quasi affine system corresponding to $\mathcal{F}_A(\Psi)$. We will now prove that the affine system is a frame if and only if the corresponding quasi-affine system is a frame, and the frame bounds are the same. This equivalence was originally discovered by Ron and Shen [21] for $A \in GL_n(\mathbb{Z})$ and expanding, under a decay assumption on ψ that was later removed by Chui, Shi and Stöckler in [5]. Our proof, which is adapted from Laugesen [19, Thm. 7.1], generalizes this result to matrices which are expanding on a subspace of $\mathbb{R}^n$.

Theorem 2.16. *Let $A \in GL_n(\mathbb{Z})$, where $B = A^T$ is expanding on a subspace F of $\mathbb{R}^n$. The affine systems $\mathcal{F}_A(\Psi) = \{D_A^j\, T_k\, \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ is a frame if and only if $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (refoo), is also a frame with the same frame bounds, where $j_0 \in \mathbb{Z}$ and*

$$R_j = \begin{cases} A^{-j+j_0} & j < j_0 \\ I, & j \geq j_0. \end{cases} \tag{2.58}$$

Proof. As in the proof of Theorem 2.13, it suffices to prove the case $j_0 = 0$. Also, it suffices to prove the theorem for f in a dense subspace for $L^2(\mathbb{R}^n)$; then the extension to $f \in L^2(\mathbb{R}^n)$ follows from a standard density argument.

By Theorem 2.13, if $\mathcal{F}_A(\Psi)$ is a frame, then $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is also a frame with the same frame bounds.

Conversely, assume that $\mathcal{F}_A^{\{R_j\}}(\Psi)$ a frame, let $J \in \mathbb{Z}$, $J \geq 0$, and, for $j < 0$, let V_j be a complete set of distinct representatives of the quotient group $A^j\mathbb{Z}^n/\mathbb{Z}^n$. Then, using the change of indices $j \to j - J$, we have:

$$\sum_{\ell=1}^{L} \sum_{j<0} \sum_{k\in\mathbb{Z}^n} |\langle f, |\det A|^{j/2} D_A^{j-J} T_{A^j k} \psi^\ell \rangle|^2 =$$

$$= \sum_{\ell=1}^{L} \sum_{j<0} |\det A|^j \sum_{v\in V_j} \sum_{k\in\mathbb{Z}^n} |\langle f, D_A^{j-J} T_{v+k} \psi^\ell \rangle|^2$$

$$= \sum_{\ell=1}^{L} \sum_{j<0} |\det A|^j \sum_{v\in V_j} \sum_{k\in\mathbb{Z}^n} |\langle T_{A^{J-j}v} f, D_A^{j-J} T_k \psi^\ell \rangle|^2$$

$$= \sum_{\ell=1}^{L} \sum_{j<-J} |\det A|^{j+J} \sum_{v\in V_{j+J}} \sum_{k\in\mathbb{Z}^n} |\langle T_{A^{-j}v} f, D_A^{j} T_k \psi^\ell \rangle|^2. \tag{2.59}$$

Since $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a Bessel system, by Proposition 1.2, equation (2.23) is satisfied and we can apply Proposition 2.8 (see the Remark following its proof), which gives us, for each $f \in \mathcal{D}_E$ and any $x \in \mathbb{R}^n$:

$$\sum_{\ell=1}^{L} \sum_{j<-J} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, D_A^{j} T_k \psi^\ell \rangle|^2 = \sum_{\ell=1}^{L} \sum_{j<-J} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, e^{2\pi i B^j m\cdot x}, \tag{2.60}$$

with absolute convergence of the sum, where $\hat{w}_{j,\ell}(m)$ is given by (2.22), $\mathcal{D}_E$ is given by (1.6) and E is the complementary space to F. Thus, from (2.59) and (2.60) with $x = A^{-j}v$ we obtain that, for any $f \in \mathcal{D}_E$,

$$\sum_{\ell=1}^{L} \sum_{j<0} \sum_{k\in\mathbb{Z}^n} |\langle f, |\det A|^{j/2} D_A^{j-J} T_{A^j k} \psi^\ell \rangle|^2 = \sum_{\ell=1}^{L} \sum_{j<-J} \sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)\, |\det A|^{j+J} \sum_{v\in V_{j+J}} e^{2\pi i m\cdot v}. \tag{2.61}$$

Since the cardinality of V_{j+J} is exactly $|\det A|^{j+J}$ and $\sum_{m\in\mathbb{Z}^n} \hat{w}_{j,\ell}(m)$ converges absolutely, then the expression (2.61) converges to zero when J approaches infinity. Using (2.61) and letting $N^2_{\{R_j\}}(f)$ be the functional given by (2.24), we obtain that, for any $f \in \mathcal{D}_E$,

$$\lim_{J\to\infty} N^2_{\{R_j\}}(D_A^J f) = \lim_{J\to\infty} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle D_A^J f, |\det R_j|^{-1/2} D_A^j T_{R_j^{-1}k} \psi^\ell \rangle|^2$$

$$= \lim_{J\to\infty} \Big(\sum_{\ell=1}^{L} \sum_{j<0} \sum_{k\in\mathbb{Z}^n} |\langle f, |\det A|^{j/2} D_A^{j-J} T_{A^j k} \psi^\ell \rangle|^2 +$$

$$+ \sum_{\ell=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{Z}^n} |\langle f, D_A^{j-J} T_k \psi^\ell \rangle|^2 \Big)$$

$$= \lim_{J\to\infty} \sum_{\ell=1}^{L} \sum_{j\geq 0} \sum_{k\in\mathbb{Z}^n} |\langle f, D_A^{j-J} T_k \psi^\ell \rangle|^2.$$

Thus, comparing this quantity with the functional

$$N^2(f) = \lim_{J\to\infty} \sum_{\ell=1}^{L} \sum_{j\ge -J} \sum_{k\in\mathbb{Z}^n} |\langle f, D_A^j\, T_k\, \psi^\ell\rangle|^2 = \lim_{J\to\infty} \sum_{\ell=1}^{L} \sum_{j\ge 0} \sum_{k\in\mathbb{Z}^n} |\langle f, D_A^{j-J}\, T_k\, \psi^\ell\rangle|^2,$$

we have that, for any $f \in \mathcal{D}_E$,

$$N^2(f) = \lim_{J\to\infty} N^2_{\{R_j\}}(D_A^J f). \tag{2.62}$$

Since the $\|D_A^J f\| = \|f\|$ and $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a frame, then there are constants $0 < \alpha \le \beta < \infty$ such that

$$\alpha\, \|f\|^2 \le N^2_{\{R_j\}}(D_A^J f) \le \beta\, \|f\|^2,$$

for any $f \in \mathcal{D}_E$. By (2.62), the same inequalities hold for $N^2(f)$. $\square$

