HW 10

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) Let

\[ f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x} & \text{if } x \neq 0; \\
  0 & \text{if } x = 0. 
\end{cases} \]

(a) Use the chain rule and the product rule to show that \( f \) is differentiable at each \( x \neq 0 \) and find \( f'(x) \).

(b) Use the definition to show that \( f \) is differentiable at \( x = 0 \) and find \( f'(0) \).

(c) Show that \( f' \) is not continuous at \( x = 0 \).

(d) Let \( g(x) = x^2 \) if \( x \leq 0 \) and \( g(x) = x^2 \sin \frac{1}{x} \) if \( x > 0 \). Determine whether or not \( g \) is differentiable at \( x = 0 \). If it is, find \( g'(0) \).

(2) Let \( f(x) = x^2 \) if \( x \) is rational and \( f(x) = 0 \) if \( x \) is irrational.

(a) Prove that \( f \) is continuous at exactly one point, namely at \( x = 0 \).

(b) Prove that \( f \) is differentiable at exactly one point, namely at \( x = 0 \).

(3) Use the mean value theorem to establish the following inequalities

(a) \( e^x > 1 + x \), for \( x > 0 \).

(b) \( \frac{x-1}{x} < \ln x < x - 1 \), for \( x > 1 \).

(f) \( \sin x \leq x \), for \( x \geq 0 \).

(4) A differentiable function \( f \) is said to be increasing on an interval \( I \) if \( x_1 < x_2 \) in \( I \) implies that \( f(x_1) \leq f(x_2) \). Prove that \( f \) is increasing on \( I \) iff \( f'(x) \geq 0 \) for all \( x \in I \).