Exercises:

(1) Consider the linear transformation:

\[ T: P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R}), T(p(x)) = 2p'(x) + \int_0^x p(t)dt \]

Prove that \( T \) is one-to-one but not onto.

The map \( T \) cannot be onto because the dimension of the co-domain is larger than the dimension of the domain.

To show \( T \) is 1-1, we will show that \( N(T) = \{0\} \).

Let \( p(x) = a_0 + a_1x + a_2x^2 \in N(T) \), then

\[ 2p'(x) + \int_0^x p(t)dt = 2a_1 + 4a_2x + a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} = 0 \]

The equation above implies that \( a_0 = a_1 = a_2 = 0 \) so that \( N(T) = \{0\} \).

(2) Let \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be given by

\[ T(a_1, a_2) = (a_1 + a_2, a_1 - a_2). \]

(a) Write \( [T]_{\beta}^\gamma \) with \( \beta = \{(1, 0), (0, 1)\} \) and \( \gamma = \{(1, 0), (0, 1)\} \).

(b) Write \( [T]_{\tilde{\beta}}^\tilde{\gamma} \) with \( \beta = \{(1, 0), (0, 1)\} \) and \( \tilde{\gamma} = \{(1, 2), (1, 1)\} \).

(a) Let \( v_1 = (1, 0), v_2 = (0, 1) \). then

\[ T(v_1) = (1, 1) = 1(1, 0) + 1(0, 1), \rightarrow [T(v_1)]_\gamma = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ T(v_2) = (1, -1) = 1(1, 0) - 1(0, 1), \rightarrow [T(v_2)]_\gamma = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Hence \( [T]_{\beta}^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \)

(b) Let \( v_1 = (1, 0), v_2 = (0, 1) \). then

\[ T(v_1) = (1, 1) = 0(1, 2) + 1(1, 1), \rightarrow [T(v_1)]_{\tilde{\gamma}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ T(v_2) = (1, -1) = -2(1, 2) + 3(1, 1), \rightarrow [T(v_2)]_{\tilde{\gamma}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \]

Hence \( [T]_{\beta}^{\tilde{\gamma}} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \)

(3) Let \( T: P_1(\mathbb{R}) \rightarrow P_1(\mathbb{R}) \) and \( U: P_1(\mathbb{R}) \rightarrow \mathbb{R}^2 \) be the linear transformations defined by

\[ T(p(x)) = p'(x) + 2p(x), \quad U(a + bx) = (a + b, a) \]
Let $\beta$ and $\gamma$ be the standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$, respectively. Find $[T]_\beta$, $[U]_\beta^\gamma$ and $[U \circ T]_\beta^\gamma$.

**The standard ordered bases are** $\beta = \{1, x\}$, $\gamma = \{(1,0), (0,1)\}$. **We have**

\[
T(1) = 2 = 2(1) + 0(x) \rightarrow [T(v_1)]_\beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad T(x) = 1 + 2x = 1(1) + 2(x) \rightarrow [T(v_2)]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\]

**Hence** $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

Similarly, we have

\[
U(1) = (1,1) = 1(1,0) + 1(0,1) \rightarrow [U(v_1)]_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad U(x) = (1,0) = 1(1,0) + 0(0,1) \rightarrow [U(v_2)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

**Hence** $[U]^\gamma_\beta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

**It follows that** $[U \circ T]^\gamma_\beta = [U]^\gamma_\beta [T]_\beta = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

(4) For the following pairs of vector spaces $V$ and $W$, define an explicit isomorphism or explain why no isomorphism exists between such spaces.

(a) $V = \mathbb{R}^2$, $W = M^{1,1}$
(b) $V = \mathbb{R}^4$, $W = M^{2,2}$
(c) $V = \mathbb{R}^4$, $W = P_4(\mathbb{R})$
(d) $V = \mathbb{R}^4$, $W = P_3(\mathbb{R})$
(e) $V = \mathbb{R}^2$, $W = \mathbb{C}$ (space of complex numbers)

(a) **Since dim($V$) = 2 and dim($W$) = 1, there is no isomorphism between V and W.**

(b) **For $(a,b,c,d) \in \mathbb{R}^4$, let $T(a,b,c,d) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ This map is linear and invertible.**

(c) **Since dim($V$) = 4 and dim($W$) = 5, there is no isomorphism between V and W.**

(d) **For $(a,b,c,d) \in \mathbb{R}^4$, let $T(a,b,c,d) = a + bx + cx^2 + dx^3$. This map is linear and invertible.**

(e) **For $(a,b) \in \mathbb{R}^2$, let $T(a,b) = a + ib$. This map is linear and invertible.**

(5) Let $\beta' = \{(3,1), (2,4)\}$, $\beta = \{(1,1), (1,-1)\}$. Find the change of coordinates matrix $Q = [I_V]^\beta_{\beta'}$.

Let $\beta' = \{(3,1), (2,4)\} := \{v_1, v_2\}$. Hence

\[
I(v_1) = (3,1) = a_1(1,1) + a_2(1,-1) = (a_1 + a_2, a_1 - a_2)
\]

Solving the system, one finds $a_1 = 2, a_2 = 1$. **Similarly we have**

\[
I(v_2) = (2,4) = b_1(1,1) + b_2(1,-1) = (b_1 + b_2, b_1 - b_2)
\]

Solving the system, one finds $b_1 = 3, b_2 = -1$

**Thus, we have that** $Q = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$.