2.3.3 Class (II).

We will now examine the case of oversampling matrices $\{R_j\}$ in the class (II), defined in Theorem 2.4. Given a dilation matrix $A = B^T \in GL_n(\mathbb{Q})$, we will consider the matrices $R_j = S_j^T$ satisfying

$$S_j\, \mathbb{Z}^n = B^{-j}\, \mathbb{Z}^n \bigcap \mathbb{Z}^n, \quad j \in \mathbb{Z}. \tag{2.63}$$

From (2.63), it is clear that $S_j\, \mathbb{Z}^n \subseteq B^{-j}\, \mathbb{Z}^n$, which implies that $R_j\, A^j\, \mathbb{Z}^n \subseteq \mathbb{Z}^n$, and, thus, the matrices R_j given by (2.63) are in class (II). For example, in the one-dimensional case, with $A = a = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $p > q \ge 2$, $(p,q) = 1$, equation (2.63) becomes $R_j\, \mathbb{Z} = S_j\, \mathbb{Z} = (\frac{p}{q})^{-j}\mathbb{Z} \bigcap \mathbb{Z}$, and, thus, we have

$$R_j = \begin{cases} \alpha_j\, p^{-j}, & j < 0 \\ \beta_j\, q^j, & j \ge 0, \end{cases}$$

where $\alpha_j = \pm 1$ and $\beta_j = \pm 1$. Under these assumptions on R_j, the oversampled affine systems $\mathcal{F}_a^{\{R_j\}}(\psi)$, given by (2.6), are the quasi affine systems discussed in Section 2.1. In higher dimensions, the oversampled systems $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6) with R_j given by (2.63), are the n-dimensional quasi affine systems for rational dilation matrices introduced by Bownik [2]. The following theorem shows that, in this case, $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a PF for $L^2(\mathbb{R}^n)$ if and only if the corresponding affine system $\mathcal{F}_A(\Psi)$ has the same property. This result is a generalization of a similar result in [2, Thm.3.4], where the dilation matrices are simply expanding and not expanding on a subspace.

Theorem 2.17. *Let $B = A^T \in GL_n(\mathbb{Q})$ be expanding on a subspace $F \subseteq \mathbb{R}^n$. For every $j \in \mathbb{Z}$, let $R_j = S_j^T$ be defined by (2.63). Then the affine system $\mathcal{F}_A(\Psi)$, given by (1.10), is a Parseval frame for $L^2(\mathbb{R}^n)$, if and only if the corresponding oversampled affine system $\mathcal{F}_A^{\{R_j\}}(\Psi)$, given by (2.6), is a Parseval frame for $L^2(\mathbb{R}^n)$.*

Proof. It is clear that $S_j \in GL_n(\mathbb{Z})$ for each $j \in \mathbb{Z}$. Fix $j \in \mathbb{Z}$ and let $P_j = B^j S_j$. Since

$$S_j \mathbb{Z}^n = B^{-j} \mathbb{Z}^n \bigcap \mathbb{Z}^n,$$

it follows that

$$P_j \mathbb{Z}^n = \mathbb{Z}^n \bigcap B^j \mathbb{Z}^n,$$

which shows that $P_j \in GL_n(\mathbb{Z})$, for each $j \in \mathbb{Z}$, and that $P_0\mathbb{Z}^n = \mathbb{Z}^n$. We can apply Theorem 2.4. With the assumptions that we made for $\{S_j\}$, we have

$$\Lambda = \bigcup_{j\in\mathbb{Z}} B^j S_j \mathbb{Z}^n = \bigcup_{j\in\mathbb{Z}} P_j \mathbb{Z}^n = \mathbb{Z}^n$$

(using the observation that $P_j \mathbb{Z}^n \subseteq \mathbb{Z}^n$ and $P_0\mathbb{Z}^n = \mathbb{Z}^n$). For any $m \in \Lambda = \mathbb{Z}^n$ we have

$$\mathcal{P}'_m = \{j \in \mathbb{Z} : \, S_j^{-1} B^{-j} m \in \mathbb{Z}^n\} = \{j \in \mathbb{Z} : \, P_j^{-1} m \in \mathbb{Z}^n\}. \tag{2.64}$$

Thus, from Theorem 2.4 it follows that the system $\mathcal{F}_A^{\{R_j\}}(\Psi)$ is a Parseval frame if and only if

$$\sum_{\ell=1}^{L} \sum_{j\in\mathcal{P}'_m} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \, m \in \mathbb{Z}^n.$$

In order to complete the proof we need to show that the corresponding affine system $\mathcal{F}_A(\Psi)$ has the same characterization equation. To do this, it suffices to show that the set

$$\mathcal{P}_m = \{j \in \mathbb{Z} : \, B^{-j} m \in \mathbb{Z}^n\} = \{j \in \mathbb{Z} : \, S_j P_j^{-1} m \in \mathbb{Z}^n\}, \tag{2.65}$$

which appears in Theorem 1.4 (in the characterization equation of $\mathcal{F}_A(\Psi)$) is equal to $\mathcal{P}'_m$. Fix $m \in \mathbb{Z}^n$. Since $S_j \in GL_n(\mathbb{Z})$, then $P_j^{-1} m \in \mathbb{Z}^n$ implies $S_j P_j^{-1} m \in \mathbb{Z}^n$, and so $\mathcal{P}'_m \subseteq \mathcal{P}_m$. For the other direction, let $N = S_j P_j^{-1} m \in \mathbb{Z}^n$. Since

$$S_j \mathbb{Z}^n = B^{-j} \mathbb{Z}^n \bigcap \mathbb{Z}^n = S_j P_j^{-1} \mathbb{Z}^n \bigcap \mathbb{Z}^n,$$

then $N \in S_j\mathbb{Z}^n$, and so $P_j^{-1} m = S_j^{-1} N \in \mathbb{Z}^n$. This shows that $\mathcal{P}_m \subseteq \mathcal{P}'_m$, and thus $\mathcal{P}_m = \mathcal{P}'_m$, for each $m \in \mathbb{Z}^n$.

