(1)[4Pts] Find the general solution and solve the following IVP

\[6y' - 2y = xy^4, \quad y(0) = -2\]

First get the differential equation in the proper form and then write down the substitution.

\[6y^{-4}y' - 2y^{-3} = x \quad \Rightarrow \quad v = y^{-3}, \quad v' = -3y^{-4}y'\]

Plugging the substitution into the differential equation gives,

\[-2v' - 2v = x \quad \Rightarrow \quad v' + v = -\frac{1}{2}x, \quad e^{h(x)} = e^{\int 1dx} = e^x\]

Again, we’ve rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

\[v(x) = -\frac{1}{2}(x - 1) + ce^{-x}\]

Now back substitute to get back into \(y\)’s.

\[y^{-3} = -\frac{1}{2}(x - 1) + ce^{-x} \Rightarrow y(x) = -\left(\frac{2}{x - 1 + 2ce^{-x}}\right)^{\frac{1}{3}}\]

Now we need to apply the initial condition and solve for \(c\).

\[-\frac{1}{8} = \frac{1}{2} + c \quad \Rightarrow \quad c = -\frac{5}{8}\]

Plugging in \(c\) and solving for \(y\) gives,

\[y(x) = -\frac{2}{(4x - 4 + 5e^{-x})^{\frac{1}{3}}}\]

(2)[4Pts] Find the general solution and solve the following IVP

\[y' + \frac{1}{x}y - \sqrt{y} = 0, \quad y(1) = 0\]

Let’s first get the differential equation into proper form.

\[y' + \frac{1}{x}y = y^{\frac{1}{2}} \quad \Rightarrow \quad y^{-\frac{1}{2}}y'^2 + \frac{1}{x}y^{\frac{1}{2}} = 1\]

The substitution is then,

\[v = y^{\frac{1}{2}} \quad v' = \frac{1}{2}y^{-\frac{1}{2}}y'\]
Now plug the substitution into the differential equation to get,
\[ 2v' + \frac{1}{x}v = 1 \Rightarrow v' + \frac{1}{2x}v = \frac{1}{2}, \quad e^{h(x)} = e^{\int \frac{1}{2x} dx} = x^{\frac{1}{2}} \]

As we’ve done with the previous examples we’ve done some rearranging and given the integrating factor needed for solving the linear differential equation. Solving this gives us,
\[ v(x) = \frac{1}{3}x + cx^{-\frac{1}{2}} \]

In terms of \( y \) this is,
\[ y^{\frac{1}{2}} = \frac{1}{3}x + cx^{-\frac{1}{2}}, \Rightarrow y(x) = \left( \frac{1}{3}x + cx^{-\frac{1}{2}} \right)^2 \]

Applying the initial condition and solving for \( c \) gives,
\[ 0 = \frac{1}{3} + c \Rightarrow c = -\frac{1}{3} \]

Plugging in for \( c \) and solving for \( y \) gives us the solution.
\[ y(x) = \left( \frac{1}{3}x - \frac{1}{3}x^{-\frac{1}{2}} \right)^2 = \frac{x^3 - 2x^{\frac{3}{2}} + 1}{9x} \]

(3)[4Pts] Find the general solution of the following problem
\[ xyy' + 4x^2 + y^2 = 0 \]

Let’s first divide both sides by \( x^2 \) to rewrite the differential equation as follows,
\[ \frac{y}{x}y' = -4 - \frac{y^2}{x^2} = -4 - \left( \frac{y}{x} \right)^2 \]

Now, this is not in the officially proper form as we have listed above, but we can see that everywhere the variables are listed they show up as the ratio, \( y/x \) and so this is really as far as we need to go. So, let’s plug the substitution into this form of the differential equation to get,
\[ v(v + xv') = -4 - v^2 \]

Next, rewrite the differential equation to get everything separated out.
\[ vxv' = -4 - 2v^2 \]
\[ xv' = -\frac{4 + 2v^2}{v} \]
\[ \frac{v}{4 + 2v^2} dv = -\frac{1}{x} dx \]

Integrating both sides gives,
\[ \frac{1}{4} \ln (4 + 2v^2) = -\ln(x) + c \]

