HW 1

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [3Pts] Prove that
\[ \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1) \quad n \in \mathbb{N}. \]

**Solution:** For \( n = 1 \) it is easily verified.
Now assume it works for \( n = k \).
For \( n = k + 1 \), we have
\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2 \\
= \frac{1}{6} k(k + 1)(2k + 1) + (k + 1)^2 \\
= \frac{1}{6} (k + 1)(k(2k + 1) + 6(k + 1)) \\
= \frac{1}{6} (k + 1)(2k^2 + 7k + 6) \\
= \frac{1}{6} (k + 1)(k + 2)(2k + 3) \\
= \frac{1}{6} (k + 1) ((k + 1) + 1) (2(k + 1) + 1) \quad \text{QED.}
\]

(2) [3Pts] Prove that, for any \( n \in \mathbb{N} \), the number \( 9^n - 4^n \) is divisible by 5.

**Solution:** For \( n = 1 \), \( 9 - 4 = 5 \), hence it is true.
Now assume it is true for \( n = k \); that is, there is an \( m \in \mathbb{N} \) such that \( 9^k - 4^k = 5m \).
So, for \( n = k + 1 \) we get
\[
9^{k+1} - 4^{k+1} = 9 \cdot 9^k - 4 \cdot 4^k \\
= 9(9^k - 4^k) + 9 \cdot 4^k - 4 \cdot 4^k \\
= 9 \cdot 5 \cdot m + 5 \cdot 4^k \\
= 5(9 \cdot m + 4^k) \quad \text{QED.}
\]
(3) [3Pts] Prove that, for any \( n \geq 4 \) the following inequality holds
\[ n^2 \leq 2^n. \]

**Solution:** It satisfies for \( n = 4 \) since \( 4^2 = 2^4 = 16 \).
Now assume it is true for \( n = k > 4 \). That is, \( k^2 \leq 2^k \).
Then for \( n = k + 1 \) we have
\[
(k + 1)^2 = k^2 + 2k + 1 \leq 2^k + 2k + 1.
\]
We observe \(^1\) that, for \( k > 4 \), \( 2k + 1 < 2k + k < 3k < k^2 \). Hence, going back to the prior inequality and using the inductive step we have
\[
(k + 1)^2 = k^2 + 2k + 1 \leq 2^k + k^2 \leq 2^k + 2^k = 2^{k+1} \quad \text{QED.}
\]

(4) [1Pts] Let \( x, y \in \mathbb{R} \) and \( \epsilon > 0 \). Prove that if \( |x - y| \leq \epsilon \), then \( |x| \leq |y| + \epsilon \).

**Solution:** Using the triangle inequality, we have that
\[
|x| = |x - y + y| \leq |x - y| + |y| \leq \epsilon + |y| \quad \text{QED.}
\]

\(^1\) Alternative argument: We can show that \( 2k + 1 < k^2 \) for \( k > 0 \) by observing that the quadratic equation \( k^2 - 2k - 1 \) has roots at \( 1 \pm \sqrt{2} \) hence its positive for \( k > 1 + \sqrt{2} \).