Since the sets (2.64) and (2.65) are equal, it follows that the systems $\mathcal{F}_A^{\{R_j\}}(\Psi)$ and $\mathcal{F}_A(\Psi)$ have the same characterizing equations, and this completes the proof. □

2.3.4 Special case: quasi-affine systems $a \in \mathbb{Q}$

Let us consider the one-dimensional case where $A = a = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $p > q > 1$, $(p,q) = 1$ and let

$$R_j = \begin{cases} s\,p^{-j}, & j < 0 \\ s\,q^{j}, & j \geq 0, \end{cases} \tag{2.66}$$

where $s \in \mathbb{Z}$. The following theorem, which is *not* a consequence of Theorem 2.17, shows that, in this case, if the affine system $\mathcal{F}_a(\psi)$ is a frame, then also the corresponding oversampled system is a frame. As is the case for similar results that we have proved in Sections 2.3.1 and 2.3.2, here, again, the idea of the proof consists in expressing $N^2_{\{R_j\}}(f)$ as an appropriate average over $N^2(T_v\, f)$, where v ranges over a finite set.

Theorem 2.18. *Let $a = \frac{p}{q}$, where $p, q \in \mathbb{Z}$, $p > q \geq 2$, $(p,q) = 1$, and $s \in \mathbb{Z}$, with $s \geq 1$, $(s,p) = 1$, $(s,q) = 1$. If the affine system $\mathcal{F}_a(\psi) = \{D_a^j\, T_k\, \psi : j \in \mathbb{Z},\, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$, then $\mathcal{F}_a^{\{R_j\}}(\psi) = \{R_j^{-1/2}\, D_a^j\, T_{R_j^{-1}k}\, \psi : j \in \mathbb{Z},\, k \in \mathbb{Z}\}$, where $\{R_j\}$ is given by (2.66), is also a frame for $L^2(\mathbb{R})$, and the frame bounds are the same.*

Proof. It suffices to prove the theorem for f in a dense subspace of $L^2(\mathbb{R})$. By Proposition 2.8, for each $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (1.6) with $E = \{0\}$, and for any $x \in \mathbb{R}$ we have

$$N^2(T_x\, f) = \sum_{j,k \in \mathbb{Z}} |\langle T_x\, f, D_a^j\, T_k\, \psi \rangle|^2 = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{w}_j(m)\, e^{2\pi i a^j m \cdot x}, \tag{2.67}$$

where $\hat{w}_j(m)$ is given by (2.22), and the sum converges absolutely.

Denote by V_j, with $j \in \mathbb{Z}$, a complete set of distinct representatives of the quotient group $\mathbb{Z}/(\frac{q^j}{p^j}\,\mathbb{Z} \cap \mathbb{Z})$. Observe that $(q/p)^j\,\mathbb{Z} \cap \mathbb{Z} = p^{-j}\,\mathbb{Z}$ if $j < 0$ and $(q/p)^j\,\mathbb{Z} \cap \mathbb{Z} = q^j\,\mathbb{Z}$ if $j \geq 0$, and so the order of the group is $|V_j| = p^{-j}$ if $j < 0$ and $|V_j| = q^j$ if $j \geq 0$.

If $j \geq 0$, it follows from Lemma 2.15 (it suffices to let $u = k/q^j$ in the lemma) that:

$$\frac{1}{|V_j|} \sum_{v(j) \in V_j} e^{2\pi i \frac{k}{q^j} \cdot v(j)} = \begin{cases} 1 & \text{if } k \in q^j\,\mathbb{Z} \\ 0 & \text{if } k \in \mathbb{Z} \setminus q^j\,\mathbb{Z}. \end{cases} \tag{2.68}$$

This implies that

$$\frac{1}{|V_j|} \sum_{v(j) \in V_j} e^{2\pi i m \frac{p^j}{q^j} \cdot v(j)} = \begin{cases} 1 & \text{if } m \in q^j\,\mathbb{Z} \\ 0 & \text{if } m \in \mathbb{Z} \setminus q^j\,\mathbb{Z}. \end{cases} \tag{2.69}$$

In fact, if $m \in q^j\mathbb{Z}$, then $k = m\,p^j \in p^j q^j\,\mathbb{Z} \subset q^j\mathbb{Z}$. On the other hand, if $m \in \mathbb{Z} \setminus q^j\mathbb{Z}$, then $k = m\,p^j \in \mathbb{Z}$, but $k = mp^j \notin q^j\mathbb{Z}$ since $(p,q) = 1$. Now we will argue as in the proof of Proposition 2.14. By the Third Homomorphism Theorem [10], for any $0 \leq j \leq J$ the quotient group $(\mathbb{Z}/q^J\mathbb{Z})/(q^j\mathbb{Z}/q^J\mathbb{Z})$ is isomorphic to $\mathbb{Z}/q^j\mathbb{Z}$. This implies that $\mathbb{Z}/q^J\mathbb{Z} = \bigcup_{v(j)\in V_j}\big(v(j)+q^j\mathbb{Z}/q^J\mathbb{Z}\big)$, and, thus, each $v(J)\in V_J$ is of the form $v(J)=v(j)+q^j\,v(J-j)$, where $v(j)\in V_j$ and $v(J-j)\in V_{J-j}$ (notice that $q^j\,(\mathbb{Z}/q^J\mathbb{Z}\simeq q^j\,\mathbb{Z}/q^{J-j}\mathbb{Z})$). Since V_{J-j} has cardinality q^{J-j}, V_J is made up of as many copies of V_j. Therefore, from (2.69) we have that, for each $0\le j\le J$,

$$\frac{1}{|V_J|}\sum_{v\in V_J} e^{2\pi i m\frac{p^j}{q^j}\cdot v} = \begin{cases} 1 & \text{if } m\in q^j\,\mathbb{Z}\\ 0 & \text{if } m\in\mathbb{Z}\setminus q^j\,\mathbb{Z}.\end{cases} \tag{2.70}$$

If $j<0$ the same argument carries through with the roles of p and q reversed, and we obtain that, for each $-J\le j<0$,

$$\frac{1}{|V_J|}\sum_{v\in V_J} e^{2\pi i m\frac{p^j}{q^j}\cdot v} = \begin{cases} 1 & \text{if } m\in p^{-j}\,\mathbb{Z}\\ 0 & \text{if } m\in\mathbb{Z}\setminus p^{-j}\,\mathbb{Z}.\end{cases} \tag{2.71}$$

