HW 2

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [3Pts] Let $S, T$ be nonempty subsets of $\mathbb{R}$ and suppose that $S \subset T$. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T$$

Proof. For every $t \in T$, by definition it is $\inf T \leq t$.
Since $S \subset T$, then for every $s \in S$, it is $\inf T \leq s$. This shows that $\inf T$ is a lower bond of $S$, hence $\inf T \leq \inf S$.
Since for every $s \in S$, it is $\inf S \leq s$, then $\inf S \leq \sup S$.
For every $t \in T$, by definition it is $t \leq \sup T$. Since $S \subset T$ then $s \leq \sup T$ for all $S \in S$. This shows that $\sup T$ is an upper bound of $S$, hence $\sup S \leq \sup T$.
Combining these observations, we conclude that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$  

(2) [3Pts] Let $S$ be a nonempty and bounded subset of $\mathbb{R}$. Prove that $M = \sup S$ is unique.

Proof. Suppose that there exists another number $M_1 = \sup S$ with $M_1 \neq M$. Then either $M_1 > M$ or $M_1 < M$. If $M_1 > M$ then $M_1$ would not be $\sup S$ since it could not be the least upper bound of $S$. Similarly, if $M_1 < M$ then $M$ would not be $\sup S$ since it could not be the least upper bound of $S$. Thus it must be $M = M_1$.

(3) [3Pts] Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\sup S = 1$ and find the accumulation points of $S$ is any. Justify your answer.

Proof. $1 - \frac{1}{n} \leq 1$, for all $n$, hence 1 is an upper bound of $S$. To show that 1 is the least upper bound, observe that, if $M = 1 - \epsilon$, for some $\epsilon > 0$, by the Archimedean property there exists some $n \in \mathbb{N}$ such that $1 - \frac{1}{n} > 1 - \epsilon$, so that $M$ cannot be an upper bound. Hence $\sup S = 1$. 


1 is an accumulation point of $S$ since, for any interval of the form $(1 - \epsilon, 1 + \epsilon)$, with $\epsilon > 0$, the Archimedean property implies that there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ so that $1 + \epsilon > 1 - \frac{1}{n} > 1 - \epsilon$. $S$ has no other accumulation points. For any point $x_n = 1 - \frac{1}{n}$, the distance to the closest point is $\frac{1}{n(n+1)}$, so that the deleted neighborhood $N(x_n, r_n)$ with $r_n < \frac{1}{n(n+1)}$ has empty intersection with the set $S$.

(4) [3Pts] Let $X \in \mathbb{R}$ be nonempty and $f, g$ be bounded functions defined on $X$. Prove that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$ 

Give examples to show that the inequality can be either an equality or a strict inequality.

Proof. For any $x \in X$, $f(x) \leq \sup\{f(y) : y \in X\}$ and $g(x) \leq \sup\{g(y) : y \in X\}$. Hence for any $x \in X$,

$$f(x) + g(x) \leq \sup\{f(y) : y \in X\} + g(x) \leq \sup\{g(y) : y \in X\}.$$ 

This shows that the right hand side is an upper bound of the set $\{f(x) + g(x) : x \in X\}$. Hence

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$ 

Example (equality). Set $f(x) = x$, $g(x) = 1$, for $x \in [0, 1]$ Then

$$2 = \sup\{f(x) + g(x) : x \in [0, 1]\}$$

$$= \sup\{f(x) : x \in [0, 1]\} + \sup\{g(x) : x \in [0, 1]\}$$

$$= 1 + 1.$$ 

Example (inequality). Set $f(x) = x$, $g(x) = -x$, for $x \in [0, 1]$ Then

$$0 = \sup\{f(x) + g(x) : x \in [0, 1]\}$$

$$= \sup\{f(x) : x \in [0, 1]\} + \sup\{g(x) : x \in [0, 1]\}$$

$$= 1 + 0.$$ 

(5) [3Pts] Let $S \subset \mathbb{R}$ be nonempty. Show that $S$ is bounded if and only if there exists a closed bounded interval $I$ such that $S \subset I$.

Proof. If $S$ is bounded then let $m = \inf S$ and $M = \sup S$. It follows that $S$ is contained in the interval $I = [m, M]$.

Conversely, suppose that $S \subset I = [a, b]$, where $a, b \in \mathbb{R}$. It follows that $a \leq \inf S$ and $\sup S \leq b$. Hence $S$ is bounded.