Homework #4

You must justify all steps to get credit for your work.

Please submit the HW via CASA or email your completed assignment as a single PDF file to jshi24@CougarNet.UH.EDU.

(1)[5Pts] Consider the following linear second-order differential equation

\[ xy'' - (x + 1)y' + y = 0, \quad x > 0 \]

(a) Show that \( y_1 = e^x \) and \( y_2 = x + 1 \) are solutions of the differential equation above.

(b) Use the method of variation of parameters to find a particular solution of

\[ xy'' - (x + 1)y' + y = x^2 e^{2x}, \quad x > 0 \]

(c) Write the general solution of the non-homogeneous equation in part (b)

(a)

\[ y_1 = e^x \Rightarrow y_1' = e^x, \quad y_1'' = e^x \Rightarrow xy''_1 - (x + 1)y_1' + y_1 = e^x(x - (x + 1) + 1) = 0. \]

\[ y_2 = x + 1 \Rightarrow y_2' = 1, \quad y_2'' = 0 \Rightarrow xy''_2 - (x + 1)y_2' + y_2 = 0 - (x + 1) + (x + 1) = 0. \]

(b) Let’s first get the differential equation into proper form:

\[ y'' - \frac{x + 1}{x}y' + \frac{1}{x}y = xe^{2x}, \quad x > 0. \]

By (a), we compute the Wronskian

\[ W(x) = y_1y_2' - y_2y_1' = -xe^x. \]

By the method of variation of parameters, a particular solution is

\[ z(x) = uy_1 + vy_2, \]

where

\[ u' = \frac{-y_2f}{W}, \quad v' = \frac{y_1f}{W} \quad \text{and} \quad f = xe^{2x}. \]

We compute

\[ u(x) = \int \frac{-y_2f}{W} \, dx = \int \frac{-(x + 1)e^{2x}}{-xe^x} \, dx = xe^x, \]

\[ v(x) = \int \frac{y_1f}{W} \, dx = \int \frac{e^x xe^{2x}}{-xe^x} \, dx = -\frac{1}{2} e^{2x}. \]

Hence

\[ z(x) = uy_1 + vy_2 = \frac{1}{2} (x - 1)e^{2x}. \]

(c)

\[ y(x) = Ae^x + B(x + 1) + \frac{1}{2} (x - 1)e^{2x}. \]
(2)[4Pts] Find the general solution of the following differential equation

\[ y'' - 3y' + 2y = x^2 \]

We solve the homogeneous equation first. The characteristic equation is

\[ p(r) = r^2 - 3r + 2 = 0, \quad r = 2, r = 1. \]

Hence the solution to the homogeneous problem is

\[ y_h(x) = Ae^{2x} + Be^x. \]

Based on the form of the function on the right hand side, the particular solution should have the form

\[ z = A_0 + A_1x + A_2x^2. \]

We have

\[ z' = A_1 + 2A_2x, \quad \text{and} \quad z'' = 2A_2. \]

Substitution into the differential equation gives

\[ 2A_2 - 3(A_1 + 2A_2x) + 2(A_0 + A_1x + A_2x^2) = x^2. \]

Hence we have

\[
\begin{align*}
A_0 &= \frac{7}{4} \\
A_1 &= \frac{3}{2} \\
A_2 &= \frac{1}{2}
\end{align*}
\]

It follows that

\[ z = \frac{1}{4}(2x^2 + 6x + 7). \]

Thus the general solution is

\[ y = Ae^{2x} + Be^x + \frac{1}{4}(2x^2 + 6x + 7). \]

(3)[4Pts] Find the general solution of the following differential equation

\[ y'' - y' - 2y = e^{-x} + x^2 + \cos(x) \]

We solve the homogeneous equation first. The characteristic equation is

\[ p(r) = r^2 - r - 2 = 0, \quad r = -1, 2. \]

Hence the solution to the homogeneous problem is

\[ y_h = Ae^{-x} + Be^{2x}. \]

This implies that \( z = Ce^{-x} \) is the solution of the reduced equation. Thus, to find a particular solution of

\[ y'' - y' - 2y = e^{-x}, \]
we need to choose a solution of the form 
\[ z_1 = Cxe^{-x}. \]