From (2.67), for any $J\ge 0$, we have:

$$\begin{aligned}\frac{1}{|V_J|}\sum_{v\in V_J} N^2(T_v\,f) &= \frac{1}{|V_J|}\sum_{v\in V_J}\sum_{-J\le j<0}\sum_{m\in\mathbb{Z}}\hat{w}_j(m)\,e^{2\pi i a^j m\cdot v}\\ &+\frac{1}{|V_J|}\sum_{v\in V_J}\sum_{0\le j\le J}\sum_{m\in\mathbb{Z}}\hat{w}_j(m)\,e^{2\pi i a^j m\cdot v}\\ &+\frac{1}{|V_J|}\sum_{v\in V_J}\sum_{|j|>J}\sum_{m\in\mathbb{Z}}\hat{w}_j(m)\,e^{2\pi i a^j m\cdot v}\\ &= I_1(J)+I_2(J)+I_3(J),\end{aligned} \tag{2.72}$$

where $I_1(J)$ is the sum for $-J\le j<0$, $I_2(J)$ is the sum for $0\le j\le J$ and $I_3(J)$ is the sum for $|j|>J$. Since the sum (2.72) converges absolutely, then

$$\lim_{J\to\infty} I_3(J)=0. \tag{2.73}$$

By (2.71),

$$I_1(J)=\sum_{-J\le j<0}\sum_{m\in\mathbb{Z}}\hat{w}_j(p^{-j}m), \tag{2.74}$$

and by (2.70)

$$I_2(J)=\sum_{0\le j\le J}\sum_{m\in\mathbb{Z}}\hat{w}_j(q^j m). \tag{2.75}$$

Thus, using (2.73), (2.74), and (2.75) into (2.72), it follows that

$$\lim_{J\to\infty}\frac{1}{|V_J|}\sum_{v\in V_J} N^2(T_v\,f) = \lim_{J\to\infty}\Big(\sum_{-J\le j<0}\sum_{m\in\mathbb{Z}}\hat{w}_j(p^{-j}m)+\sum_{0\le j\le J}\sum_{m\in\mathbb{Z}}\hat{w}_j(q^j m)\Big). \tag{2.76}$$

Write

$$\widetilde{R}_j = \begin{cases} p^{-j}, & j < 0 \\ q^j, & j \geq 0, \end{cases} \tag{2.77}$$

so that $R_j = s\,\widetilde{R}_j$ for all $j \in \mathbb{Z}$. By Proposition 2.9 we have that, for any $f \in \mathcal{D}$,

$$N^2_{\{\widetilde{R}_j\}}(f) = \sum_{j,m\in\mathbb{Z}} \hat{w}_{j,\ell}(\widetilde{R}_j\, m),$$

where $N^2_{\{\widetilde{R}_j\}}(f)$ is given by (2.24), $\hat{w}_{j,\ell}(m)$ is given by (2.22) and the series converges absolutely. Using (2.77) in this expression we have

$$N^2_{\{\widetilde{R}_j\}}(f) = \sum_{m\in\mathbb{Z}} \sum_{j<0} \hat{w}_{j,\ell}(p^{-j}\, m) + \sum_{m\in\mathbb{Z}} \sum_{j\geq 0} \hat{w}_{j,\ell}(q^j\, m).$$

Comparing this equation with (2.76) we deduce

$$N^2_{\{\widetilde{R}_j\}}(f) = \lim_{J\to\infty} \frac{1}{|V_J|} \sum_{v\in V_J} N^2(T_v\, f). \tag{2.78}$$

Now let U be a complete set of representatives of the quotient group $s^{-1}\mathbb{Z}/\mathbb{Z}$. An argument similar to the one used in Theorem 2.12 shows that

$$N^2_{\{R_j\}}(f) = \lim_{K\to\infty} \frac{1}{s} \sum_{u\in U} N^2(T_{p^K q^K u}\, f), \tag{2.79}$$

for all $f \in \mathcal{D}$. Notice that to prove (2.79) it is necessary to use the assumptions $(p, q) = 1$, $(s, p) = 1$ and $(s, q) = 1$. Finally, combining (2.78) and (2.79) we obtain that, for any $f \in \mathcal{D}$,

$$N^2_{\{R_j\}}(f) = \lim_{J,K\to\infty} \frac{1}{s} \sum_{u\in U} \frac{1}{|V_J|} \sum_{v\in V_J} N^2(T_{v+p^K q^K u}\, f). \tag{2.80}$$

The proof now follows from (2.80) as in the last step of the proof of Theorem 2.6. □

2.3.5 Co-affine Systems

Another possible choice of matrices in class (II), defined in Theorem 2.4, is given by the matrices $R_j = A^{-j}$, for each $j \in \mathbb{Z}$, where $A \in GL_n(\mathbb{R})$ is the dilation matrix and $B = A^T$ is expanding on a subspace $F \subset \mathbb{R}^n$. In fact, if $R_j = A^{-j}$, for each $j \in \mathbb{Z}$, then $R_j\, A^j = I \in GL_n(\mathbb{Z})$ and this trivially shows that the matrices R_j are in class (II). Under these assumptions, since $D_A^j\, T_{A^j k} = T_k\, D_A^j$, it follows that the oversampled affine systems $\mathcal{F}_A^{\{R_j\}}(\Psi)$ are the **co-affine system**

$$\mathcal{F}_A^{\{A^{-j}\}}(\Psi) = \widetilde{\mathcal{F}}_A(\Psi) = \{|\det A|^{j/2}\, T_k\, D_A^j\, \psi^\ell : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \ell = 1, \ldots, L\}.$$

However, since $|\det R_j| = |\det A|^{-j}$ is not bounded below, it follows from Proposition 2.1 that this system cannot be Bessel and, thus, cannot be a frame for $L^2(\mathbb{R}^n)$. This situation is also investigated in [9].

3 Oversampling of the shift-invariant systems

In this section, we consider families of the form

$$\Phi_C^{\{g_p\}} = \{T_{Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}, \tag{3.1}$$

where $\{g_p : p \in \mathcal{P}\} \in L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. They are special cases of the families $\Phi_{\{C_p\}}^{\{g_p\}}$, given by (1.3), where $C_p = C$, for each $p \in \mathcal{P}$. It is clear that, unlike the more general systems given by (1.3), the families $\Phi_C^{\{g_p\}}$ are invariant with respect to $C\,\mathbb{Z}^n$ translations, and so, we will refer to these systems as **shift-invariant systems**. As we shall see, this invariance makes the study of these systems easier and their properties simpler than the general systems $\Phi_{\{C_p\}}^{\{g_p\}}$ where C_p depends on $p \in \mathcal{P}$.

In [17], we characterize those $\{g_p : p \in \mathcal{P}\}$ such that $\Phi_C^{\{g_p\}}$ is a Parseval frames for $L^2(\mathbb{R}^n)$. We obtain the following result (cf. [17, Th.3.1]), that can also be found in [20]. Observe that this characterization is simpler than Theorem 1.1 since the L.I.C. is not needed in this case.

