MATH 4377/6308 - Advanced linear algebra I - Summer 2024
Homework 1

Exercises:

1. Let $A = \{1, 2, 5\}$, $B = \{4, 5\}$, $C = \{4, 6\}$. Explicitly write down the sets:
   
   \[ A \cup B, \quad A \cap (B \cup C), \quad B \cap (A \setminus B), \quad A \times C. \]

   **SOLUTION:**
   \[ A \cup B = \{1, 2, 4, 5\}, \quad A \cap (B \cup C) = \{5\}, \quad B \cap (A \setminus B) = \emptyset, \quad A \times C = \{(1, 4), (1, 6), (2, 4), (2, 6), (5, 4), (5, 6)\} \]

2. Let $x, y \in \mathbb{Z}$. Prove if the following relations are equivalence relations or not:

   a) $x \sim y$ if and only if $x - y < 10$.
   b) $x \sim y$ if and only if $x \cdot y \geq 0$.
   c) $x \sim y$ if and only if $x - y$ is even.

   **SOLUTION:**
   
   (a) No. Symmetry fails
   (b) No. Transitivity fails. There are $x, y, z \in \mathbb{Z}$ s.t. $x \cdot y \geq 0$ and $y \cdot z \geq 0$, but $x \cdot z < 0$
   (c) Yes. (i) $x - x$ is even; (ii) if $x - y = 2m$, then $y - x = 2(-m)$; (iii) if $x - y = 2m$ and if $y - z = 2n$, then $x - z = 2(m + n)$

3. Give an example of a set $A$ and a relation on $A$ which is reflexive and transitive but not symmetric.

   **SOLUTION:**
   
   $x, y \in \mathbb{Z}$, with $x \sim y$ if and only if $x \leq y$
   In this case, $x \leq x$ holds, $x \leq y$ and $y \leq z$ implies $x \leq z$. However $x \leq y$ does not imply $y \leq x$

4. Let $f : \{0, 1, 2, 3, 4\} \rightarrow \mathbb{N}$, $n \rightarrow n^3 + n$.
   
   a) Find domain, codomain, and range of $f$.
   b) Is $f$ one-to-one?
   c) Is $f$ onto?

   **SOLUTION:**
   
   (a) domain: $\{0, 1, 2, 3, 4\}$, codomain: $\mathbb{N}$, range: $\{0, 2, 2^2 + 2, 3^3 + 3+, 4^4 + 4\}$
   (b) yes. $n^3 + n = m^3 + m$ implies $n = m$
   (c) no. There is no $n$ in the domain of $f$ such that $f(n) = 1$

5. Let $f : [0, 2\pi] \rightarrow [-1, 1]$ be defined by $f(x) = \sin(x)$.
   
   a) Is $f$ one-to-one? Is $f$ onto?
   b) Find an interval $S$, such that $f|_S$ is both one-to-one and onto.
SOLUTION:
(a) \( f \) is not 1-1 since \( f(0) = f(\pi) \). \( f \) is onto.
(b) \( f \) one-to-one and onto in the interval \([\pi/2, 3\pi/2]\)

6. Let \( z = 1 + i2, w = 1 - i3 \). Write: \( z, z + w, zw, \frac{1}{w} \) in the form \( a + ib \). Finally write \( |z| \).

SOLUTION:
\( z = 1 - i2, z + w = 2 - i, zw = 7 - 3, \frac{1}{w} = \frac{1}{10}(1 + 3i), |z|^2 = 5, |z| = \sqrt{5} \)

7. Let \( x, y \in \mathbb{Z} \). Let \( x \sim y \) if and only if \( y + 4x \) is an integer multiple of 5. Prove that \( \sim \) is an equivalence relation.

SOLUTION:
(i) \( x + 4x = 5x \) is an integer multiple of 5; (ii) if \( y + 4x = 5m \), then \( y = 5m - 4x \); hence \( x + 4y = x + 20m - 16x = 20m - 15x = 5(4m - 3x) \) which is also a multiple of 5; (iii) if \( y + 4x = 5m \) and if \( z + 4y = 5n \), then (using these two equations to express \( z \) and \( 4x \)) \( z + 4x = (5n - 4y) + (5m - y) = 5(n + m) - 5y = 5(n + m - y) \), which is a multiple of 5.