Microlocal analysis of singularities from directional multiscale representations

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Abstract The classical wavelet transform is a remarkably effective tool for the analysis of pointwise regularity of functions and distributions. During the last decade, the emergence of a new generation of multiscale representations has extended the classical wavelet approach leading to the introduction of a class of generalized wavelet transforms - most notably the shearlet transform – which offer a much more powerful framework for microlocal analysis. In this paper, we show that the shearlet transform enables a precise geometric characterization of the set of singularities of a large class of multidimensional functions and distributions, going far beyond the capabilities of the classical wavelet transform. This paper generalizes and extends several results that previously appeared in the literature and provides the theoretical underpinning for advanced applications from image processing and pattern recognition including edge detection, shape classification and feature extraction.

1 Introduction

How do you detect the location of a jump discontinuity in a function? One possible approach consists in using as probes a collection of well-localized functions of the form $\psi_{a,t}(x) = a^{-n/2}\psi(a^{-1}(x-t))$, $a > 0$, $t \in \mathbb{R}^n$, where $\psi \in L^2(\mathbb{R}^n)$. We assume that ψ is chosen such that $\hat\psi \in C_c^\infty(\mathbb{R}^n)$, with $0 \notin \operatorname{supp} \hat\psi$. Since ψ has rapid decay

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in space-domain, the functions $\psi_{a,t}$ are mostly concentrated around t, with the size of the essential support controlled by the scaling parameter a. We can then analyze the local regularity of a function or distribution f via the mapping

$$f \to \langle f, \psi_{a,t} \rangle, \quad a > 0, t \in \mathbb{R}^n.$$

To illustrate this approach, let us consider as a prototype of a jump discontinuity the one-dimensional Heaviside function $f(x) = 1$ if $x \geq 0$ and $f(x) = 0$ otherwise. Using Plancherel theorem and the distributional Fourier transform of f, a direct calculation using the analyzing functions $\psi_{a,t}$ with $n = 1$ shows that [1]

$$\begin{aligned}
\langle f, \psi_{a,t} \rangle &= \langle \hat{f}, \hat{\psi}_{a,t} \rangle \\
&= \sqrt{a} \int_{\mathbb{R}} \hat{f}(\xi)\, \overline{\hat{\psi}(a\xi)}\, e^{-2\pi i \xi t}\, d\xi \\
&= \sqrt{a} \int_{\mathbb{R}} \frac{1}{2\pi i \xi}\, \overline{\hat{\psi}(a\xi)}\, e^{-2\pi i \xi t}\, d\xi \\
&= \sqrt{a} \int_{\mathbb{R}} \hat{\gamma}(\eta)\, e^{-2\pi i \eta \frac{t}{a}}\, d\eta,
\end{aligned}$$

where $\hat{\gamma}(\eta) = \frac{1}{2\pi i \eta}\, \overline{\hat{\psi}(\eta)}$. If $t = 0$, the calculation above shows that $|\langle f, \psi_{a,t} \rangle| \approx \sqrt{a}$, provided that $\int \hat{\gamma}(\eta) d\eta \neq 0$. On the other hand, if $t \neq 0$, an application of the Inverse Fourier Transform theorem yields that $\langle f, \psi_{a,t} \rangle = \sqrt{a}\, \gamma(-t/a)$. Since $\hat{\gamma} \in C_c^\infty(\mathbb{R})$, γ has rapid decay in space-domain, implying that $\langle f, \psi_{a,t} \rangle$ decays rapidly to 0, as $a \to 0$; that is, for any $N \in \mathbb{N}$, there is a constant $C_N > 0$ such that $|\langle f, \psi_{a,t} \rangle| \leq C_N a^N$, as $a \to 0$.

In summary, *the elements $\langle f, \psi_{a,t} \rangle$ exhibit rapid asymptotic decay, as $a \to 0$, for all $t \in \mathbb{R}$ except at the location of the singularity $t = 0$, where $\langle f, \psi_{a,t} \rangle$ behaves as $O(\sqrt{a})$.*

The mapping $f \to \langle f, \psi_{a,t} \rangle$ is the classical *continuous wavelet transform* and this simple example illustrates its ability to detect local regularity information of functions and distributions through its asymptotic decay at fine scales (cf.[16, 17, 18, 22]).