Theorem 3.1. *Let $\{g_p\}_{p\in\mathcal{P}} \subset L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. Then $\Phi_C^{\{g_p\}}$, given by (3.1), is a PF for $L^2(\mathbb{R}^n)$ if and only if*

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{g}_p(\xi + C^I m)} = |\det C| \;\; \delta_{m,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{3.2}$$

for each $u \in \mathbb{Z}^n$, where $C^I = (C^T)^{-1}$ and δ is the Kronecker delta in $\mathbb{Z}^n$.

In the same paper, we also deduce the following result (cf. [17, Prop.4.1]), which is similar to Proposition 2.7, except that the L.I.C. is not needed in this case.

Proposition 3.2. *Let $\{g_p\}_{p\in\mathcal{P}}$ be a countable collection in $L^2(\mathbb{R}^n)$ and $C \in GL_n(\mathbb{R})$. Assume that*

$$\sum_{p\in\mathcal{P}} |\hat{g}_p(\xi)|^2 \le B \tag{3.3}$$

for a.e $\xi \in \mathbb{R}^n$, for some $B > 0$. Then, for each $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (1.6) with $E = \varnothing$, the function $w(x) = N^2(T_x f) = \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{Ck}\, g_p\rangle|^2$ is continuous and coincides pointwise with the absolutely convergent series $\sum_{m\in\mathbb{Z}^n} \hat{w}(C^I m)\, e^{2\pi i C^I m\cdot x}$, where

$$\hat{w}(C^I m) = \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I m)} \Big(\sum_{p\in\mathcal{P}} \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C^I m) \Big)\, d\xi, \tag{3.4}$$

and the integral in (3.4) converges absolutely.

Define the **oversampled shift-invariant system** corresponding to $\Phi_C^{\{g_p\}}$ as the family

$$\Phi_{R^{-1}C}^{\{g_p\}} = \{|\det R|^{-1/2}\, T_{R^{-1}Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}, \tag{3.5}$$

where $R \in GL_n(\mathbb{R})$. Using the same approach as in the case of affine systems, we obtain conditions such that, if the shift-invariant system $\Phi_C^{\{g_p\}}$ is a frame, then the corresponding oversampled system $\Phi_{R^{-1}C}^{\{g_p\}}$ is also a frame with the same frame bounds. The following result can also be found in [15], where the proof does not involve the use of the Fourier series expansion. The proof that we present, on the other hand, illustrates how the the approach that we used in the case of affine systems simplifies in the case of shift-invariant systems.

Theorem 3.3. *If the shift-invariant system $\Phi_C^{\{g_p\}}$, $C \in GL_n(\mathbb{R})$, is a frame and $W = C^{-1}\,RC \in GL_n(\mathbb{Z})$, then $\Phi_{R^{-1}C}^{\{g_p\}}$ is also a frame with the same frame bounds.*

Proof. Since $\Phi_C^{\{g_p\}}$ is a frame, then it is Bessel with Bessel constant β, for some $\beta > 0$. By Proposition 3.2, condition (3.3) is satisfied (with $B = \beta|\det C|$), and, thus, we can apply Proposition 3.2, which gives the absolutely convergent series

$$N^2(T_x\, f) \;=\; \sum_{p\in\mathcal{P}}\sum_{k\in\mathbb{Z}^n} |\langle T_x f, T_{Ck}\, g_p\rangle|^2 = \sum_{m\in\mathbb{Z}^n} \hat{w}(C^I m)\, e^{2\pi i C^I m\cdot x}, \tag{3.6}$$

for all $f \in \mathcal{D}$ and $x \in \mathbb{R}^n$, where $\hat{w}(C^I m)$ is given by (3.4). Replacing the matrix C by $R^{-1}C$ in (3.6) and letting $S = R^T$, Proposition 3.2 also gives

$$N_R^2(T_x\, f) = \frac{1}{|\det R|}\sum_{k\in\mathbb{Z}^n}\sum_{p\in\mathcal{P}} |\langle T_x f, T_{R^{-1}Ck}\, g_p\rangle|^2 = \sum_{m\in\mathbb{Z}^n} \hat{w}(SC^I m)\, e^{2\pi i SC^I m\cdot x},$$

which is an absolutely convergent series for all $f \in \mathcal{D}$ and $x \in \mathbb{R}^n$.

Let V be a complete set of distinct representatives of the group $W^{-1}\,\mathbb{Z}^n/\mathbb{Z}^n$, where $W = C^{-1}RC \in GL_n(\mathbb{Z})$. By Lemma 2.11,

$$\frac{1}{|\det W|}\sum_{v\in V} e^{2\pi i m\cdot v} \;=\; \begin{cases} 1 & \text{if } m \in W^T\mathbb{Z}^n \\ 0 & \text{if } m \in \mathbb{Z}^n \setminus W^T\mathbb{Z}^n. \end{cases}$$

Observe that $m \in W^T\mathbb{Z}^n = C^T SC^I\mathbb{Z}^n$ if and only if $C^I m \in SC^I\mathbb{Z}^n$. Thus, the above expression is equivalent to

$$\frac{1}{|\det W|}\sum_{v\in V} e^{2\pi i C^I m\cdot v} \;=\; \begin{cases} 1 & \text{if } C^I m \in SC^I\mathbb{Z}^n \\ 0 & \text{if } C^I m \in \mathbb{Z}^n \setminus SC^I\mathbb{Z}^n. \end{cases} \tag{3.7}$$

By applying (3.7) to (3.6), it follows that

$$\begin{aligned} \frac{1}{|\det W|}\sum_{v\in V} N^2(T_v\, f) &= \sum_{m\in\mathbb{Z}^n} \hat{w}(C^I m)\frac{1}{|\det W|}\sum_{v\in V} e^{2\pi i C^I m\cdot v} \\ &= \sum_{m\in\mathbb{Z}^n} \hat{w}(SC^I m) = N_R^2(f), \end{aligned} \tag{3.8}$$

for all $f \in \mathcal{D}$. Since $\Phi_C^{\{g_p\}}$ is a frame, there are $0 < \alpha \leq \beta < \infty$ such that $\alpha \|f\|^2 \leq N^2(f) \leq \beta \|f\|^2$, for all $f \in L^2(\mathbb{R}^n)$, and, as a consequence, $\alpha \|f\|^2 \leq \frac{1}{|\det W|} \sum_{v \in V} N^2(T_v f) \leq \beta \|f\|^2$ for any $f \in L^2(\mathbb{R}^n)$. Using (3.8), it follows that

$$\alpha \|f\|^2 \leq N_R^2(f) \leq \beta \|f\|^2,$$

for all $f \in \mathcal{D}$. These inequalities can be extended to all $f \in L^2(\mathbb{R}^n)$ by the usual density argument. Therefore, $\Phi_{R^{-1}C}^{\{g_p\}}$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds α and β. □

