HW 4

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) [4 Pts] Use the definition of convergence to prove the following:
(a) For any real number $k$, $\lim_{n \to \infty} \frac{k}{n} = 0$
We need to show that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that
$$\left| \frac{k}{n} \right| < \epsilon$$
provided $n > N$. For that, let $N = \left\lceil \frac{|k|}{\epsilon} \right\rceil$. Then for all $n > N$ we have that
$$\left| \frac{k}{n} \right| < \frac{|k|}{N} < \epsilon.$$

(b) $\lim_{n \to \infty} \frac{3n+1}{n+2} = 3$.
We need to show that, given $\epsilon > 0$, there exists $N = N(\epsilon)$ such that
$$\left| \frac{3n+1}{n+2} - 3 \right| = \frac{3}{n+2} < \epsilon$$
provided $n > N$. For that, choose, $N = \left\lceil \frac{3}{\epsilon} \right\rceil$. Then $\frac{3}{n+2} < \frac{3}{n} < \epsilon$ if $n > N$.

(2) [3 Pts] Show that the sequence $a_n = \cos \frac{n\pi}{3}$ is divergent.

Arguing by contradiction, suppose that $\lim a_n = a$. It then follows by definition that there exists an $N \in \mathbb{N}$ such that
$$\left| \cos \frac{n\pi}{3} - a \right| < 1, \quad \text{for all } n > N.$$
If we take $n = 6m$, then the inequality above implies that $|\cos(2m\pi) - a| < 1$, that is $|1 - a| < 1$ so that $0 < a < 2$. If instead we take $n = 3(2m - 1)$, then the inequality above implies that $|\cos((2m - 1)\pi) - a| < 1$, that is $|1 + a| < 1$ so that $-2 < a < 0$. Since the two conditions on $a$ cannot be satisfied at the same time, then we have a contradiction.

(3) [3 Pts]
(a) Let $(s_n)$ be a sequence such that $\lim_{n \to \infty} s_n = 0$ and $(t_n)$ be a bounded sequence. Prove that the sequence $(s_n t_n)$ is convergent.

(b) Show by example that the boundedness of $(t_n)$ is necessary in part (a). That is, produce an example to show that the sequence $(s_n t_n)$ may diverge if $(t_n)$ is not bounded.
(a) Proof. Since \((t_n)\) is bounded, there is an \(M > 0\) such that \(t_n < M\) for all \(n \in \mathbb{N}\). Since \(\lim_{n \to \infty} s_n = 0\), given any \(\epsilon > 0\), there exists and \(N = N(\epsilon)\) such that \(|s_n| < \frac{\epsilon}{M}\) if \(n > N\). It follows that, given \(\epsilon > 0\), there exists \(N = N(\epsilon)\) such that \(|s_n t_n| < \frac{\epsilon}{M} M = \epsilon\) if \(n > N\).

(b) Consider the sequences \((s_n) = (\frac{1}{n})\) and \((t_n) = (n^2)\). Then \((s_n t_n) = (n)\) and this sequence is not convergent.

(4)[3 Pts] Prove or give a counterexamples:

(a) If \((s_n)\) and \((t_n)\) are divergent sequences, then \((s_n + t_n)\) diverges.
FALSE. Let \((s_n) = (-1)^n\) and \((t_n) = (-1)^{n+1}\). \((s_n + t_n) = 0\) convergent.

(b) If \((s_n)\) and \((t_n)\) are divergent sequences, then \((s_n t_n)\) diverges.
FALSE. Let \((s_n) = (-1)^n\) and \((t_n) = (-1)^n\). \((s_n t_n) = 1\) convergent.

(c) If \((s_n)\) and \((s_n + t_n)\) are convergent sequences, then \((t_n)\) converges.
TRUE by Limit Theorems. \((t_n) = (s_n + t_n) - (s_n)\) convergent since it is an algebraic sum of convergent sequences.

(5)[3 Pts] Prove that if \((x_n)\) is a convergent sequence, \(|x_n|\) is also convergent. Is the converse true?

Proof.
Since \((x_n)\) converges, \(\lim x_n = s\). Hence, given any \(\epsilon > 0\), there exists an \(N = N(\epsilon)\) such that \(|x_n - s| < \epsilon\) if \(n > N\).

Since \(|x_n| \leq |x_n - s| + |s|\) and \(|s| \leq |s - x_n| + |x_n|\), it follows that \(||x_n| - |s|| \leq |x_n - s|\). It follows that \(||x_n| - |s|| < \epsilon\) if \(n > N\). Hence \(|x_n|\) converges and \(\lim |x_n| = |s|\).

The converse is not true. Consider \((x_n) = (-1)^n\). In this case, \(|x_n| = 1\) is convergent but \((x_n)\) is not convergent.

(6)[3 Pts] Suppose that \((x_n)\) is a convergent sequence and \((y_n)\) is a sequence such that, for any \(\epsilon > 0\), there exists an \(M > 0\) such that \(|x_n - y_n| < \epsilon\) for all \(n > M\). Does it follow that \((y_n)\) converge? Prove it or find a counterexample.

Proof.
Since \((x_n)\) converges, \(\lim x_n = s\). Hence, given any \(\epsilon > 0\), there exists an \(N_1 = N_1(\epsilon)\) such that \(|x_n - s| < \epsilon\) if \(n > N_1\). Since, for any \(n \in \mathbb{N}\),

\[|y_n - s| = |y_n - x_n + x_n - s| \leq |y_n - x_n| + |x_n - s|,\]

it follows that \(|y_n - s| < 2\epsilon\) if \(n > N = \max\{N_1, M\}\). This proves that \(\lim y_n = s\).