Test #2

You must show your work and report all steps of your solution to get credit for your work.

(1)[4Pts] Consider the following differential equation

\[ y'' - 4y' + 5y = 0 \]

(a) Find the general solution.
(b) Solve the IVP with \( y(0) = 0 \) and \( y'(0) = -1 \).

\( a) \) Characteristic polynomial:

\[ p(r) = r^2 - 4r + 5 \]

Roots: \( r_{1,2} = 2 \pm i \)

General solution:

\[ y(x) = e^{2x}(A \cos x + B \sin x) \]

(b)

\[ 0 = y(0) = A \implies A = 0 \]

\[ -1 = y'(0) = (2B - A)e^{2x} \sin x + (2A + B)e^{2x} \cos x \bigg|_{x=0} = 2A + B \quad \overset{A=0}{\implies} \quad B = -1 \]

Hence, the IVP solution is

\[ y(x) = -e^{2x} \sin x \]

(2)[4Pts] Find the general solution of the following differential equation

\[ y'' + 4y = \sin(2x) \]

We first find the solution of the homogeneous problem. The characteristic polynomial is

\[ r^2 + 4 = 0 \]

Roots are \( r = \pm 2i \), hence the solution of the homogeneous problem is

\[ y_h(x) = A \cos(2x) + B \sin(2x) \]

Since \( \sin(2x) \) is already a solution of the homogeneous problem, we look for a particular solution of the form

\[ y_p(x) = x(C_1 \cos(2x) + C_2 \sin(2x)) \]

The method of substitution yields: \( C_1 = -1/4, \ C_2 = 0 \). Hence \( y_p(x) = -\frac{1}{4}x \cos(2x) \) and, thus, the general solution is

\[ y(x) = A \cos(2x) + B \sin(2x) - \frac{1}{4}x \cos(2x) \]
Without computing the constants, give the form of a particular solution for
\[ y'' - 4y' + 5y = x - 7e^{2x} + e^{2x} \sin x \]

From the solution of problem 1, we have that the solution of the homogeneous problem is
\[ y(x) = e^{2x}(A \cos x + B \sin x) \]
The forcing term is the sum of 3 functions:
\[ y_1(x) = x, \quad y_2(x) = -7e^{2x}, \quad y_3(x) = e^{2x} \sin x \]
y_1 and y_2 are not contained in the solution of the homogeneous problem. Hence, the form of particular solutions corresponding to these terms are
\[ y_{p1}(x) = A + Bx, \quad y_{p2}(x) = Ce^{2x} \]
Since \( e^{2x} \sin x \) is already a solution of the homogeneous problem, here we look for a particular solution of the form
\[ y_{p3}(x) = xe^{2x}(D \cos x + E \sin x) \]
Hence, combining the 3 terms, a particular solution is of the form
\[ y_p = A + Bx + Ce^{2x} + xe^{2x}(D \cos x + E \sin x) \]

Find the general solution of the following non-homogeneous differential equation
\[ y''' - 8y = -16x \]
(Hint: \( r = 2 \) is a root of the characteristic polynomial.)

Characteristic polynomial
\[ p(r) = r^3 - 8 = (r - 2)(r^2 + 2r + 4) \]
whose roots are
\[ r_1 = 2, \quad r_{2,3} = -1 \pm i\sqrt{3} \]
Hence the homogeneous solution is
\[ y_h(x) = C_1e^{2x} + e^{-x}\left( C_2 \cos(\sqrt{3}x) + C_3 \sin(\sqrt{3}x) \right) \]
The non-homogeneous term \(-16x\) is not contained in the homogeneous solution so we look for a particular solution of the form \( y_p = A + Bx \). By substitution, we find \( B = 2 \), hence the general solution is
\[ y(x) = C_1e^{2x} + e^{-x}\left( C_2 \cos(\sqrt{3}x) + C_3 \sin(\sqrt{3}x) \right) + 2x \]

Consider the following differential equations modeling harmonic motion with forced vibration. For each equation, explain if (i) the equation models a damped or undamped system and if (ii) the solution is in resonance (this occurs when the amplitude of vibrations increases without bound) or not.
(a) \( y'' + 4y = \sin(x) \)

(b) \( y'' + 4y = \sin(2x) \)

(c) \( y'' + 2y' + 3y = \sin(2x) \)

(a-b) They model an **undamped** system, since there is no first derivative term modeling friction. Characteristic polynomial

\[
p(r) = r^2 + 4
\]

whose roots are \( r = \pm 2i \). Hence the homogeneous solution is

\[
y_h(x) = C_1 \cos(2x) + C_2 \sin(2x) = A \sin(2x + \phi)
\]

(a) The forcing term is NOT contained in the homogeneous solution, hence there is **no resonance**.

(b) The forcing term is contained in the homogeneous solution so there is **resonance**. In fact, a particular solution of the form \( y_p(x) = Bx \sin(2x + \psi) \)

(c) It models a **damped** system, since there is a first derivative term modeling friction. Characteristic polynomial:

\[
p(r) = r^2 + 2r + 3
\]

whose roots are \( r = -1 \pm i\sqrt{2} \). Hence the homogeneous solution is

\[
y_h(x) = e^{-x}(C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)) = Ae^{-x} \sin(\sqrt{2}x + \phi)
\]