Outline

- Section 1.4 - Linear Combinations and System of Linear Equations
- Section 1.5 - Linear Dependance/Independence
- Section 1.6 - Bases and Dimension
Linear Combinations and System of Linear Equations

Section 1.4
**Linear combination**

**Definition**

Let $V$ be a vector space over field $F$ and $S$ a nonempty subset of $V$. We call $v \in V$ a **linear combination** of vectors in $S$ if there exist vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n \in F$ such that

$$v = a_1 u_1 + \ldots + a_n u_n$$

**Exercise:** Take $V = \mathbb{R}^3$ and $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Write $(3, 4, 1)$ as a linear combination of vectors in $S$. 
Exercise: Write \((3, 1, 2)\) as a linear combination of \((1, 0, 1), (0, 1, 1), (1, 2, 1)\).
Exercise: Write \((3, 1, 2)\) as a linear combination of \((1, 0, 1), (0, 1, 1), (1, 2, 1)\).

To solve this problem, we need to solve

\[ x_1(1, 0, 1) + x_2(0, 1, 1) + x_3(1, 2, 1) = (3, 1, 2) \]

which is gives a system of linear equations:

\[
\begin{align*}
  x_1 + x_3 &= 3 \\
  x_2 + 2x_3 &= 1 \\
  x_1 + x_2 + x_3 &= 2
\end{align*}
\]
You can simplify the solution of a system of linear equations by performing any of these \textit{elementary row operations}:

- Add a constant multiple of one equation to another.
- Multiply an equation by a nonzero scalar.
- Interchange the order of any two equations.

These three operations DO NOT change the solution of the system!
Solving systems of linear equations

From the system of linear equations

\[ \begin{align*}
  x_1 + x_3 &= 3 \\
  x_2 + 2x_3 &= 1 \\
  x_1 + x_2 + x_3 &= 2
\end{align*} \]

we write the augmented matrix

\[
\begin{pmatrix}
  1 & 0 & 1 & 3 \\
  0 & 1 & 2 & 1 \\
  1 & 1 & 1 & 2
\end{pmatrix}
\]

We will apply elementary row operations until we obtain a simplified matrix which is equivalent to the original one.
Solving systems of linear equations

\[ r_1 \leftrightarrow r_3, \quad r_2 \leftrightarrow r_3 \]

\[
\begin{pmatrix}
1 & 0 & 1 & 3 \\
0 & 1 & 2 & 1 \\
1 & 1 & 1 & 2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 1 & 2 \\
1 & 0 & 1 & 3 \\
0 & 1 & 2 & 1 \\
\end{pmatrix}
\]

\[ r_2 \rightarrow r_2 - r_1 \]

\[
\begin{pmatrix}
1 & 1 & 1 & 2 \\
1 & 0 & 1 & 3 \\
0 & 1 & 2 & 1 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 1 & 2 \\
0 & -1 & 0 & 1 \\
0 & 1 & 2 & 1 \\
\end{pmatrix}
\]

\[ r_2 \rightarrow -r_2, \quad r_3 \rightarrow r_3 - r_2; \text{ then } r_3 \rightarrow \frac{1}{2}r_3 \]

\[
\begin{pmatrix}
1 & 1 & 1 & 2 \\
0 & -1 & 0 & 1 \\
0 & 1 & 2 & 1 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 1 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 2 & 2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 1 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

The last matrix is a row-echelon form matrix.
Solving systems of linear equations

The row-echelon form matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

corresponds to the system

\[
\begin{align*}
x_1 + x_2 + x_3 &= 2 \\
x_2 &= -1 \\
x_3 &= 1
\end{align*}
\]

which is easily solved: \(x_1 = 2, x_2 = -1, x_3 = 1\).

Thus we solve the original linear combination problem as

\[
(3, 1, 2) = 2 (1, 0, 1) - (0, 1, 1) + (1, 2, 1)
\]
**Exercise:** Write $(3, 1, 2)$ as a linear combination of $(1, 0, 0)$, $(0, 1, 0)$, $(1, 2, 0)$. 
Exercise: Write $(3, 1, 2)$ as a linear combination of $(1, 2, -1),(1, 6, -3), (0, 1, 2),(1, 2, 1)$. 

