**Math 3339** Name: SOLUTION

## HW #2

Please, write clearly and justify all your steps, to get proper credit for your work.

(1) Two cards are drawn successively and without replacement from a 52-card deck of playing cards. Compute the probability of drawing:

(i) two hearts;

$$P(\heartsuit, \heartsuit) = P(\heartsuit \text{ on first draw})P(\heartsuit \text{ on second draw}|(\heartsuit \text{ on first draw})) = \frac{13}{52}\frac{12}{51}$$

(ii) a heart on the first draw, a club on the second draw;

$$P(\heartsuit, \clubsuit) = P(\heartsuit \text{ on first draw})P(\clubsuit \text{ on second draw}|(\heartsuit \text{ on first draw})) = \frac{13}{52}\frac{13}{51}$$

(iii) a heart on the first draw, an ace on the second draw.

$$\begin{aligned} P(\heartsuit, A) &= P(\heartsuit \text{ on first draw, no ace})P(A \text{ on second draw}|(\heartsuit \text{ on first draw, no ace})) \\ &+ P(\text{Ace of } \heartsuit \text{ on first draw})P(A \text{ on second draw}|(\text{Ace of } \heartsuit \text{ on first draw})) \\ &= \frac{12}{52}\frac{4}{51} + \frac{1}{52}\frac{3}{51} \end{aligned}$$

(2) From a regular deck of 52 playing cards, cards are drawn successively and without replacement. Compute the probability that the third spade appears on the sixth draw.

$$P(2\spadesuit \text{ on 5 draws})P(\spadesuit \text{ on sixth draw}|(2\spadesuit \text{ on 5 draws})) = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}\frac{11}{47}$$

(3) A survey organization asked respondents from 3 different geographical regions what they views were on a certain topic. The answer are reported below.

| | East | Midwest | West |
|---|---|---|---|
| Pessimistic | 100 | 90 | 110 |
| Optimistic | 40 | 70 | 90 |
| *Total* | 140 | 160 | 200 |

(i) What is the probability that a randomly selected respondent is pessimistic?

$$P(\text{pessimistic}) = \frac{300}{500} = \frac{3}{5}$$

(ii) What is the conditional probability that a respondent from the Midwest is optimistic?

$$P(\text{optimistic}|\text{midwest}) = \frac{70}{160} = \frac{7}{16}$$

(iii) What is the conditional probability that a respondent who is optimistic comes from the Midwest?

$$P(\text{midwest}|\text{optimistic}|) = \frac{70}{200} = \frac{7}{20}$$

(iv) Are the views of the respondents independent on the geographical regions? Justify your answer. If no, with the same marginal totals, specify what the numbers for the West region would have been, had the two factors been independent.

*Not independent.* $P(west\&optimistic|) = \frac{90}{500} = \frac{9}{50}$ *but this is different from* $P(west)P(optimistic|) = \frac{200}{500}\frac{200}{500} = \frac{4}{25} = \frac{8}{50}$.

*If respondents were independent of the geographical region, then*

$P(west\&pessimistic) = P(west)P(pessimistic) = \frac{200}{500}\frac{300}{500} = \frac{120}{500}$

$P(west\&optimistic) = P(west)P(optimistic) = \frac{200}{500}\frac{200}{500} = \frac{80}{500}$

*Hence, the numbers in the last column would be 120 and 80, respectively*

(4) A disease in a bat population has a prevalence of 0.1%. To test for the presence of the disease, a screening test was designed with a reported sensitivity of 92% and a reported specificity of 85%. We a randomly selected bat from this population and perform the screening test. What is the probability that the bat is affected by the disease if the test returns positive? What is the probability that the bat is not affected by the disease if the test returns negative?

**Solution.** From the word problem we have $P(D^+) = 0.001$, hence $P(D^-) = 1 - P(D^+) = 0.999$.

We also have $P(T^+|D^+) = 0.92$, $P(T^-|D^-) = 0.85$, hence we derive $P(T^+|D^-) = 1 - P(T^-|D^-) = 0.15$ and $P(T^-|D^+) = 1 - P(T^+|D^+) = 0.08$

Hence

$$\begin{aligned} P(D^+|T^+) &= \frac{P(T^+|D^+)\,P(D^+)}{P(T^+|D^+)\,P(D^+) + P(T^+|D^-)\,P(D^-)} \\ &= \frac{(0.92)\,(0.001)}{(0.92)\,(0.001) + (0.15)\,(0.999)} = 0.0061 \\ P(D^-|T^-) &= \frac{P(T^-|D^-)\,P(D^-)}{P(T^-|D^-)\,P(D^-) + P(T^-|D^+)\,P(D^+)} \\ &= \frac{(0.85)\,(0.999)}{(0.85)\,(0.999) + (0.08)\,(0.001)} = 0.9999 \end{aligned}$$

(5) The surfaces of human red blood cells ("erythrocytes") are coated with antigens that are classified into four disjoint blood types: $O, A, B$, and $AB$. Each type is associated with blood serum antibodies for the other types, that is

1. Type $O$ blood contains both $A$ and $B$ antibodies. (This makes Type $O$ the "universal donor", but capable of receiving only Type $O$.)
2. Type $A$ blood contains only $B$ antibodies.
3. Type $B$ blood contains only $A$ antibodies.

4. Type $AB$ blood contains neither $A$ nor $B$ antibodies. (This makes Type $AB$ the "universal recipient", but capable of donating only to Type $AB$.)

In addition, blood is also classified according to the presence (+) or absence (-) of Rh factor. Hence there are eight distinct blood groups corresponding to this joint classification system: $O^+, O^-, A^+, A^-, B^+, B^-, AB^+, AB^-$. According to the American Red Cross, the U.S. population has the following blood group relative frequencies

| **Blood type** | Rh+ | Rh- | *Totals* |
|---|---|---|---|
| O | 0.384 | 0.077 | 0.461 |
| A | 0.323 | 0.065 | 0.388 |
| B | 0.094 | 0.017 | 0.111 |
| AB | 0.032 | 0.007 | 0.039 |
| *Totals* | 0.833 | 0.166 | 0.999 |

From these table, we can calculate, for instance, the following probabilities:
$P(\text{Blood type } O^+) = 0.384$,
$P(\text{Blood type } O) = P(\text{Blood type } O^+) + P(\text{Blood type } O^-) = 0.461$,
$P(A \text{ antibodies}) = P(\text{Blood type } O) + P(\text{Blood type } B) = 0.461 + 0.111 = 0.572$.
$P(B \text{ antibodies}) = P(\text{Blood type } O) + P(\text{Blood type } A) = 0.461 + 0.388 = 0.849$.
Answer the following questions:
(a) Is having $A$ antibodies independent of having $B$ antibodies?
(b) Is having $B$ antibodies independent of $Rh^+$?

**Solution.**
(a)
$P(A \text{ and } B \text{ antibodies}) = P(\text{Blood type } O) = 0.461$
This is different from $P(A \text{ antibodies})\, P(B \text{ antibodies}) = (0.572)(0.849) = 0.486$.
Hence having $A$ antibodies in NOT independent of having $B$ antibodies.

(b)
$P(B\text{antibodies and } Rh^+) = P(\text{Blood type } A^+) + P(\text{Blood type } O^+) = 0.323 + 0.384 = 0.707$
This is equal to $P(B \text{ antibodies})\, P(Rh^+) = (0.849)(0.833) = 0.707$.
Hence having $B$ antibodies and $Rh^+$ in independent.