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Mini-Workshop: Shearlets

Organised by
Gitta Kutyniok, Osnabrück
Demetrio Labate, Houston

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Abstract. Over the last 20 years, multiscale methods and wavelets have revolutionized the field of applied mathematics by providing an efficient means for encoding isotropic phenomena. Directional multiscale systems, particularly shearlets, are now having the same dramatic impact on the encoding of multivariate signals. Since its introduction about five years ago, the theory of shearlets has rapidly developed and gained wide recognition as the superior way of achieving a truly unified treatment in both the continuum and digital setting. By now, shearlet analysis has reached maturity as a research field, with deep mathematical results, efficient numerical methods, and a variety of high-impact applications. The main goal of the Mini-Workshop *Shearlets* was to gather the world's experts in this field in order to foster closer interaction, attack challenging open problems, and identify future research directions.

Mathematics Subject Classification (2000): 42C15, 42C40.

Introduction by the Organisers
Shearlets: The First Five Years

The Mini-Workshop *Shearlets*, organized by Gitta Kutyniok (Osnabrück) and Demetrio Labate (Houston) was held October 4th–October 8th, 2010. This meeting was attended by 16 participants whose background ranged from the theory of group representations over approximation theory to image analysis. This unique selection provided the ideal setting for a vivid and fertile discussion of the theory and applications of shearlets, a novel multiscale approach particularly designed for multivariate problems.

Multivariate problems in applied mathematics are typically governed by anisotropic phenomena such as singularities concentrated on lower dimensional embedded manifolds or edges in digital images. Wavelets and multiscale methods, which were extensively exploited during the past 25 years for a wide range of both theoretical and applied problems, have been shown to be suboptimal for the encoding of anisotropic features. To overcome these limitations, several intriguing approaches such as ridgelets, contourlets, and curvelets, have since then been proposed, which all provide optimally sparse approximations of anisotropic features. Among those, shearlets are unique in encompassing the mathematical framework of affine systems and are, to date, the only approach capable of achieving a truly unified treatment in both the continuum and digital setting. This includes a precise mathematical analysis of sparse approximation properties in both settings as well as numerically efficient discrete transforms. Therefore shearlets are regarded as having the same potential impact on the encoding of multivariate signals as traditional wavelets did about 20 years ago for univariate problems.

Shearlet systems are designed to efficiently encode anisotropic features such as singularities concentrated on lower dimensional embedded manifolds. To achieve optimal sparsity, shearlets are scaled according to a parabolic scaling law encoded in the *parabolic scaling matrix* A_a, $a > 0$, and exhibit directionality by parameterizing slope encoded in the *shear matrix* S_s, $s \in \mathbb{R}$, defined by

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \qquad \text{and} \qquad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

respectively. Hence, shearlet systems are based on three parameters: $a > 0$ being the *scale parameter* measuring the resolution level, $s \in \mathbb{R}$ being the *shear parameter* measuring the directionality, and $t \in \mathbb{R}^2$ being the *translation parameter* measuring the position. This parameter space $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ can be endowed with the group operation

$$(a, s, t) \cdot (a', s', t') = (aa', s + s'\sqrt{a}, t + S_s A_a t'),$$

leading to the so-called *shearlet group* $\mathbb{S}$, which can be regarded as a special case of the general affine group. The *continuous shearlet systems* arise from the unitary group representation

$$\sigma : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^2)), \quad (\sigma(a, s, t)\psi)(x) = a^{-3/4}\psi(A_a^{-1} S_s^{-1}(x - t))$$

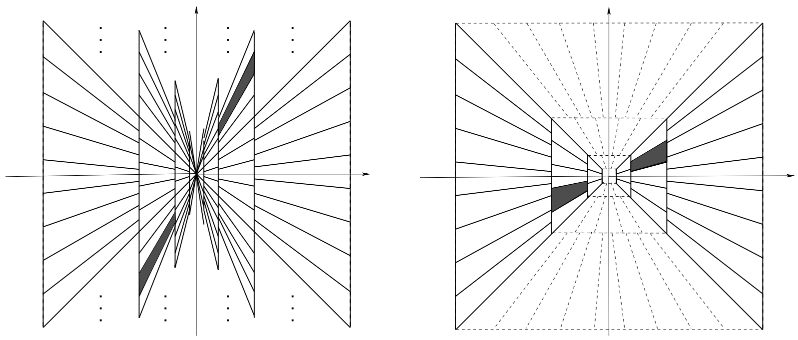

FIGURE 1. Left to Right: Frequency tiling of a discrete shearlet system; Frequency tiling of a cone-adapted discrete shearlet system.

and are defined by

$$\{\psi_{a,s,t} = \sigma(a,s,t)\psi = a^{-3/4}\psi(A_a^{-1}S_s^{-1}(\,\cdot\, - t)) : (a,s,t) \in \mathbb{S}\}.$$

For appropriate choices of the *shearlet* $\psi \in L^2(\mathbb{R}^2)$, the *Continuous Shearlet Transform*

$$\mathcal{SH}_\psi : f \to \mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{ast} \rangle,$$

is a linear isometry from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{S})$. Alternatively, rather than defining the shearing parameter s on $\mathbb{R}$, the domain can be restricted to, say, $|s| \leq 1$. This gives rise to the so-called *Cone-adapted Continuous Shearlet Transform*, which allows an equal treatment of all directions in contrast to a slightly biased treatment by the Continuous Shearlet Transform. In fact, it could be proven that the Cone-adapted Continuous Shearlet Transform resolves the wavefront set of distributions and can be applied to precisely characterize edges in images. Notice that, although directions are treated slightly biased, the Continuous Shearlet Transform has the advantage of being equipped with a simpler mathematical structure. This allows the application of group theoretic methodologies to, for instance, discretize the set of parameters through coorbit theory.