To solve this problem, we need to solve:

\[ x_1 (1, 2, -1) + x_2 (1, 6, -3) + x_3 (0, 1, 2) + x_4 (1, 2, 1) = (3, 1, 2) \]

which gives a system of linear equations:

\[ \begin{align*}
    x_1 + x_2 + x_4 &= 3 \\
    2x_1 + 6x_2 + 2x_4 &= 1 \\
    -x_1 - 3x_2 + 2x_3 + x_4 &= 2
\end{align*} \]
**Exercise:** Write \((3, 1, 2)\) as a linear combination of \((1, 2, -1),(1, 6, -3), (0, 1, 2),(1, 2, 1)\).

To solve this problem, we need to solve

\[
x_1(1, 2, -1) + x_2(1, 6, -3) + x_3(0, 1, 2) + x_4(1, 2, 1) = (3, 1, 2)
\]

which is gives a system of linear equations:

\[
\begin{align*}
x_1 + x_2 + x_4 &= 3 \\
2x_1 + 6x_2 + x_3 + 2x_4 &= 1 \\
-x_1 - 3x_2 + 2x_3 + x_4 &= 2
\end{align*}
\]
Example

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 3 \\
2 & 6 & 1 & 2 & 1 \\
-1 & -3 & 2 & 1 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 & 1 & 3 \\
0 & 4 & 1 & 0 & -5 \\
0 & -2 & 2 & 2 & 5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 & 1 & 3 \\
0 & 1 & -1 & -1 & -5/2 \\
0 & 0 & 5 & 4 & 5
\end{pmatrix}
\]

Hence we have the solution:

\[
5x_3 = 5 - 4x_4, \quad x_2 = -5/2 + x_3 + x_4, \quad x_1 = 3 - x_2 - x_4
\]

Choosing any value of \( x_4 \in \mathbb{R} \), we find a solution of the linear combination problem.
Definition
Let $V$ be a vector space and $S$ a nonempty subset of $V$. We call span($S$) the set of all vectors in $V$ that can be written as a linear combination of vectors in $S$.

**Exercise:** Let $S = \{(1, 0, 0), (0, 1, 0), (2, 1, 0)\}$. What is span($S$)?
Theorem

The span of any subset $S$ of a vector space $V$ is a subspace of $V$.

Proof
Theorem

The span of any subset $S$ of a vector space $V$ is a subspace of $V$.

Proof

Solution: need to show that span $S$ is closed under the operations of addition and scalar multiplication.
Exercise: Does $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ span $\mathbb{R}^3$?
Exercise: Does $S = \{(1, 2), (2, 1)\}$ span $\mathbb{R}^2$?
Exercise: Does $S = \{(1, 2), (2, 1)\}$ span $\mathbb{R}^2$?

To solve this problem, we need to verify that, for any $(a, b) \in \mathbb{R}^2$ we can solve

$$x_1(1, 2) + x_2(2, 1) = (a, b)$$

This gives the system of linear equations:

$$x_1 + 2x_2 = a$$
$$2x_1 + x_2 = b$$
Exercise: Does $S = \{(1, 2), (2, 1)\}$ span $\mathbb{R}^2$?

To solve this problem, we need to verify that, for any $(a, b) \in \mathbb{R}^2$ we can solve

$$x_1(1, 2) + x_2(2, 1) = (a, b)$$

This gives the system of linear equations:

$$x_1 + 2x_2 = a$$
$$2x_1 + x_2 = b$$

This is equivalent to the row-reduced system

$$x_1 + 2x_2 = a$$
$$-3x_2 = b - 2a$$

showing that the system has always a solution.
**Exercise:** Does $S = \{(1, 2)\}$ span $\mathbb{R}^2$?