We need to do a little rewriting using basic logarithm properties in order to be able to easily solve this for \( v \).
\[ \ln \left( 4 + 2v^2 \right)^{\frac{1}{4}} = \ln(x)^{-1} + c \]
Now exponentiate both sides and do a little rewriting

\[(4 + 2v^2)^{\frac{1}{4}} = e^{\ln(x)^{-1} + c} = e^c e^{\ln(x)^{-1}} = \frac{c}{x}\]

Note that because \(c\) is an unknown constant then so is \(e^c\) and so we may as well just call this \(c\) as we did above. Finally, let’s solve for \(v\) and then plug the substitution back in and we’ll play a little fast and loose with constants again.

\[4 + 2v^2 = \frac{c^4}{x^4} = \frac{c}{x^4}\]

\[v^2 = \frac{1}{2} \left( \frac{c}{x^4} - 4 \right)\]

\[y^2 = \frac{1}{2} \left( \frac{c - 4x^4}{x^4} \right)\]

\[y^2 = \frac{1}{2} x^2 \left( \frac{c - 4x^4}{x^4} \right) = \frac{c - 4x^4}{2x^2}\]

(4)[4Pts] Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of tea obeys Newton’s law of cooling. If the coffee has a temperature of 200F when freshly poured, and 1 min later has cooled to 190F in a room at 70F. Determine when the coffee reaches a temperature of 150F.

Let \(T\) be the temperature of the object, and \(T_s\) the temperature of the surroundings. We can write Newton’s law of cooling in equation form as follows:

\[\frac{dT}{dt} = -k(T - T_s)\]

Here \(k\) is a still-unknown constant, greater than zero. Let’s double-check the sign: if the object has a higher temperature than its surroundings, then \(T > T_s\), so \(dT/dt\) is negative, so the object is cooling, which is what we expect. We solve the equation by separating the variables.

\[\int \frac{1}{T - T_s} dT = \int -k dt\]

\[\ln |T - T_s| = -kt + C\]

\[|T - T_s| = Ae^{-kt}\]

\[T - T_s = Ae^{-kt} \quad \text{ (replacing } A \text{ with } \pm A)\]

\[T = Ae^{-kt} + T_s\]

In the given problem (with units degrees Fahrenheit), we have \(T_s = 70\) and \(T(0) = 200\), so we get \(A = 130\). Since \(T(1) = 190\),

\[190 = 130e^{-k} + 70\]

\[120 = 130e^{-k}\]

\[120/130 = e^{-k}\]

\[k = \ln(120/130) \approx 0.08\]
Finally, we solve for $t$ in $T(t) = 150$.

\[
150 = 130e^{-0.08t} + 70 \\
80 = 130e^{-0.08t} \\
\frac{80}{130} = e^{-0.08t} \\
\ln(\frac{80}{130}) = -0.08t \\
t = 6.066 \text{ min}.
\]

(5)[4Pts] The size of a certain bacterial colony increases at a rate proportional to the size of the colony. Suppose the colony occupied an area of 0.25 square centimeters initially, and after 8 hours it occupied an area of 0.35 square centimeters.

(a) Estimate the size of the colony $t$ hours after the initial measurement.

(b) What is the expected size of the colony after 12 hours?

(c) Find the doubling time of the colony

(a). Let $P(t)$ denote the size of the colony $t$ hours after the initial measurement. Since $P(0) = 0.25$ and $P(8) = 0.35$, we have

\[
P(t) = P(0)e^{kt} = 0.25e^{kt} \\
P(8) = 0.25e^{8k} = 0.35
\]

Thus

\[
e^{8t} = \frac{7}{5} \implies k = \frac{\ln(7/5)}{8} \approx 0.0421
\]

and

\[
P(t) = 0.25e^{0.0421t} \text{ or } 0.25 \times \left(\frac{7}{5}\right)^{t/8}
\]

(b).

\[
P(12) = 0.25e^{0.0421\times12} \approx 0.414 \text{ cm}^2
\]

(c). The doubling time is

\[
T = \frac{\ln 2}{k} \approx \frac{\ln 2}{0.0421} \approx 16.464 \text{ hours}.
\]