We will now apply Theorem 3.3 to the Gabor systems. Let M_y, $y \in \mathbb{R}^n$, be the **modulation operator**, defined by $M_y f(x) = e^{2\pi i y \cdot x} f(x)$. The **Gabor systems** generated by $G = \{g^1, g^2, \ldots, g^L\} \subset L^2(\mathbb{R}^n)$ are the families of the form

$$\mathcal{G}_{B,C}(G) = \{T_{Ck} M_{Bm} g^\ell : m, k \in \mathbb{Z}^n, \ \ell = 1, 2, \cdots, L\}, \tag{3.9}$$

where $B, C \in GL_n(\mathbb{R})$. The corresponding **oversampled Gabor system** $\mathcal{G}_{B,C}^R(G)$ are defined as the collections

$$\mathcal{G}_{B,C}^R(G) = \{|\det R|^{-1/2} T_{R^{-1}Ck} M_{Bm} g^\ell : m, k \in \mathbb{Z}^n, \ \ell = 1, 2, \cdots, L\},$$

where $B, C, R \in GL_n(\mathbb{R})$. An elementary application of Theorem 3.3 gives the following result.

Corollary 3.4. *If the Gabor system $\mathcal{G}_{B,C}(G)$ is a frame and $C^{-1} RC \in GL_n(\mathbb{Z})$, then $\mathcal{G}_{B,C}^R(G)$ is also a frame with the same frame bounds.*

Proof. Let $\mathcal{P} = \{(m, \ell) : m \in \mathbb{Z}^n, \ \ell = 1, \cdots, L\}$, $g_p = g_{m,\ell} = M_{Bm} g^\ell$ for any $p = (m, \ell) \in \mathcal{P}$. Under these assumptions, the system $\{T_{Ck} g_p : p \in \mathcal{P}\}$ is exactly the Gabor system given by (3.9). The proof now follows immediately from Theorem 3.3. □

4 Dilation–oversampling of the affine systems

So far we have considered the oversampled systems obtained by using a larger collection of translations. In this section, we examine the case where we increase the number of dilations of an affine system.

Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and $A \in GL_n(\mathbb{R})$ be of the form $A = e^E$, where $E \in GL_n(\mathbb{R})$. Let $\mathcal{F}_A(\Psi)$ be the affine system given by (1.10). For $M \in \mathbb{N}$, define the **dilation–oversampled** affine systems relative to $\mathcal{F}_A(\Psi)$ as the collections

$$\mathcal{F}_{A,M}(\Psi) = \{\frac{1}{\sqrt{M}} D_{A^{j/M}} T_k \psi^\ell : j \in \mathbb{Z}, \ k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}. \tag{4.1}$$

We obtain the following result which shows that the dilation–oversampled affine systems $\mathcal{F}_{A,M}(\Psi)$ preserve the frame property of the corresponding affine systems $\mathcal{F}_A(\Psi)$.

Theorem 4.1. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, and $A \in GL_n(\mathbb{R})$ be of the form $A = e^E$, where $E \in GL_n(\mathbb{R})$. If the affine system $\mathcal{F}_A(\Psi)$, given by (1.10), is a frame for $L^2(\mathbb{R}^n)$, then the dilation–oversampled affine system $\mathcal{F}_{A,M}(\Psi)$, given by (4.1) is also a frame for $L^2(\mathbb{R}^n)$, and the frame bounds are the same.*

Proof. Suppose that the affine system $\mathcal{F}_A(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$. Then there are constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \, \|f\|^2 \leq N^2(f) \leq \beta \, \|f\|^2, \tag{4.2}$$

for all $f \in L^2(\mathbb{R}^n)$, where $N^2(f)$ is given by (2.21). Then, for all $f \in L^2(\mathbb{R}^n)$ we have:

$$\begin{aligned} N_M^2(f) &= \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle f, \frac{1}{\sqrt{M}} \, D_{A^{j/M}} \, T_k \, \psi^\ell \rangle|^2 \\ &= \frac{1}{M} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \sum_{m\in\mathbb{Z}} \sum_{u=0}^{M-1} |\langle f, D_{A^m} \, D_{A^{u/M}} \, T_k \, \psi^\ell \rangle|^2 \\ &= \frac{1}{M} \sum_{u=0}^{M-1} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \sum_{m\in\mathbb{Z}} |\langle D_{A^{-u/M}} \, f, D_{A^m} \, T_k \, \psi^\ell \rangle|^2 \\ &= \frac{1}{M} \sum_{u=0}^{M-1} N^2(D_{A^{-u/M}} \, f). \end{aligned} \tag{4.3}$$

Since $\|f\| = \|D_{A^{-u/M}} \, f\|$, for any $f \in L^2(\mathbb{R}^n)$, it follows from (4.2) that

$$\alpha \, \|f\|^2 \leq N^2(D_{A^{-u/M}} \, f) \leq \beta \, \|f\|^2,$$

for any $f \in L^2(\mathbb{R}^n)$ and all $u = 0, 1, \ldots, M-1$. Thus, from (4.3) we have that

$$\alpha \, \|f\|^2 \leq N_M^2(f) \leq \beta \, \|f\|^2,$$

for any $f \in L^2(\mathbb{R}^n)$. $\square$

5 Wave Packets

In this section, we examine those function systems generated by the combined action of translations, modulations and dilations on a finite family of functions. Systems of this form have been considered by several authors, including [6], [16], [14], and have been applied, for example, to decompose the symbol or the kernel of some classes of singular integral operators (see, for example, [6], [8, Ch. 3], [18]). In this paper, we will only consider discrete systems and will refer to such systems as **wave packet systems**. Our terminology generalizes the one introduced by Córdoba and Fefferman, where the wave packets are the families of functions obtained by applying dilations, modulations and translations to the Gaussian function.

Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $S \subset \mathbb{Z} \times \mathbb{R}^n$ be a countable set. We define the **wave packet systems** generated by Ψ relative to the dilation matrices $A = \{A_j\} \subset GL_n(\mathbb{R})$ and to $S \subseteq \mathbb{Z} \times \mathbb{R}^n$ as the set

$$\mathcal{W}_{A,S}(\Psi) = \big\{D_{A_j}\, T_k\, M_\nu\, \psi^\ell : \, k \in \mathbb{Z}^n, \, (j,\nu) \in S, \, \ell = 1, \ldots, L\big\}. \tag{5.1}$$

Special cases of such $\mathcal{W}_{A,S}(\Psi)$ are the affine systems $\mathcal{F}_A(\Psi)$, where $A_j = A^j$, $S = \mathbb{Z} \times \{0\}$, and the Gabor systems $\mathcal{G}_B(G)$, where $S = \{0\} \times \mathbb{Z}^n$. This simple observation already suggests that the wave packet systems provide greater flexibility than the affine or the Gabor systems. We are interested in characterizing the families $\Psi \subset L^2(\mathbb{R}^n)$ such that the system $\mathcal{W}_{A,S}(\Psi)$ is a reproducing system for $L^2(\mathbb{R}^n)$. While it is well known that such reproducing systems exist in the special cases given by the affine and Gabor systems, it is not obvious that for more general sets $S \subseteq \mathbb{Z} \times \mathbb{R}^n$ there exist families $\Psi \subset L^2(\mathbb{R}^n)$ such that the collection $\mathcal{W}_{A,S}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$. The following one-dimensional example, whose idea was suggested to us by D. Speegle, shows that such "nontrivial" wave packet systems do exist.

Example 5.1. Let $\psi \in L^2(\mathbb{R})$ be such that $\hat{\psi}(\xi) = \chi_{[1,2)}(\xi)$, $\xi \in \mathbb{R}$. Let $\mathbb{Z}_+ = \{j \in \mathbb{Z} : j \geq 0\}$ and

$$S = \{(j,0) : j \in \mathbb{Z}_+\} \bigcup \{(j,-3) : j \in \mathbb{Z}_+\} \bigcup \big\{(0,-1),(0,-2)\big\}. \tag{5.2}$$

We will show that the wave packet system $\mathcal{W}_{2,S}(\psi)$, given by (5.1) with dilations $A_j = 2^j$ and S given by (5.2), is an orthonormal basis (ONB) for $L^2(\mathbb{R})$.

Observe that

$$\big(D_2^j\, T_k\, M_\nu\, \psi\big)^\wedge = D_2^{-j}\, M_{-k}\, T_\nu\, \hat{\psi}.$$

If $(j,\nu) = (0,-1)$ and $(j,\nu) = (0,-2)$, we have:

$$\{M_{-k}\, T_{-1}\, \hat{\psi}(\xi) : k \in \mathbb{Z}\} = \{e^{-2\pi i k\cdot\xi}\, \chi_{[0,1)}(\xi) : k \in \mathbb{Z}\},$$

and

$$\{M_{-k}\, T_{-2}\, \hat{\psi}(\xi) : k \in \mathbb{Z}\} = \{e^{-2\pi i k\cdot\xi}\, \chi_{[-1,0)}(\xi) : k \in \mathbb{Z}\}.$$

The combination of these two systems forms an ONB for $L^2([-1,1))$. For $(j,\nu) = (j,0), j \in \mathbb{Z}_+$ we have the system

$$\{D_2^{-j}\, M_{-k}\, \hat{\psi}(\xi) : k \in \mathbb{Z}, j \in \mathbb{Z}_+\} = \{2^{-j/2}\, e^{-2\pi i 2^{-j} k\cdot\xi}\, \chi_{[2^j,2^{j+1})}(\xi) : k \in \mathbb{Z}, j \in \mathbb{Z}_+\}.$$

This gives an ONB for $L^2([1,\infty))$. For $(j,\nu) = (j,-3), j \in \mathbb{Z}_+$, a similar calculation shows that the system

$$\{D_2^{-j}\, M_{-k}\, T_{-3}\, \hat{\psi}(\xi) : k \in \mathbb{Z}, j \in \mathbb{Z}_+\} = \{2^{-j/2}\, e^{-2\pi i 2^{-j} k\cdot\xi}\, \chi_{[-2^{j+1},-2^j)}(\xi) : k \in \mathbb{Z}, j \in \mathbb{Z}_+\}$$

is an ONB for $L^2((-\infty,-1))$. Combining all these systems, we have that the collection $\{D_2^{-j} M_k T_\nu \hat{\psi} : k \in \mathbb{Z}, (j,\nu) \in S\}$ is an ONB for $L^2(\mathbb{R})$ and, as a consequence, $\mathcal{W}_{2,S}(\psi)$ is also an ONB for $L^2(\mathbb{R})$.

We will now turn our attention to the characterization of the wave packet systems. Using Theorem 1.1, we obtain the following characterization of all $\Psi \subset L^2(\mathbb{R}^n)$ such that the system $\mathcal{W}_{A,S}(\Psi)$, given by (5.1), is a PF for $L^2(\mathbb{R}^n)$.

Theorem 5.2. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $A \in GL_n(\mathbb{R})$ and $S \subset \mathbb{Z}\times\mathbb{R}^n$ be a countable set. Assume the L.I.C.:*

$$L(f) = \sum_{(j,\nu)\in S} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + B_j\, m)|^2\, |\hat{\psi}^\ell(B_j^{-1}\xi - \nu)|^2\, d\xi < \infty \tag{5.3}$$

for all $\ell = 1, \ldots, L$ and for any $f \in \mathcal{D}$, where $\mathcal{D} = \mathcal{D}_E$ is given by (1.6) with $E = \{0\}$. Then the system $\mathcal{W}_{A,S}(\Psi)$, given by (5.1), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{(j,\nu)\in\mathcal{P}_\alpha} \hat{\psi}^\ell(B_j^{-1}\xi - \nu)\, \overline{\hat{\psi}^\ell(B_j^{-1}(\xi+\alpha) - \nu)} = \delta_{\alpha,0} \quad \textit{for a.e. } \xi \in \mathbb{R}, \tag{5.4}$$

where $B_j = A_j^T$, $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} B_j\, \mathbb{Z}^n$ and, for each $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{(j,\nu) \in S : B_j^{-1}\, \alpha \in \mathbb{Z}^n\}$.

Proof. Let $\mathcal{P}$, $\{g_p\}_{p\in\mathcal{P}}$ and $\{C_p\}_{p\in\mathcal{P}}$ be defined by

$$\mathcal{P} = \big\{(j,\nu,\ell) : (j,\nu) \in S, \text{ and } \ell = 1, \ldots, L\big\},$$

$$g_p(x) = g_{(j,\nu,\ell)}(x) = D_{A_j}\, M_\nu\, \psi^\ell(x), \qquad C_p = C_{(j,\nu,\ell)} = A_j^{-1}. \tag{5.5}$$

With these assumptions, it follows that

$$T_{C_p k}\, g_p = T_{A_j^{-1}\, k}\, D_{A_j}\, M_\nu\, \psi^\ell = D_{A_j}\, T_k\, M_\nu\, \psi^\ell,$$

and so the collection $\{T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the wave packet system $\mathcal{W}_{A,S}(\Psi)$. We can now apply Theorem 1.1.

