Math 4332/6313 – Spring 2017 Name:

Test #1

This is a closed book test. Please, write clearly and justify all your steps, to get proper credit for your work. You can cite general theorems from the book if needed.

(1) [3 Pts] Let $C_b[0,1]$ be the space of all bounded continuous functions on $[0,1]$. For $f \in C_b[0,1]$, let $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Prove that this function is a norm on $C_b[0,1]$.

(2) [3 Pts] Let $(V, \langle , \rangle)$ be an inner product space. Prove that the inner product is continuous, that is, if $(x_n), (y_n) \in V$ and $x_n \to x$, $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(3)[3 Pts] Show that the unit ball of the space $C[-1,1]$, with norm $\|\cdot\|_\infty$, is not compact. Follow the hint below:

1. Consider the sequence (f_n) in $C[-1,1]$ where $f_n(x) = 0$ for $x \in [-1,0]$, $f_n(x) = nx$ for $x \in [0,1/n]$, $f_n(x) = 1$ for $x \in [1/n,1]$. Prove that $\|f_n - f_m\|_\infty \geq 1/2$ if $m \geq 2n$.
2. Show that (f_n) has no convergent subsequence.
3. Conclude that the unit ball of $C[-1,1]$ is not compact.

(4)[3 Pts] For each of the following statements, either prove it (you can use the theorems discussed in class) or give a counterexample (in this case, you need to show how your counterexample disproves the statement).

(a) For f in $C^1[a,b]$ let $\rho(f) = \sup_{x\in[a,b]} |f'(x)|$. $\rho(f)$ is a norm on $C^1[a,b]$.

(b) Let V be a finite dimensional vector space with two norms $\|\cdot\|$ and $|||\cdot|||$, and let (x_n) be a sequence in V such that $\lim_{n\to\infty} \|x - x_n\| = 0$. Then $\lim_{n\to\infty} |||x - x_n||| = 0$.

(c) A bounded subset of a normed space is compact.

TEST 1 SOLUTION

(1) Let $f \in C_b[0,1]$

POSITIVITY: If $f=0$, then $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)| = 0$

If $\|f\|_\infty = 0$ then $f(x)=0$ $\forall x$. Otherwise, if $\exists x_0 \in [0,1]$ s.t. $f(x_0)\neq 0$, then $\sup|f(x)|>0$.

HOMOGENEITY: $\|\alpha f\|_\infty = \sup|\alpha f(x)| \le |\alpha| \sup|f(x)| = |\alpha|\,\|f\|_\infty$

TRIANGLE INEQ. $\|f+g\|_\infty = \sup|f(x)+g(x)| \le \sup(|f(x)|+|g(x)|) \le \sup|f(x)| + \sup|g(x)| = \|f\|_\infty + \|g\|_\infty$

(2) $|\langle x_n,y_n\rangle - \langle x,y\rangle| = |\langle x_n,y_n\rangle - \langle x_n,y\rangle + \langle x_n,y\rangle - \langle x,y\rangle| \le$

$\le |\langle x_n,y_n\rangle - \langle x_n,y\rangle| + |\langle x_n,y\rangle - \langle x,y\rangle| =$

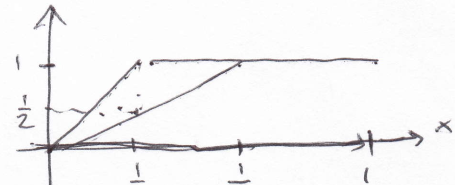
$= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \le \|x_n\|\,\|y_n-y\| + \|x_n-x\|\,\|y\|$

Since $(x_n)\to x$, then $\exists M>0$ s.t. $\|x_n\|\le M$

Given $\varepsilon>0$, by hyp. $\exists N\in\mathbb{N}$ s.t. $\|x_n - x\| < \frac{\varepsilon}{2\|y\|}$ and $\|y-y_n\| < \frac{\varepsilon}{2M}$, if $n>N$

Hence $|\langle x_n,y_n\rangle - \langle x,y\rangle| \le M\cdot\frac{\varepsilon}{2M} + \|y\|\frac{\varepsilon}{2\|y\|} = \varepsilon$

(3) • (1) Let $m=2n$, then at $x=\frac{1}{2n}$ $f_{2n}(x) - f_n(x) = 1 - \frac{n}{2n} = \frac{1}{2}$



If $m\ge 2n$, $f_m(\frac{1}{2n}) = 1$, hence $f_m(x) - f_n(x) = \frac{1}{2}$

This shows that $\sup|f_m(x) - f_n(x)| \ge \frac{1}{2}$ if $m\ge 2n$

• (2) Arguing by contradiction, suppose that the seq. (f_n) has a convergent subsequence (f_{n_k}). Since it is a convergent subsequence, it is Cauchy, hence, given $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $|f_{n_k} - f_{n_\ell}| < \varepsilon$ for all $n_k, n_\ell > N$.

However this is NOT true by observation above if $n_\ell \ge 2n_k$. Contradiction.

• (3) As we have shown that the unit ball of $C[-1,1]$ contains a sequence with no convergent subsequence, then it follows that this space is NOT compact.

(4) (a) [F] Consider $f(x)=c\neq 0$ on $[a,b]$.

In this case $\rho(f) = \sup|f'(x)| = 0$ even though $f\neq 0$, hence ρ is NOT a NORM.

(b) [T] By the properties of finite-dim vector spaces, all norms are equivalent, that is $\exists\ 0<c_1\le c_2<\infty$ s.t.

$$c_1\|x\| \le |||x||| \le c_2\|x\| \quad \forall x\in V$$

This IMPLIES that $|||x_n - x||| \le c_2\|x_n - x\|$, hence convergence in $\|\cdot\|$ IMPLIES convergence in $|||\cdot|||$

(c) [F] Consider the open set $(0,1)$ in $\mathbb{R}$. This set is bounded but NOT COMPACT as Heine-Borel theorem requires that any compact subset of $\mathbb{R}$ be closed and bounded.