5.1 Eigenvalues and Eigenvectors

- Diagonalization
- Eigenvalues and Eigenvectors
- Characteristic Polynomial
- Properties
A linear operator $T$ on a finite-dimensional vector space $V$ is diagonalizable if there is an ordered basis $\beta$ for $V$ such that $[T]_\beta$ is a diagonal matrix. A square matrix $A$ is diagonalizable if $L_A$ is diagonalizable.
5.1 Eigenvalues and Eigenvectors

**Definition**

Let $T$ be a linear operator on a vector space $V$. A nonzero vector $v \in V$ is an eigenvector of $T$ if there exists a scalar eigenvalue $\lambda$ corresponding to the eigenvector $v$ such that $T(v) = \lambda v$.

Let $A \in M_{n \times n}(F)$. A nonzero vector $v \in F^n$ is an eigenvector of $A$ if $v$ is an eigenvector of $L_A$; that is, if $Av = \lambda v$ for some scalar eigenvalue $\lambda$ of $A$ corresponding to the eigenvector $v$. 
5.1

Eigenvalues and Eigenvectors: Example

Example

Let

\[
A = \begin{bmatrix}
0 & -2 \\
-4 & 2 \\
\end{bmatrix},
\]

\[
u = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix}, \text{ and } v = \begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}.
\]

Examine the images of \(u\) and \(v\) under multiplication by \(A\).

Solution

\[
Au = \begin{bmatrix}
0 & -2 \\
-4 & 2 \\
\end{bmatrix} \begin{bmatrix}
1 \\
1 \\
\end{bmatrix} = \begin{bmatrix}
-2 \\
-2 \\
\end{bmatrix} = -2u
\]

\(u\) is called an eigenvector of \(A\) since \(Au\) is a multiple of \(u\).

\[
Av = \begin{bmatrix}
0 & -2 \\
-4 & 2 \\
\end{bmatrix} \begin{bmatrix}
-1 \\
1 \\
\end{bmatrix} = \begin{bmatrix}
-2 \\
6 \\
\end{bmatrix} \neq \lambda v
\]

\(v\) is not an eigenvector of \(A\) since \(Av\) is not a multiple of \(v\).
Example

Show that 4 is an eigenvalue of \( A = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \) and find the corresponding eigenvectors.

\textit{Solution:} Scalar 4 is an eigenvalue of \( A \) if and only if \( Ax = 4x \) has a nontrivial solution.

\[
Ax - 4x = 0
\]

\[
Ax - 4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0
\]

\[
(A - 4I)x = 0.
\]

To solve \( (A - 4I)x = 0 \), we need to find \( A - 4I \) first:

\[
A - 4I = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -4 & -2 \end{bmatrix}
\]
Eigenvalues and Eigenvectors: Example

Now solve \((A-4I)x = 0\):

\[
\begin{bmatrix}
-4 & -2 & 0 \\
-4 & -2 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & \frac{1}{2} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[\Rightarrow \quad x = \begin{bmatrix}
-\frac{1}{2}x_2 \\
x_2 \\
\end{bmatrix} = x_2 \begin{bmatrix}
-\frac{1}{2} \\
1 \\
\end{bmatrix}.
\]

Each vector of the form \(x_2 \begin{bmatrix}
-\frac{1}{2} \\
1 \\
\end{bmatrix}\) is an eigenvector corresponding to the eigenvalue \(\lambda = 4\).

Eigenspace for \(\lambda = 4\)

The set of all solutions to \((A-\lambda I)x = 0\) is called the eigenspace of \(A\) corresponding to \(\lambda\).
**Theorem (5.1)**

A linear operator $T$ on a finite-dimensional vector space $V$ is diagonalizable if and only if there exists an ordered basis $\beta$ for $V$ consisting of eigenvectors of $T$. If $T$ is diagonalizable, $\beta = \{v_1, \cdots, v_n\}$ is an ordered basis of eigenvectors of $T$, and $D = [T]_\beta$, then $D$ is a diagonal matrix and $D_{jj}$ is the eigenvalue corresponding to $v_j$ for $1 \leq j \leq n$. 
To diagonalize a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.
Theorem (5.2)

