**Test #1**

This is a closed-book, no-notes test. Please, write clearly and justify your arguments using the material covered in class to get credit for your work.

(1)[4 Pts] Use an argument by induction to prove that, if \( x \geq 0 \), then 
\[
(1 + x)^n \geq 1 + nx \quad \text{for all } n \in \mathbb{N}.
\]

*Proof.*  
- For \( n = 1 \), \((1 + x)^1 \geq 1 + 1 \cdot x = 1 + x\).  
- Assume \((1 + x)^n \geq 1 + nx\) for some \( n \in \mathbb{N} \).  
- We now derive the case \( n + 1 \). Using the statement above for \( n \), we observe that 
\[
(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1+nx+nx^2 \geq 1+nx = 1+(n+1)x.
\]
This shows that statement is true for \( n + 1 \), hence it is true for all \( n \in \mathbb{N} \). \( \Box \)

(2)[6 Pts] Let \( A, B \) be subsets of \( \mathbb{R} \).

(a) State the definition of *boundary* for a subset \( S \) of \( \mathbb{R} \).

(b) Prove that \( \text{bd}(A \cup B) \subseteq \text{bd}(A) \cup \text{bd}(B) \).

(c) Show that the converse containment \( \text{bd}(A) \cup \text{bd}(B) \subseteq \text{bd}(A \cup B) \) may fail by giving a counterexample, i.e., find sets \( A, B \) such that \( \text{bd}(A) \cup \text{bd}(B) \nsubseteq \text{bd}(A \cup B) \).

(a) The boundary of \( S \) is the set of its boundary points. A boundary point of \( S \) is a point \( x \) such that for any \( \epsilon > 0 \) we have that \( N(x, \epsilon) \cap S \neq \emptyset \) and \( N(x, \epsilon) \cap S^c \neq \emptyset \).

(b) *Proof.* We will show that, for any \( x \in \text{bd}(A \cup B) \) then \( x \in \text{bd}(A) \) or \( x \in \text{bd}(B) \).

If \( x \in \text{bd}(A \cup B) \), then for any \( \epsilon > 0 \) we have that \( N(x, \epsilon) \cap (A \cup B) \neq \emptyset \) and \( N(x, \epsilon) \cap (A \cup B)^c \neq \emptyset \). Note that \((A \cup B)^c = (A^c \cap B^c)\). Hence, for any \( \epsilon > 0 \) we have that \( N(x, \epsilon) \cap A \neq \emptyset \) and \( N(x, \epsilon) \cap A^c \neq \emptyset \) or \( N(x, \epsilon) \cap B \neq \emptyset \) and \( N(x, \epsilon) \cap B^c \neq \emptyset \). That is, we have that \( x \in \text{bd}(A) \) or \( x \in \text{bd}(B) \). \( \Box \)

(c) Let \( A = [1, 3] \) and \( B = (2, 4) \). Observe that \( \text{bd}(A) = \{1, 3\} \), \( \text{bd}(B) = \{2, 4\} \) and \( \text{bd}(A \cup B) = \text{bd}([1, 4]) = \{1, 4\} \). Hence in this case \( \text{bd}(A \cup B) \) is strictly contained in \( \text{bd}(A) \cup \text{bd}(B) \) so that \( \text{bd}(A) \cup \text{bd}(B) \nsubseteq \text{bd}(A \cup B) \).
(3)[5 Pts] Find a set (or sets) satisfying the description below, or explain why they do not exist.

(a) A set $S \in \mathbb{R}$ that is neither open nor closed.

$S = [1, 2)$

(b) A set $S \in \mathbb{R}$ that has a maximum, a minimum and is not closed.

$S = [1, 2) \cap (3, 4]$. Note that $\min S = 1$ and $\max S = 4$ but $S$ is not closed.

(c) A collection of open sets $A_n$ such that $\bigcap_n A_n$ is not open.

$A_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap_n A_n = \{0\}$, closed set.

(d) A collection of open sets $A_n$ such that $\bigcup_n A_n$ is not open.

Not possible. By theorem in class, the union of any collection of open sets is open.

(e) An unbounded set containing no accumulation points.

The set $\mathbb{N}$ is unbounded and contains no accumulation points.