You must show your work and justify your steps to receive credit.

Problems:
(1) [4Pts] Let \( x, y \in \mathbb{Z} \). Prove that the following relation is an equivalence relation or show that it is not: 
\( x \sim y \) if and only if \( x - y \) is a multiple of 3

**SOLUTION:**
Yes.
(i) \( x - x = 0 \) is a multiple of 3 with multiplicative constant 0;
(ii) if \( x - y = 5m \), then \( y - x = 5(-m) \), which is also a multiple of 5;
(iii) if \( x - y = 5m \) and if \( y - z = 5n \), then \( x - z = 5(m + n) \)

(2) [6Pts] For each one of the statements below, construct an example or explain why such example does not exist.

a) A subspace of \( \mathbb{R}^3 \) of dimension 1.
b) A non-trivial subset of \( \mathbb{R}^3 \) that is not a subspace (non-trivial means it should not be the empty set).
c) A linearly independent set of 2 vectors in \( \mathbb{R}^3 \).
d) A linearly independent set of 4 vectors in \( \mathbb{R}^3 \).
e) A spanning set of \( \mathbb{R}^3 \) that is not a basis.
f) An infinite dimensional vector space (that is, a vector spaces with no basis of finite cardinality).

**SOLUTION.**
(a) The set \( \{(a, a, a) : a \in \mathbb{R}\} \)
b) The set \( \{(1,1,1)\} \).
c) The set \( \{(1,0,0),(0,1,0)\} \)
d) Not possible since a basis in \( \mathbb{R}^3 \) has dimension 3 and any set of more than 3 elements is linearly dependent.
e) The set \( \{(1,0,0),(0,1,0),(0,0,1),(2,0,0)\} \).
f) The space of continuous function with domain in \( \mathbb{R} \).

(3) [6Pts] Determine if the following subsets of the vector space of 2 \( \times \) 2 matrices with real entries are subspaces:

a) \( S = \left\{ \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\} \)
b) \( R = \left\{ \begin{bmatrix} a & b-a \\ a-b & b \end{bmatrix} : a, b \in \mathbb{R} \right\} \)

**SOLUTION.**
a) This is not a subspace. Note that:
\[
\begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & \frac{1}{a'} \end{bmatrix} = \begin{bmatrix} a + a' & b + b' \\ 0 & \frac{1}{a} + \frac{1}{a'} \end{bmatrix} \neq \begin{bmatrix} a + a' & (a + a')(b + b') \\ 0 & \frac{1}{a + a'} \end{bmatrix}
\]
This shows that the matrix resulting from adding two matrices in \( S \) does not belong to \( S \).

(b) This is a subspace.

(i) For any \( \alpha \in \mathbb{R} \), \( \alpha \begin{bmatrix} a & b-a \\ a-b & b \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha(b-a) \\ \alpha(a-b) & \alpha b \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{b}-\tilde{a} \\ \tilde{a}-\tilde{b} & b \end{bmatrix} \), for \( \tilde{a}, \tilde{b} \in \mathbb{R} \).

(ii) \( \begin{bmatrix} a & b-a \\ a-b & b \end{bmatrix} + \begin{bmatrix} a' & b'-a' \\ a'-b' & b' \end{bmatrix} = \begin{bmatrix} a + a' & (b + b') - (a + a') \\ (a + a' - (b + b')) & b + b' \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{b}-\tilde{a} \\ \tilde{a}-\tilde{b} & b \end{bmatrix} \), for \( \tilde{a}, \tilde{b} \in \mathbb{R} \).

(4) [6Pts] Let \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) be given by
\[
T(a_1, a_2, a_3, a_4) = (a_1 + a_2 + a_3, -a_1 + 2a_2 + a_4, 3a_2 + a_3 + a_4)
\]

(a) Find the nullity and the rank of \( T \).
(b) Find bases for the null space and the range of \( T \).

**SOLUTION.**

(a) The null space is determined by the equations
\[
\begin{align*}
    a_1 + a_2 + a_3 &= 0, \\
    -a_1 + 2a_2 + a_4 &= 0, \\
    3a_2 + a_3 + a_4 &= 0
\end{align*}
\]
This gives
\[
\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
The solution is: \( a_3 = -a_1 - a_2, a_4 = a_1 - 2a_2 \), with \( a_1, a_2 \in \mathbb{R} \). This shows that nullity = 2. By the dimension theorem, we also derive that \( \text{rank}(T) = 2 \).

(b) From the equations of the nullspace, choosing the free parameters as \( (a_1, a_2) = (1, 0) \) and \( (a_1, a_2) = (0, a) \) we have that a basis for the null space is
\[
B = \{(1, 0, -1, 1), (0, 1, -1, -2)\}
\]
The range is determined by the equations
\[
\begin{align*}
    a_1 + a_2 + a_3 &= x_1, \\
    -a_1 + 2a_2 + a_4 &= x_2, \\
    3a_2 + a_3 + a_4 &= x_3
\end{align*}
\]
This gives
\[
\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]
Hence the range of \( T \) satisfies the condition \( x_3 - x_1 - x_2 = 0 \) A basis for the range is
\[
D = \{(1, 0, 1), (0, 1, 1)\}.
\]

(5) [5Pts] Let \( L = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \). Let \( G = \left\{ \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \). You can assume without proof that \( G \) spans \( M^{2\times 2} \). Find a subset \( H \subset G \) such that \( H \cup L \) spans \( M^{2\times 2} \). You need to justify that the set you build spans \( M^{2\times 2} \).
Note that the first two matrices of $G$ are already in the span of $L$ since
\[
\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}
\]
Because $G$ spans $M^{2 \times 2}$, and the first two matrices are already in the span of $L$, then we need to choose the other two matrices of $G$ to form the set $H$, that is $H = \{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \}$ so that $H \cup L$ spans $M^{2 \times 2}$.

(6) [5Pts] Let $V$ be a finite-dimensional vector space and $V_0$ be a proper subspace of $V$ (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

**SOLUTION.**

Suppose that $\dim V_0 = m$ and that $B = \{v_1, \ldots, v_m\}$ be a basis of $V_0$. Since $V_0 \subset V$, we also have that $S \subset V$.

Since $V_0 \neq V$ and $V_0 \subset V$, it follows that there is a vector $u \in V$ that is not in $\text{span}B$. Hence the set $E = \{u, v_1, \ldots, v_m\} \subset V$ is linearly independent. It follows that $\dim V \geq m + 1$. 