Let $A \in M_{n \times n}(F)$. Then a scalar $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda I_n) = 0$. 
Characteristic Polynomial

**Definition**

Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tl_n)$ is called the characteristic polynomial of $A$. 
Definition

Let \( T \) be a linear operator on an \( n \)-dimensional vector space \( V \) with ordered basis \( \beta \). We define the characteristic polynomial \( f(t) \) of \( T \) to be the characteristic polynomial of \( A = [T]_\beta \):

\[
f(t) = \det(A - tl_n).
\]
Properties

Theorem (5.3)

Let $A \in M_{n \times n}(F)$.

(a) The characteristic polynomial of $A$ is a polynomial of degree $n$ with leading coefficient $(-1)^n$.

(b) $A$ has at most $n$ distinct eigenvalues.
Theorem (5.4)

Let $T$ be a linear operator on a vector space $V$, and let $\lambda$ be an eigenvalue of $T$. A vector $v \in V$ is an eigenvector of $T$ corresponding to $\lambda$ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$. 
5.2 Diagonalizability

- Diagonalizability
- Multiplicity
- Direct Sums
Diagonalizability

Theorem (5.5)

Let $T$ be a linear operator on a vector space $V$, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of $T$. If $v_1, \cdots, v_k$ are the corresponding eigenvectors, then $\{v_1, \cdots, v_k\}$ is linearly independent.

Corollary

Let $T$ be a linear operator on an $n$-dimensional vector space $V$. If $T$ has $n$ distinct eigenvalues, then $T$ is diagonalizable.
**Definition**

A polynomial $f(t)$ in $P(F)$ splits over $F$ if there are scalars $c$, $a_1$, $a_2$, \ldots, $a_n$ in $F$ such that 

\[ f(t) = c(t - a_1)(t - a_2) \cdots (t - a_n). \]

**Theorem (5.6)**

The characteristic polynomial of any diagonalizable operator splits.
5.2 

Multiplicity

Definition
Let $\lambda$ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The (algebraic) multiplicity of $\lambda$ is the largest positive integer $k$ for which $(t - \lambda)^k$ is a factor of $f(t)$.

Definition
Let $T$ be a linear operator on a vector space $V$, and let $\lambda$ be an eigenvalue of $T$. Define $E_\lambda = \{x \in V : T(x) = \lambda x\} = N(T - I_V)$. The set $E_\lambda$ is the eigenspace of $T$ corresponding to the eigenvalue $\lambda$. The eigenspace of a square matrix $A$ is the eigenspace of $L_A$. 
Theorem (5.7)

Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $\lambda$ be an eigenvalue of $T$ having multiplicity $m$. Then $1 \leq \dim(E_\lambda) \leq m$. 
Lemma
Let $T$ be a linear operator, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of $T$. For $i = 1, \cdots, k$, let $v_i \in E_{\lambda_i}$. If

$$v_1 + v_2 + \cdots + v_k = 0,$$

then $v_i = 0$ for all $i$.

Theorem (5.8)
Let $T$ be a linear operator on a vector space $V$, and let $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of $T$. For $i = 1, \cdots, k$, let $S_i$ be a finite linearly independent subset of the eigenspace $E_{\lambda_i}$. Then $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of $V$. 
Theorem (5.9)

Let $T$ be a linear operator on a finite-dimensional vector space $V$ such that the characteristic polynomial of $T$ splits. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $T$. Then

(a) $T$ is diagonalizable if and only if the multiplicity of $\lambda_i$ is equal to $\dim(E_{\lambda_i})$ for all $i$.

(b) If $T$ is diagonalizable and $\beta_i$ is an ordered basis for $E_{\lambda_i}$, for each $i$, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for $V$ consisting of eigenvectors of $T$. 
Test for Diagonalization

Let $T$ be a linear operator on an $n$-dimensional vector space $V$. Then $T$ is diagonalizable if and only if both of the following conditions hold.

