

**HW 5**

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) Prove that

$$\lim_{n \to \infty} \sqrt{n^2 + 1} - n = 0$$

**Proof.** Observe that

$$|\sqrt{n^2 + 1} - n| = \frac{1}{\sqrt{n^2 + 1} + n} < \frac{1}{n}$$

Given any $\epsilon > 0$, let $N > \frac{1}{\epsilon}$, then

$$|\sqrt{n^2 + 1} - n| < \frac{1}{n} < \epsilon, \quad \text{if } n > N.$$  

This proves that $\lim_{n \to \infty} \sqrt{n^2 + 1} - n = 0$.

(2) Prove that if $\lim_{n \to \infty} s_n = \infty$ and if $(t_n)$ is a bounded sequence, then

$$\lim_{n \to \infty} (s_n + t_n) = \infty$$

**Proof.** Since $(t_n)$ is bounded, there exists an $N_1$ such that $t_n > L$ if $n > N_1$. Note that $L$ can be a negative number.

Since $\lim_{n \to \infty} s_n = \infty$, given any $M > 0$, there exists an $N_2$ such that $s_n > M - L$ if $n > N_2$.

Hence, provided $n > \max N_1, N_2$, we have that $s_n + t_n > M$. Since $M$ is arbitrary, this proves that $\lim_{n \to \infty} (s_n + t_n) = \infty$.

(3) Prove that if $\lim_{n \to \infty} s_n = \infty$ and $\lim_{n \to \infty} t_n = L > 0$, then

$$\lim_{n \to \infty} (s_nt_n) = \infty$$

**Proof.** Since $\lim_{n \to \infty} t_n = L$, there exists an $N_1$ such that $|t_n - L| < L/2$ if $n > N_1$. Hence, $t_n > L/2$, is $n > N_1$.

Since $\lim_{n \to \infty} s_n = \infty$, given any $M > 0$, there exists an $N_2$ such that $s_n > 2M/L$ if $n > N_2$.

Hence, provided $n > \max N_1, N_2$, we have that $s_nt_n > M$. Since $M$ is arbitrary, this proves that $\lim_{n \to \infty} (s_nt_n) = \infty$. 
(4) Prove that the sequence below is monotone and bounded. Next find its limit.

\[ s_1 = 1, \quad s_{n+1} = \frac{1}{5}(s_n + 7), \quad n \geq 1. \]

**Proof.** Note that \( s_1 = 1, \ s_2 = \frac{1}{5}(1 + 7) = \frac{8}{5}. \)

**Claim:** \( s_n \leq 2. \) **Proof by induction:**

\( s_1 = 1 < 2. \)

Assume \( s_n \leq 2. \)

Then \( s_{n+1} = \frac{1}{5}(s_n + 7) \leq s_n + 1 = \frac{1}{5}(2 + 7) = \frac{9}{5} < 2. \)

**Claim:** \( s_{n+1} \geq s_n. \) **Proof by induction:**

\( s_2 < s_1. \)

Assume \( s_{n+1} \geq s_n. \)

Then \( s_{n+2} = \frac{1}{5}(s_{n+1} + 7) \geq \frac{1}{5}(s_n + 7) = s_{n+1}. \)

Since \((s_n)\) is monotone and bounded, then it is convergent. Thus

\[ s = \lim s_{n+1} = \lim \frac{1}{5}(s_n + 7) = \frac{1}{5}(s + 7). \]

**Hence**

\[ 5s = s + 7 \quad \text{and} \quad s = \frac{7}{4}. \]

(5) Let \((a_n)\) and \((b_n)\) be monotone sequences. Prove or give a counterexample.

(a) The sequence \((c_n)\) given by \(c_n = a_n + b_n\) is monotone.

**FALSE.** Let \((a_n) = (1, 2, 2, 2, \ldots)\) and \((b_n) = (2, 2, 1, 1, \ldots).\) Then \((c_n) = (a_n + b_n) = (3, 4, 3, 3, \ldots)\) is not monotone.

(b) The sequence \((c_n)\) given by \(c_n = a_n b_n\) is monotone.

**FALSE.** Let \((a_n) = (1, 2, 2, 2, \ldots)\) and \((b_n) = (2, 2, 1, 1, \ldots).\) Then \((c_n) = (a_n b_n) = (2, 4, 2, 2, \ldots)\) is not monotone.