Discrete shearlet system are obtained by appropriate sampling of the continuous shearlet systems presented above. Specifically, for $\psi \in L^2(\mathbb{R}^2)$, a *(discrete) shearlet system* is a collection of functions of the form

$$(1) \qquad \{\psi_{j,k,m} = 2^{3j/4}\psi(S_k A_{2^j}\,\cdot - m) : j \in \mathbb{Z}, k \in K \subset \mathbb{Z}, m \in \mathbb{Z}^2\},$$

where K is a carefully chosen indexing set of shears. Notice that the shearing matrix S_k maps the digital grid $\mathbb{Z}^2$ onto itself, which is the key idea for deriving a unified treatment of the continuum and digital setting. The discrete shearlet system defines a collection of waveforms at various scales j, orientations controlled by k, and locations dependent on m. In particular, if $K = \mathbb{Z}$ in (1), the shearlet system contains elements oriented along all possible slopes as illustrated in Figure 1. This particular choice is in accordance with the continuous shearlet systems generated by a group action. To avoid the already mentioned biased treatment of directions which the discrete systems inherit, the *cone-adapted discrete shearlet systems* were introduced as

$$\{\phi(\cdot - m) : m \in \mathbb{Z}^2\} \cup \{\psi_{j,k,m}, \tilde{\psi}_{j,k,m} : j \geq 0, |k| \leq \lceil 2^{j/2} \rceil, m \in \mathbb{Z}^2\},$$

where $\tilde{\psi}_{j,k,m}$ is generated from $\psi_{j,k,m}$ by interchanging both variables, and $\psi, \tilde{\psi}$, and ϕ are L^2 functions. Figure 1 illustrates a typical frequency tiling associated with a cone-adapted shearlet system. A suitable choice of the shearlets ψ and $\tilde{\psi}$ generates well-localized shearlet systems which form frames or even Parseval frames.

Over the last years, an abundance of results on the theory and applications of shearlets have been derived by a constantly growing community of researchers. One main goal of this workshop was to discuss the state of the art of this vivid research area. The talks which were delivered by the participants covered the following topics:

(1) *(Cone-adapted) Continuous Shearlet Systems.* Novel results for continuous shearlet systems were presented by S. Dahlke and G. Teschke, who exploited their group structure through a coorbit theory approach to derive feasible discretization of the shearlet parameters as well as associated function spaces even for the 3D setting. F. DeMari's talk then revealed intriguing properties of the set of groups the shearlet group belongs to. Focusing on cone-adapted continuous shearlet systems instead, the microlocal properties of such systems were presented by P. Grohs, and their application to the characterization of edges for 2D and 3D data was discussed by K. Guo.

(2) *(Cone-adapted) Discrete Shearlet Systems.* Recently, compactly supported discrete shearlet systems which provide optimally sparse approximations of anisotropic features were introduced for both 2D and 3D signals to allow superior spatial localization. These novel results were presented by J. Lemvig and W. Lim.

(3) *Numerical Implementations and Applications.* Different efficient numerical implementations of the shearlet transform have been proposed in the past, but further improvements are desirable to achieve additional computational efficiency and features such as locality. W. Lim presented a new fast shearlet transform in his talk which is extremely competitive for applications such as denoising and data separation. A subdivision approach towards a shearlet multiresolution analysis with associated fast decomposition algorithm was discussed by T. Sauer. G. Easley and V. Patel then showed that the shearlet approach is extremely competitive in a wide range of applications from signal and image processing including edge detection, halftoning and image deconvolution.

A further main objective of the workshop was to foster interaction in order to attack a number of open problems and identify future directions of this area of research. During our discussions, the following topics and problems have emerged as main themes to be investigated within the next five years:

- *Shearlet Smoothness Spaces.* For problems arising in the theory of partial differential equations and in approximation theory, it is essential to precisely understand the nature of the spaces defined using shearlets as building blocks and their relation to classical function spaces.
- *Shearlet Constructions and Applications in 3D.* While the theory of shearlets is well understood in the bivariate case, the extension to higher dimensions is still far from being complete. Open problems in this direction include, in particular, the analysis of corner and irregular surface points using shearlets.
- *Construction of "good" Shearlet Systems.* Several results are known, by now, for compactly supported shearlet systems, which though do not form tight frames. Thus, it would be highly desirable to construct well localized shearlet systems which are compactly supported, form a tight frame or even an orthonormal system, and provide provably optimal sparse approximations of anisotropic features.
- *Numerical Implementations.* Starting with the bivariate situation, one main goal is to derive a complete analog of the fast wavelet transform in the sense of a fast algorithm with associated multiresolution structure paralleling the continuum setting. Furthermore, as the theory for sparse 3D shearlet representations is emerging, numerical implementations for the trivariate case are also in demand. The higher complexity of such data poses a particular difficulty.

The organizers:

G. Kutyniok and D. Labate