Math 3339

Test #2

(1)[6 Pts] An experimental medication was given to patients with a certain medical condition. Suppose that the probability that a patient with the underlying condition will experience severe side effects if given that medication is 8.5%. What is the probability that, of 35 randomly chosen such patients,
(a) more than 3 will experience severe side effects?
(b) exactly 3 will experience severe side effects?
(c) How many of the 35 patients do you expect will experience severe side effects?

Number of patients with severe side effects: $X \sim \text{bin}(n = 35, p = 0.085)$
(a) $P(X > 3) = 1 - P(X \leq 3) = 1 - \text{pbinom}(3, 35, 0.085) = 0.347$
(b) $P(X = 3) = \text{dbinom}(3, 35, 0.085) = 0.234$
(c) $E(X) = 35 \times 0.085 = 2.975$

(2)[8 Pts] Let $X$ and $Y$ have the following joint p.d.f.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1 0.14 0.32</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.39 0.15</td>
<td></td>
</tr>
</tbody>
</table>

(a) Calculate the marginal densities. Are $X$ and $Y$ independent?
(b) Compute the means and variances.
(c) Are $X$ and $Y$ positively correlated? negatively correlated? uncorrelated?
(d) Let $Z = 1 - X + 2Y$. Find the mean and the variance of $Z$.

(a) > p <- matrix(c(.14,.39,.32,.15),ncol=2)
> px <- apply(p,2,sum)
[1] 0.53 0.47
> py <- apply(p,1,sum)
[1] 0.46 0.54
That is: $f_1(x) = (0.53, 0.47)$ and $f_2(y) = (0.46, 0.54)$

$X, Y$ not independent since $f(1, 1) = 0.14 \neq f_1(1) \times f_2(1) = 0.53 \times 0.46 = 0.244$
(b-c) > x <- c(1,2)
> y <- c(1,2)
> EX = sum(px*x) = 1.47
> EY = sum(py*y) =1.54
> VarX = sum(px*x*x)-EX*EX = 0.249
> VarY = sum(py*y*y)-EY*EY = 0.248
> A=0
> for(i in 1:2)for(j in 1:2)A <- A+p[i,j]*x[i]*y[j]
> EXY<-A
> EXY = 2.16
> COVXY = EXY-EX*EY = -0.104
Hence: $\mu_X = 1.47$, $\mu_Y = 1.54$, $\sigma^2_X = 0.249$, $\sigma^2_Y = 0.248$, $\sigma_{XY} = -0.104$

$X$ and $Y$ are negatively correlated.
(d) $\mu_Z = E[Z] = 1 - E[X] + 2E[Y] = 1 - 1.47 + 2 \times 1.54 = 2.61$
$\text{var}(Z) = \text{var}(X) + 4 \text{var}(Y) - 4 \text{cov}(X,Y) = 0.249 + 4 \times 0.248 + 4 \times 0.104 = 1.657$
(3) [6 Pts] Let $X$ be a normal random variable with mean $\mu = 9$ and standard deviation $\sigma = 4$.

(i) Calculate the probability $P(X > 11.5)$

(ii) Calculate the probability $P(|X - 9| \leq 5)$

(iii) Find the value $x_c$ such that $P(X > x_c) = 0.8810$

\[ X \sim N(\mu = 9, \sigma = 4) \]

Using R:

(i) $P(X > 11.5) = 1 - P(X \leq 11.5) = 1 - \text{pnorm}(11.5, \text{mean} = 9, \text{sd} = 4) = 0.266$

(ii) $P(|X - 9| \leq 5) = P(4 \leq X \leq 14) = \text{pnorm}(14, \text{mean} = 9, \text{sd} = 4) - \text{pnorm}(4, \text{mean} = 9, \text{sd} = 4) = 0.789$

(iii) $P(X > x_c) = 1 - P(X \leq x_c) = 0.8810 \Rightarrow P(X \leq x_c) = 1 - 0.8810 = 0.1190 \Rightarrow x_c = \text{qnorm}(0.119, \text{mean} = 9, \text{sd} = 4) = 4.280$

(4) [4 Pts] Let $\overline{X}$ be the mean of a random sample of size $n = 75$ from an uniform distribution on the interval $-2 \leq x \leq 6$. Use the central limit theorem to approximate the probability $P(2.2 < \overline{X} < 2.6)$.

\[ E[X] = \frac{a + b}{2} = 2 = E[\overline{X}] \]

\[ \text{var}[X] = \frac{(b - a)^2}{12} = \frac{16}{3} \Rightarrow \sigma^2_{\overline{X}} = \frac{\text{var}[X]}{n} = \frac{16}{225} \Rightarrow \sigma_{\overline{X}} = \frac{4}{15} \]

\[ P(2.2 < \overline{X} < 2.6) = \text{pnorm}(2.6, \text{mean} = 2, \text{sd} = 4/15) - \text{pnorm}(2.2, \text{mean} = 2, \text{sd} = 4/15) = 0.214 \]

(5) [6 Pts] Let $X$ be a continuous random variable with pdf $f(x) = 6x(1-x)$ for $0 \leq x \leq 1$.

(i) Compute the mean and variance of $X$.

(ii) Let $\overline{X}$ be the mean of a random sample of size $n = 50$ from the pdf given above. Compute the mean and variance of $\overline{X}$.

(iii) Compute the probability $P(0.45 < \overline{X} < 0.55)$.

(i) By direct calculation

\[ E[X] = \int_0^1 6x(1-x)dx = \int_0^1 (6x^2 - 6x^3)dx = \frac{6}{3} - \frac{6}{4} = \frac{1}{2} = 0.5 \]
\[ \text{var}[X] = \int_0^1 6x^3(1-x)dx - \frac{1}{4} = \int_0^1 (6x^3 - 6x^4)dx - \frac{1}{4} = \frac{6}{4} - \frac{6}{5} - \frac{1}{4} = \frac{1}{20} = 0.05 \]

(ii) From (i), it follows that

\[ \mu_{\overline{X}} = E[\overline{X}] = \frac{1}{2}, \quad \sigma^2_{\overline{X}} = \frac{\text{var}[X]}{n} = 0.001 \]

(iii) $P(0.4 < \overline{X} < 0.6) = \text{pnorm}(0.55, \text{mean} = 1/2, \text{sd} = \text{sqrt}(0.001)) - \text{pnorm}(0.45, \text{mean} = 1/2, \text{sd} = \text{sqrt}(0.001)) = 0.886$