Using the argument above, we can see that not every element in $\mathbb{R}^2$ can be written as a linear combination of $S$. 
**Exercise:** Which \((a, b, c)\) are in \(\text{span}(\{(1, 1, 2), (0, 1, 1), (2, 1, 3)\})\)?
Examples

Exercise: Which \((a, b, c)\) are in \(\text{span}\left(\{(1, 1, 2), (0, 1, 1), (2, 1, 3)\}\right)\)?

To solve this problem, we can examine the linear system

\[ x_1(1, 1, 2) + x_2(0, 1, 1) + x_3(2, 1, 3) = (a, b, c) \]

which is associated with the augmented matrix

\[
\begin{pmatrix}
1 & 0 & 2 & | & a \\
1 & 1 & 1 & | & b \\
2 & 1 & 3 & | & c \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 2 & | & a \\
0 & 1 & -1 & | & b - a \\
0 & 1 & 1 & | & c - 2a \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 2 & | & a \\
0 & 1 & -1 & | & b - a \\
0 & 0 & 2 & | & c - b - a \\
\end{pmatrix}
\]

Since the linear system can be solved for any \((a, b, c) \in \mathbb{R}^3\), then \(\text{span}\left(\{(1, 1, 2), (0, 1, 1), (2, 1, 3)\}\right) = \mathbb{R}^3\)
Linear Dependence and Linear Independence

Section 1.5
Goal: given a vector space $V$, we want to find the SMALLEST set $S \subset V$ such that $\text{span}(S) = V$. 
Linear dependence

Definition

A subset $S$ of a vector space $V$ is called \textit{linearly dependent} if there exist a finite number of vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n$, \text{NOT ALL EQUAL TO ZERO}, such that

$$a_1 u_1 + \ldots + a_n u_n = 0.$$

If the vectors in $S$ are not linearly dependent, we say that they are \textit{linearly independent}.

Remark: Linear dependence is equivalent to say that at least one vector in $S$ can be written as a linear combinations of the others. Linear independence on the other hand implies that no vector in the set can be expressed as a linear combination of the others.
Example

Let \( S = \{(1, 1, 1), (2, 2, 2)\} \).

\( S \) linearly dependent since

\[
2(1, 1, 1) = (2, 2, 2)
\]
equivalently

\[
2(1, 1, 1) - (2, 2, 2) = 0
\]

Let \( R = \{(2, 0, 0), (0, 1, 0)\} \).

\( R \) linearly independent since

\[
a_1(2, 0, 0) + a_2(0, 1, 0) = (2a_1, a_2, 0) = 0
\]

implies that \( a_1 = a_2 = 0 \), showing that \( R \) is not linearly dependent.
Remark

If \( S = \{\mathbf{u}_1, \mathbf{u}_2\} \subset V \), then \( S \) is linearly dependent if and only if there exists a constant \( \alpha \neq 0 \) such that \( \mathbf{u}_1 = \alpha \mathbf{u}_2 \).

If \( S \) consists of more than two vectors, verifying linear dependence or independence requires more work.
Example

Let $S = \{(1, 1, 1), (-2, 0, -3), (3, 1, 4)\}$.

$S$ linearly dependent since

$$(3, 1, 4) = (1, 1, 1) - (-2, 0, -3)$$

Let $R = \{(2, 0, 0), (0, 1, 0), (0, 0, 4)\}$.

$R$ linearly independent since

$$a_1(2, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 4) = (2a_1, a_2, 4a_3) = 0$$

implies that $a_1 = a_2 = a_3 = 0$, showing that $R$ is not linearly dependent.
Linear independence

Remark

Let $S$ be a subset of a vector space $V$ and let $u_1, \ldots, u_n \in S$. These vectors are *linearly independent* if and only if

$$a_1 u_1 + \ldots + a_n u_n = 0 \Rightarrow a_1, \ldots, a_n = 0.$$
Theorem

Let $V$ be a vector space. If $S_1 \subseteq S_2$ and $S_1$ is linearly dependent, then $S_2$ is linearly dependent.