Under these assumptions for $\mathcal{P}$, g_p and C_p, we have that $\Lambda = \bigcup_{p\in\mathcal{P}} C_p^I\, \mathbb{Z}^n = \bigcup_{j\in\mathbb{Z}} B_j\, \mathbb{Z}^n$, and, for $\alpha \in \Lambda$, we have $\mathcal{P}_\alpha = \{p \in \mathcal{P} : C_p^T\, \alpha \in \mathbb{Z}^n\} = \{j \in \mathbb{Z} : B_j{}^{-1}\, \alpha \in \mathbb{Z}^n\}$. Since $\hat{g}_p = (D_{A_j}\, M_\nu\, \psi^\ell)^\wedge = D_{B_j^{-1}}\, T_\nu\, \hat{\psi}^\ell$, the expression (1.7) is exactly the L.I.C. (5.3). Finally, by direct computation, from (1.8) we obtain equation (5.4). □

5.1 A very general example

Using some ideas from Example 5.1 it is possible to construct some very general wave packet systems. In fact, in the following example, the dilations do not have to be expanding and the modulations do not have to be associated with a lattice. For simplicity, we will present a one-dimensional construction.

Let $I_1 = [1, 2)$ and consider the tiling of $\mathbb{R}$ given by the union of countably many disjoint half-open intervals $\{I_j\}_{j\geq 1}$. That is:

$$\mathbb{R} = \bigcup_{j\geq 1} I_j. \tag{5.6}$$

For each interval $I_j = [c_j, d_j)$, let $a_j = d_j - c_j$ be the length of the interval. Let $x_j = c_j - a_j = 2\,c_j - d_j$. Thus $a_j^{-1}(I_j - x_j) = a_j^{-1}[a_j, 2\,a_j) = [1, 2)$, and this shows that to each interval I_j there is a uniquely associated dilation a_j and translation x_j mapping I_1 into I_j.

Consider the (one-dimensional) wave packet system

$$\mathcal{W}(\psi) = \big\{D_{a_j}\, T_k\, M_{a_j^{-1}x_j}\, \psi : \, k \in \mathbb{Z},\, j \in \mathbb{Z}_+\big\}, \tag{5.7}$$

where $\hat{\psi}(\xi) = \chi_{I_1}(\xi)$. We will now apply Theorem 5.2 to show that $\mathcal{W}(\psi)$ is an PF for $L^2(\mathbb{R})$. Since any function $\psi_{j,k} = D_{a_j}\, T_k\, M_{a_j^{-1}x_j}\ \psi$ in $\mathcal{W}(\psi)$ has norm equal one, this will also imply that $\mathcal{W}(\psi)$ is an orthonormal basis.

Since $|\hat{\psi}(a_j^{-1}(\xi - x_j))| = \chi_{I_j}(\xi)$, the left hand side of the L.I.C., given by (5.3), becomes

$$\begin{aligned} L(f) &= \sum_{j\geq 1}\sum_{m\in\mathbb{Z}} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + a_j m)|^2\, |\hat{\psi}(a_j^{-1}(\xi - x_j))|^2\, d\xi \\ &= \sum_{j\geq 1}\sum_{m\in\mathbb{Z}} \int_{\operatorname{supp}\hat{f}\cap I_j} |\hat{f}(\xi + a_j m)|^2\, d\xi \end{aligned} \tag{5.8}$$

We need to show that $L(f) < \infty$ for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (1.6). Since $\hat{f}$ is compactly supported and $a_j = |I_j| > 0$, for each fixed j there are only finitely many $m \in \mathbb{Z}$ such that the integral in (5.8) is nonzero. More precisely, if $\operatorname{supp}\hat{f} \subset (-R, R)$, with $R > 0$, then $|m| \leq 2\,R/a_j$. Furthermore, there are only finitely many intervals I_j intersecting $\operatorname{supp}\hat{f}$ (say, J of them). Thus, from (5.8) we have

$$L(f) \leq \sum_{j\geq 1} \frac{2\,R + 1}{a_j} \int_{\operatorname{supp}\hat{f}\cap \dot{I}_j} \|\hat{f}\|_\infty^2\, d\xi \leq J\,(R + 1)\|\hat{f}\|_\infty^2 < \infty,$$

which shows that condition (5.3) is satisfied. In order to show that the wave packet system $\mathcal{W}(\psi)$ is a Parseval frame it only remains to show that ψ satisfies the characterizing equations (5.4) which, in this case, are the two equations:

$$\sum_{j\in\mathcal{P}} |\hat{\psi}(a_j^{-1}(\xi - x_j))|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{5.9}$$

$$\sum_{j\in\mathcal{P}_\alpha} \hat{\psi}(a_j^{-1}(\xi - x_j))\,\overline{\hat{\psi}(a_j^{-1}(\xi+\alpha-x_j))} = 0, \quad \text{for a.e. } \xi\in\mathbb{R}, \text{ if } \alpha\neq 0. \tag{5.10}$$

Since $\hat{\psi}(a_j^{-1}(\xi - x_j)) = T_{x_j}\, D_{a_j}^{-1}\hat{\psi} = \chi_{I_j}(\xi)$, then (5.6) implies equation (5.9). Next consider equation (5.10) with $\alpha \neq 0$ and observe that

$$\hat{\psi}(a_j^{-1}(\xi+\alpha-x_j)) = \chi_{I_j}(\xi+\alpha) = \chi_{I_j}(\xi + a_j\,(a_j^{-1}\,\alpha)).$$

By the definition of $\mathcal{P}_\alpha$, we have that $a_j^{-1}\alpha \in \mathbb{Z}$ for each $j \in \mathcal{P}_\alpha$. Therefore, since $\chi_{I_j}(\xi)$ has support of length a_j, and $\alpha \neq 0$, we have

$$\hat{\psi}(a_j^{-1}(\xi - x_j))\,\overline{\hat{\psi}(a_j^{-1}(\xi+\alpha-x_j))} = \chi_{I_j}(\xi)\,\chi_{I_j}(\xi + a_j\,(a_j^{-1}\,\alpha)) = 0,$$

for each $j \in \mathcal{P}_\alpha$, and thus equation (5.10) is also satisfied.

Observe that the choice of the interval I_1 plays no special role in this example. The construction can easily be modified by choosing any initial interval I_1. Furthermore, this construction easily generalizes to higher dimensions.

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