Exercises:

(1) Consider the linear transformation:

\[ T : P_2(\mathbb{R}) \to P_3(\mathbb{R}), T(p(x)) = 2p'(x) + \int_0^x p(t)dt \]

Prove that \(T\) is one-to-one but not onto.

(2) Let \(T : \mathbb{R}^2 \to \mathbb{R}^2\) be given by

\[ T(a_1, a_2) = (a_1 + a_2, a_1 - a_2). \]

(a) Write \([T]_\beta^\gamma\) with \(\beta = \{(1, 0), (0, 1)\}\) and \(\gamma = \{(1, 0), (0, 1)\}\).

(b) Write \([T]_\tilde{\beta}^\tilde{\gamma}\) with \(\beta = \{(1, 0), (0, 1)\}\) and \(\tilde{\gamma} = \{(1, 2), (1, 1)\}\).

(3) Let \(T : P_1(\mathbb{R}) \to P_1(\mathbb{R})\) and \(U : P_1(\mathbb{R}) \to \mathbb{R}^2\) be the linear transformations defined by

\[ T(p(x)) = p'(x) + 2p(x), \quad U(a + bx) = (a + b, a) \]

Let \(\beta\) and \(\gamma\) be the standard ordered bases of \(P_1(\mathbb{R})\) and \(\mathbb{R}^2\), respectively. Find \([T]_\beta^\gamma\), \([U]_\beta^\gamma\) and \([U \circ T]_\beta^\gamma\).

(4) For the following pairs of vector spaces \(V\) and \(W\), define an explicit isomorphism or explain why no isomorphism exists between such spaces.

(a) \(V = \mathbb{R}^2, W = M^{1,1}\)

(b) \(V = \mathbb{R}^4, W = M^{2,2}\)

(c) \(V = \mathbb{R}^4, W = P_1(\mathbb{R})\)

(d) \(V = \mathbb{R}^4, W = P_3(\mathbb{R})\)

(e) \(V = \mathbb{R}^2, W = \mathbb{C}\) (space of complex numbers)

(5) Let \(\beta' = \{(3, 1), (2, 4)\}\), \(\beta = \{(1, 1), (1, -1)\}\). Find the change of coordinates matrix \(Q = [I_V]_{\beta'}^\beta\).