Proof
It follows form the definition.
**Theorem**

Let $S$ be a linearly independent subset of $V$. Let $v \in V \setminus S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.
Another theorem

**Theorem**

Let $S$ be a linearly independent subset of $V$. Let $v \in V \setminus S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Let $S = \{u_1, \ldots, u_m\}$

*Proof for $(\Leftarrow)$. If $v \in \text{span}(S)$, then $v$ is a linear combination of elements in $\{u_1, \ldots, u_m\}$, hence $\{v, u_1, \ldots, u_m\}$ is linearly dependent.

*Proof for $(\Rightarrow)$. If $S \cup \{v\}$ is linearly dependent, then there are constants $c_1, \ldots, c_m, c_{m+1}$ not all zero such that

$$c_1 u_1 + \cdots + c_m u_m + c_{m+1} v = 0$$

In this sum, it must be $c_{m+1} \neq 0$. If not, the rest of the sum would be 0 with $c_1, \ldots, c_m$ not all zero, violating the hypothesis that $S$ is linearly independent. Since $c_{m+1} \neq 0$, we can then write

$$v = -\frac{1}{c_{m+1}} (c_1 u_1 + \cdots + c_m u_m)$$

showing that $v \in \text{span}(S)$. 
A homogeneous system of equations like

\[
\begin{align*}
    x_1 + 2x_2 - 3x_3 &= 0 \\
    3x_1 + 5x_2 + 9x_3 &= 0 \\
    5x_1 + 9x_2 + 3x_3 &= 0
\end{align*}
\]

can be written as a vector equation

\[
x_1(1, 3, 5) + x_2(2, 5, 9) + x_3(-3, 9, 3) = (0, 0, 0)
\]

**Fact.** The vectors \((1, 3, 5), (2, 5, 9), (-3, 9, 3)\) are linearly independent if and only if the trivial solution \(x_1 = x_2 = x_3 = 0\) is the only solution of the homogeneous system.
The last observation implies that we can check the linear dependence or independence of a set of vectors by examining the solution set of the associated homogeneous system.

We examine the augmented matrix of the system

\[
\begin{pmatrix}
1 & 2 & -3 & 0 \\
3 & 5 & 9 & 0 \\
5 & 9 & 3 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & -3 & 0 \\
0 & -1 & 18 & 0 \\
0 & -1 & 18 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & -3 & 0 \\
0 & -1 & 18 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Since the row-reduced system has a row of zeros, then the homogeneous system has non-trivial solutions and, thus, the vectors \((1, 3, 5), (2, 5, 9), (-3, 9, 3)\) are \textit{linearly dependent}. 
Fact

Each linear dependence relation among the columns of the matrix $A$ corresponds to a nontrivial solution to $Ax = 0$.

The columns of a matrix $A$ are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.
Facts about linearly dependent/independent sets

- If a set $S$ in a vector space $V$ contains the 0 vector, then it is linearly dependent (since the linear dependence condition is always satisfied).
- The set of a single element $\{v\}$ is linearly independent if and only if $v \neq 0$ (it follows from the last property).
- If a set $S$ in the vector space $\mathbb{R}^n$ consists of $m > n$ vectors, then $S$ is linearly dependent. It follows from the observation that an homogeneous linear system $Ax = 0$ there the matrix $A$ has more columns than rows has always nontrivial solutions.
Bases and Dimension

Section 1.6
Basis

**Definition**

Let $V$ be a vector space. A (vector) *basis* $B$ of $V$ is a linearly independent subset of $V$ which satisfies $\text{span}(B) = V$. 
Example

Let \( S = \{(1, 0), (1, 1), (2, 3)\} \). Is \( S \) a basis for \( \mathbb{R}^2 \)?

Solution. No, since \( S \) contains 3 vectors in \( \mathbb{R}^2 \), then the set is linearly dependent.
Example

Let $S = \{(1, 0), (1, 1), (2, 3)\}$. Is $S$ a basis for $\mathbb{R}^2$?

Solution. No, since $A$ contains 3 vectors in $\mathbb{R}^2$, then the set is linearly dependent.
Let $S = \{(1, 0), (0, 1), (0, 2)\}$. Is $S$ a basis for $\mathbb{R}^2$?

Solution.

