Quiz 5

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1)[5Pts] Prove that the sequence below is monotone and bounded. Next find its limit.

\[ s_1 = 2, \quad s_{n+1} = \frac{1}{4}(2s_n + 7), \quad n \geq 1. \]

Proof. Note that \( s_1 = 2, \quad s_2 = \frac{1}{4}(4 + 7) = \frac{11}{4} > 2. \)

Claim: \( s_n \leq 4. \) Proof by induction:

1. \( s_1 = 2 \leq 4. \)
2. Assume \( s_n \leq 4. \)
3. Then \( s_{n+1} = \frac{1}{4}(s_n + 7) \leq \frac{11}{4} < 4. \)

Claim: \( s_{n+1} \geq s_n. \) Proof by induction:

1. \( s_2 > s_1. \)
2. Assume \( s_{n+1} \geq s_n. \)
3. Then \( s_{n+2} = \frac{1}{4}(2s_{n+1} + 7) \geq \frac{1}{4}(2s_n + 7) = s_{n+1}. \)

Since \( (s_n) \) is monotone nondecreasing and bounded above, then it is convergent. Thus

\[ s = \lim s_{n+1} = \lim \frac{1}{4}(2s_n + 7) = \frac{1}{4}(2s + 7). \]

Hence \( 4s = 2s + 7 \) and \( s = \frac{7}{2}. \)

(1) [3Pts] Prove or give a counterexample:

(a) Every monotone sequence converges.

\textit{FALSE.} \( (s_n) = (n) \) is monotone but not convergent.

(b) If \( (a_n) \) and \( (b_n) \) are monotone sequences, then \( (c_n) = (a_n + b_n) \) is also a monotone sequence.

\textit{FALSE.} Let \( (a_n) = (1, 2, 2, 2, \ldots) \) and \( (b_n) = (2, 2, 1, 1, \ldots). \) Then \( (c_n) = (a_n + b_n) = (3, 4, 3, 3, \ldots) \) is not monotone.

(c) If \( (a_n) \) and \( (b_n) \) are monotone non-decreasing sequences, then \( (c_n) = (a_n + b_n) \) is also a monotone non-decreasing sequence.

\textit{TRUE.} If \( a_{n+1} \geq a_n \) and \( b_{n+1} \geq b_n, \) then \( c_{n+1} = a_{n+1} + b_{n+1} \geq a_n + b_n = c_n. \)