Repeating the process of Problem # 2, we solve 
\[ C = -\frac{1}{3} \Rightarrow z_1 = -\frac{1}{3}xe^{-x}. \]

To find a particular solution of 
\[ y'' - y' - 2y = x^2, \]
we need to choose a solution of the form 
\[ z_2 = A_0 + A_1x + A_2x^2. \]

Repeating the process of Problem # 2, we solve 
\[ \begin{cases} A_0 = -\frac{3}{4} \\ A_1 = \frac{1}{2} \\ A_2 = -\frac{1}{2} \end{cases} \Rightarrow z_2 = \frac{1}{4}(-2x^2 + 2x - 3). \]

To find a particular solution of 
\[ y'' - y' - 2y = \cos(x), \]
we need to choose a solution of the form 
\[ z_3 = B_1 \cos(x) + B_2 \sin(x). \]

Repeating the process of Problem # 2, we solve 
\[ \begin{cases} B_1 = -\frac{3}{10} \\ B_2 = -\frac{1}{10} \end{cases} \Rightarrow z_2 = -\frac{3}{10} \cos(x) - \frac{1}{10} \sin(x). \]

Hence, combining these observations, we have that the particular solution has the form 
\[ z = z_1 + z_2 + z_3 = -\frac{1}{3}xe^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4} - \frac{3}{10} \cos(x) - \frac{1}{10} \sin(x). \]

Thus the general solution is 
\[ y = y_h + z = Ae^{-x} + Be^{2x} - \frac{1}{3}xe^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4} - \frac{3}{10} \cos(x) - \frac{1}{10} \sin(x). \]

(4)[4Pts] Find the general solution of the following differential equation 
\[ y'' - 4y = 3e^{2x} + 4e^{-x} \]

We solve the homogeneous equation first. The characteristic equation is 
\[ p(r) = r^2 - 4 = 0, \quad r = \pm 2. \]

Hence the solution to the homogeneous problem is 
\[ y_h = Ae^{-2x} + Be^{2x}. \]
This implies that \( z = Ce^{2x} \) is the solution of the reduced equation. Thus, to find a particular solution of

\[
y'' - 4y = 3e^{2x},
\]

we need to choose a solution of the form

\[ z_1 = Cxe^{2x}. \]

Repeating the process of Problem #2, we solve

\[ C = \frac{3}{4} \Rightarrow z_1 = \frac{3}{4}xe^{2x}. \]

To find a particular solution of

\[
y'' - 4y = 4e^{-x},
\]

we need to choose a solution of the form

\[ z_2 = De^{-x}. \]

Repeating the process of Problem #2, we solve

\[ D = -\frac{4}{3} \Rightarrow z_2 = -\frac{4}{3}e^{-x}. \]

Hence, combining these observations, we have that the particular solution has the form

\[ z = z_1 + z_2 = \frac{3}{4}xe^{2x} - \frac{4}{3}e^{-x}. \]

Thus the general solution is

\[ y = y_h + z = Ae^{-2x} + Be^{2x} + \frac{3}{4}xe^{2x} - \frac{4}{3}e^{-x}. \]

(5)[3Pts] Give the form of a particular solution for

\[
y'' - 4y' + 5y = 1 + x^2 + e^{2x}\cos x
\]

We solve the homogeneous equation first. The characteristic equation is

\[ p(r) = r^2 - 4r + 5 = 0, \quad r = 2 \pm i. \]

Hence the solution to the homogeneous problem is

\[ y_h = e^{2x}(A\cos(x) + B\sin(x)). \]

This implies that \( z = e^{2x}(A\cos(x) + B\sin(x)) \) is the solution of the reduced equation. Thus, to find a particular solution of

\[
y'' - 4y' + 5y = e^{2x}\cos x,
\]

we need to choose a solution of the form

\[ z_1 = xe^{2x}(A\cos(x) + B\sin(x)). \]

To find a particular solution of

\[
y'' - 4y' + 5y = 1 + x^2,
\]

we need to choose a solution of the form

\[ z_2 = C + Dx + Ex^2. \]

Hence, combining these observations, we have that the particular solution has the form

\[ z = z_1 + z_2 = xe^{2x}(A\cos(x) + B\sin(x)) + (C + Dx + Ex^2). \]