No, since $S$ contains 3 vectors in $\mathbb{R}^2$, then the set is linearly dependent.
Let $S = \{(1, 0), (0, 1), (0, 2)\}$. Is $S$ a basis for $\mathbb{R}^2$?

Solution. *No, since $S$ contains 3 vectors in $\mathbb{R}^2$, then the set is linearly dependent.*
Example

Let $S = \{(1, 0)\}$. Is $S$ a basis for $\mathbb{R}^2$?
Example

Let $S = \{(1, 0)\}$. Is $S$ a basis for $\mathbb{R}^2$?

Solution. *No, because the set does not span $S$.*
Example

Let $S = \{(1, 0), (1, 1)\}$. Is $S$ a basis for $\mathbb{R}^2$?

Solution. Yes, because the set is linearly independent and does span $\mathbb{R}^2$.

This can be seen by observing the matrix

$$
\begin{bmatrix}
1 & 1 \\
0 & 1 
\end{bmatrix}
$$

The columns are linearly independent since the matrix is reduced in row-echelon form.

The vectors span $\mathbb{R}^2$ because the matrix $Ax = \begin{bmatrix} a \\ b \end{bmatrix}$ can be solved for any $a, b \in \mathbb{R}^2$. 

D. Labate (UH)
Example

Let \( S = \{(1, 0), (1, 1)\} \). Is \( S \) a basis for \( \mathbb{R}^2 \)?

Solution. Yes, because the set is linearly independent and does span \( S \). This can be seen by observing the matrix

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

The columns are linearly independent since the matrix is reduced in row-echelon form.

The vectors span \( \mathbb{R}^2 \) because the matrix \( Ax = \begin{pmatrix} a \\ b \end{pmatrix} \) can be solved for any \( \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \).
Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Is $S$ a basis for $\mathbb{R}^3$?
Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Is $S$ a basis for $\mathbb{R}^3$?

Solution. Yes, because the set is linearly independent and does span $S$.

This basis is called the **canonical basis** of $\mathbb{R}^3$.

Similarly we define the **canonical basis** of $\mathbb{R}^n$, for any $n$. 
Theorem for bases

**Theorem**

Let $V$ be a vector space. Let $B = \{u_1, \ldots, u_n\}$ be a subset of $V$. Then

$$B \text{ is a basis of } V \iff \forall v \in V : \exists! a_1, \ldots, a_n \in F, v = a_1 u_1 + \ldots + a_n u_n.$$  

*Proof for $\Rightarrow$* If $B$ is a basis, for every $v \in V$, there are $a_1, \ldots, a_n$ such that $v = a_1 u_1 + \ldots + a_n u_n$ since $B$ spans $V$. To prove uniqueness, suppose there is another expansion $v = b_1 u_1 + \ldots + b_n u_n$. Then

$$(a_1 - b_1)u_1 + \ldots + (a_n - b_n)u_n = 0.$$ 

By the l.i., it must be $(a_i - b_i) = 0$ for all coefficients. This shows that the expansion must be unique.

*Proof for $\Leftarrow$* If for every $v \in V$, there is a unique sequence $a_1, \ldots, a_n$ such that $v = a_1 u_1 + \ldots + a_n u_n$, then $B$ spans $V$. To show that $B$ is l.i., consider the expansion of the 0 vector, that can be expressed by taking $a_1 = \ldots = a_n = 0$. By the uniqueness, this is the only expansion of the 0 vector. This also implies that $B$ is l.i.
Let $V$ be a vector space. Let $S$ be a finite subset of $V$ with $\text{span}(S) = V$. Then there exists a subset of $S$ which is a basis for $V$. In particular, $V$ has a finite basis.