- The characteristic polynomial of $T$ splits.
- The multiplicity of each eigenvalue $\lambda$ equals $n - \text{rank}(T - \lambda I)$. 
Direct Sums

Definition

The sum of the subspaces \( W_1, \ldots, W_k \) of a vector space is the set

\[
\sum_{i=1}^{k} W_i = \{ v_1 + \cdots + v_k : v_i \in W_i \text{ for } 1 \leq i \leq k \}.
\]

Definition

A vector space \( V \) is the direct sum of subspaces \( W_1, \ldots, W_k \), denoted \( V = W_1 \oplus \cdots \oplus W_k \), if

\[
V = \sum_{i=1}^{k} W_i \quad \text{and} \quad W_j \cap \sum_{i \neq j} W_i = \{0\} \text{ for each } j, 1 \leq j \leq k.
\]
Theorem (5.10)

Let $W_1, \cdots, W_k$ be subspaces of finite-dimensional vector space $V$. The following are equivalent:

(a) $V = W_1 \oplus \cdots \oplus W_k$.

(b) $V = \sum_{i=1}^{k} W_i$ and for any $v_1, \cdots, v_k$ s.t. $v_i \in W_i$ $(1 \leq i \leq k)$, if $v_1 + \cdots + v_k = 0$, then $v_i = 0$ for all $i$.

(c) Each $v \in V$ can be uniquely written as $v = v_1 + \cdots + v_k$, where $v_i \in W_i$.

(d) If $\gamma_i$ is an ordered basis for $W_i$ $(1 \leq i \leq k)$, then $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for $V$.

(e) For each $i = 1, \cdots, k$ there exists an ordered basis $\gamma_i$ for $W_i$ such that $\gamma_1 \cup \cdots \cup \gamma_k$ is an ordered basis for $V$. 
Direct Sums (cont.)

Theorem (5.11)
A linear operator $T$ on finite-dimensional vector space $V$ is diagonalizable if and only if $V$ is the direct sum of the eigenspaces of $T$. 
5.3 Matrix Limits and Markov Chains

- Matrix Limits
- Existence of Limits
5.3 Matrix Limits

Definition

Let $L, A_1, A_2, \cdots$ be $n \times p$ matrices with complex entries. The sequence $A_1, A_2, \cdots$ is said to converge to the limit $L$ if
\[
\lim_{m \to \infty} (A_m)_{ij} = L_{ij}
\]
for all $1 \leq i \leq n$ and $1 \leq j \leq p$. If $L$ is the limit of the sequence, we write $\lim_{m \to \infty} A_m = L$.

Theorem (5.12)

Let $A_1, A_2, \cdots$ be a sequence of $n \times p$ matrices with complex entries that converges to $L$. Then for any $P \in M_{r \times n}(C)$ and $Q \in M_{p \times s}(C)$,
\[
\lim_{m \to \infty} PA_m = PL \quad \text{and} \quad \lim_{m \to \infty} A_m Q = LQ.
\]
Corollary

Let \( A \in M_{n \times n}(C) \) be such that \( \lim_{m \to \infty} A^m = L \). Then for any invertible \( Q \in M_{n \times n}(C) \),

\[
\lim_{m \to \infty} (QAQ^{-1})^m = QLQ^{-1}.
\]
Consider the set consisting of the complex number 1 and the interior of the unit disk: \( S = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \text{ or } \lambda = 1 \} \).

**Theorem (5.13)**

*Let \( A \) be a square matrix with complex entries. Then \( \lim_{m \to \infty} A^m \) exists if and only if both of the following hold:*

(a) *Every eigenvalue of \( A \) is contained in \( S \).*

(b) *If 1 is an eigenvalue of \( A \), then the dimension of the eigenspace corresponding to 1 equals the multiplicity of 1 as an eigenvalue of \( A \).*
Existence of Limits (cont.)

Theorem (5.14)

Let $A \in M_{n \times n}(C)$. $\lim_{m \to \infty} A^m$ exists if

(a) Every eigenvalue of $A$ is contained in $S$.

(b) $A$ is diagonalizable.