MATH 4377/6308 - Advanced linear algebra I - Summer 2024
Quiz 5

Exercises:
(1) Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

a) If $B$ is a matrix that can be obtained by performing an elementary row operation on a matrix $A$, then $A$ can be obtained by performing an elementary row operation on $B$.

b) The rank of a matrix is equal to the number of its nonzero columns.

(a) True. Since $B = EA$ with $E$ an elementary matrix, it follows that $A = E^{-1}B$ where the inverse $V$ is also an elementary matrix.

(b) False. For example, the rank of $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is 1, which is not equal to the number of its nonzero columns.

(2) Determine the values of the parameter $k$ such that the following system of equations has unique solution, no solution or infinitely many solutions.

\[
\begin{align*}
  x + z &= k \\
  kx + 2y &= 1 \\
  -3x + y &= -k
\end{align*}
\]

Row reduce:
\[
\begin{pmatrix}
  1 & 0 & 1 & k \\
  k & 2 & 0 & 1 \\
-3 & 1 & 0 & -k
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 1 & k \\
  0 & 2 & -k & 1 - k^2 \\
  0 & 1 & 3 & 2k
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 1 & k \\
  0 & 1 & 3 & 2k \\
  0 & 2 & -k & 1 - k^2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  1 & 0 & 1 & k \\
  0 & 1 & 3 & 2k \\
  0 & 0 & -k - 6 & 1 - k^2 - 4k
\end{pmatrix}
\]

This shows that the system has no solution if $k = -6$. If $k \neq -6$, the system has unique solution.

(3) Consider the following matrices and determine if they are invertible or not. Justify your answer.

\[
\begin{pmatrix}
  0 & -1 & 1 \\
  1 & 3 & 2 \\
  1 & 0 & -1
\end{pmatrix}
\]

(a) NOT INVERTIBLE since the second row is linearly dependent with the other rows, namely $r2 = r3 - r1$.

(b) INVERTIBLE since the matrix can be transformed into a diagonal matrix (with non-zero diagonal entries) by changing the order of row 2 and row 1.