Proof
Exercise

Let $S = \{(1, 0), (1, 1), (2, 3)\}$. We have $\mathbb{R}^2 = \text{span}(S)$ but $S$ is not a basis. Find a subset of $S$ which is a basis for $\mathbb{R}^2$. 
Let $S = \{(1, 0), (0, 1), (0, 2)\}$. We have $\mathbb{R}^2 = \text{span}(S)$ but $S$ is not a basis. Find a subset of $S$ which is a basis for $\mathbb{R}^2$. 
Let \( S = \{(-1, -1, -1), (5, 5, 5), (0, 2, 2), (0, 0, 3), (0, 2, 5)\} \). Is \( S \) a basis for \( \mathbb{R}^3 \)? If not, can you find a subset of \( S \) which is a basis for \( \mathbb{R}^3 \)?
Replacement Theorem

**Question:** given a vector space $V$, which is the SMALLEST set $S \subset V$ such that span$(S) = V$?

**Theorem (Replacement Theorem)**

Let $V$ be a vector space. Let $V = \text{span}(G)$, where $G$ is a subset of $V$ of cardinality $n$. Let $L$ be a linearly independent subset of $V$ of cardinality $m$. Then the following holds.

1. $m \leq n$
2. there exists a subset $H \subseteq G$ of cardinality $n - m$ such that span$(L \cup H) = V$
In other words

Let’s consider two subsets of vector space \( V \):

- \( G = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\} \) (cardinality \( n = 5 \)), such that we have \( V = \text{span}(G) \),

- \( L = \{\mathbf{v}_1, \mathbf{v}_2\} \) (cardinality \( m = 2 \)) linearly independent.

Replacement theorem tells you that there are 2 vectors in \( G \) that can be replaced with the two vectors in \( L \) and the new set obtained by this replacement still spans \( V \).
Corollary 1

Let $V$ be a vector space with a finite basis $B = \{u_1, \ldots, u_n\}$. Then any set containing more than $n$ vectors is linearly dependent.
Corollary 1

Let $V$ be a vector space with a finite basis $B = \{u_1, \ldots, u_n\}$. Then any set containing more than $n$ vectors is linearly dependent.

Proof Suppose $S$ is a set with $p > n$ vectors. By the Replacement Theorem, $S$ cannot be a linearly independent subset of $V$. 
Corollary 2

Let $V$ be a vector space with a finite basis. Then all bases contain the same number of elements.
Corollary 2

Let \( V \) be a vector space with a finite basis. Then all bases contain the same number of elements.

\textit{Proof}. Suppose that \( B_1 \) and \( B_1 \) are two bases of \( V \).

By the definition of basis, both sets are l.i.

By Corollary 1, \( B_1 \) cannot contain more elements of \( B_2 \), otherwise it would be linearly dependent.

Similarly, by Corollary 1, \( B_2 \) cannot contain more elements of \( B_1 \), otherwise it would be linearly dependent.

Thus, \( B_1 \) and \( B_1 \) have the same number of elements.
Dimension of $V$

**Definition**
A vector space is called *finite dimensional* if there exists a basis consisting of finitely many vectors.

**Definition**
The unique cardinality of a basis of a finite dimensional vector space is called the *dimension* of $V$, denoted $\text{dim}(V)$. 
Examples

1. \( \dim(\mathbb{R}^n) = \)

2. \( \dim(M_{n \times m}) = \)
Let $P_n$ be the vector space of the polynomials of degree $n$. $\dim(P_n) =$
Corollary 2

Let $S \subset V$. If $V = \text{span}(S)$ and $\#S = \dim(V)$, then $S$ is a basis.

Proof
Corollary 3

Let $S \subset V$. If $S$ is linearly independent and $\#S = \dim(V)$, then $S$ is a basis.

Proof
Theorem

Let $V$ be a vector space. Let $W$ be a subspace of $V$. Assume $\dim V$ is finite. Then $\dim W \leq \dim V$ and equality holds if and only if $V = W$.

Proof

Immediate from Replacement Theorem.
Example

Let $W = \{(a_1, a_2, a_3) \mid a_1 + a_3 = 0 \text{ and } a_1 + a_2 - a_3 = 0\} \subset \mathbb{R}^3$. Find a basis for and the dimension of subspace $W$. 
Example

Let \( W = \{(a_1, a_2, a_3, a_4, a_5) \mid a_1 + a_3 + a_5 = 0 \text{ and } a_2 = a_4\} \subset \mathbb{R}^5 \). Find a basis for and the dimension of subspace \( W \).