The generalization of the example above to higher dimensions is straightforward. Let us consider the two-dimensional Heaviside function $H(x_1,x_2) = \chi_{\{x_1>0\}}(x_1,x_2)$ and let us proceed as in the example above. Using the analyzing functions $\psi_{a,t}$ with $n = 2$ and denoting $t = (t_1,t_2) \in \mathbb{R}^2$ we have:

$$\begin{aligned}
\langle H, \psi_{a,t} \rangle &= \langle \hat{H}, \hat{\psi}_{a,t} \rangle \\
&= a \int_{\mathbb{R}^2} \hat{H}(\xi_1,\xi_2)\, \overline{\hat{\psi}(a\xi_1,a\xi_2)}\, e^{-2\pi i(\xi_1 t_1+\xi_2 t_2)}\, d\xi_1\, d\xi_2 \\
&= a \int_{\mathbb{R}^2} \frac{\delta(\xi_2)}{2\pi i \xi_1}\, \overline{\hat{\psi}(a\xi_1,a\xi_2)}\, e^{-2\pi i(\xi_1 t_1+\xi_2 t_2)}\, d\xi_1\, d\xi_2
\end{aligned}$$

[1] Note that the distributional Fourier transform of f is $\hat{f}(\xi) = \frac{1}{2}\delta(\xi) + \frac{1}{2\pi i}\text{p.v.}\frac{1}{\xi}$, but the term $\frac{1}{2}\delta(\xi)$ gives no contribution in the computation for $\langle f, \psi_{a,t} \rangle$ since $\hat{\psi}(0) = 0$.

$$= a \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1} \overline{\hat{\psi}(a\xi_1, 0)}\, e^{-2\pi i \xi_1 t_1}\, d\xi_1$$
$$= a \int_{\mathbb{R}} \hat{\gamma}(\eta)\, e^{-2\pi i \eta \frac{t_1}{a}}\, d\eta,$$

where $\hat{\gamma}(\eta) = \frac{1}{2\pi i \eta}\, \overline{\hat{\psi}(\eta, 0)}$. A similar argument to the one above shows that the elements $\langle H, \psi_{a,t} \rangle$ exhibit rapid asymptotic decay, as $a \to 0$, at all $t \in \mathbb{R}^2$ except at the location of the singularity $t_1 = 0$, where $\langle H, \psi_{a,t} \rangle$ behaves as $O(a)$, provided that $\int \hat{\gamma}(\eta) d\eta \neq 0$.

However, even though the continuous wavelet transform is able to identify the location of the singularities also in this case, the result of this second example is not completely satisfactory since it provides no information about the orientation of the singularity line. In dimensions larger than one, when the singularity points are supported on a curve or on a higher dimensional manifold, it is useful not only to detect the singularity location but also to capture its *geometry*, such as the orientation of a discontinuity curve or boundary.

As a matter of fact, it is possible to overcome this limitation by introducing generalized versions of the continuous wavelet transform which are more capable of dealing with directional information. The idea of considering generalized (discrete or continuous) wavelet transforms with improved directional capabilities has a long history, going back to the steerable filters [8, 23] introduced for the analysis of discrete data and to the notion of directional wavelets [1]. More recently, starting with the introduction of ridgelets [2] and curvelets [3, 4], a new generation of more flexible and powerful multiscale transforms has emerged, which has led to several successful discrete applications in signal and image processing. Among such more recent generalizations of the wavelet transform, the shearlet transform [9, 20] is especially remarkable since it combines a simple mathematical structure which is derived from the general framework of affine systems together with a special ability to capture the geometry of the singularity sets of multidimensional functions and distributions. For example, in the case of the two-dimensional Heaviside function, the continuous shearlet transform is able to determine both the location and the orientation of the discontinuity line. More generally, by extending and generalizing several results derived previously by two of the authors, in this paper we show that the continuous shearlet transform provides a precise geometric description of the set of discontinuities of a large class of multivariate functions and distributions. These results provide the theoretical underpinning for improved algorithms for image analysis and feature extraction, cf. [25].

The rest of the paper is organize as follows. In Section 2, we recall the definition of the continuous shearlet transform; in Section 3, we present the shearlet analysis of jump discontinuities in the two-dimensional case; in Section 4 we illustrate the generalization of the shearlet approach to other types of singularities.

2 The continuous shearlet transform

To define the continuous shearlet transform, we recall first the definition of the 'generalized' continuous wavelet transform associated with the affine group on $\mathbb{R}^n$.

2.1 Wavelet transforms

The *affine group* $\mathscr{A}$ on $\mathbb{R}^n$ consists of the pairs $(M,t) \in GL_n(\mathbb{R}) \times \mathbb{R}^n$, with group operation $(M,t)\cdot(M',t') = (MM', t+Mt')$. The *affine systems* generated by $\psi \in L^2(\mathbb{R}^n)$ are obtained from the action of the quasi regular representation of $\mathscr{A}$ on ψ and are the collections of functions of the form

$$\left\{ \psi_{M,t}(x) = |\det M|^{-\frac{1}{2}}\, \psi(M^{-1}(x-t)) :\ (M,t) \in \mathscr{A} \right\}.$$

Let $\Lambda = \{(M,t) : M \in G, t \in \mathbb{R}^n\} \subset \mathscr{A}$, where G is a subset of $GL_n(\mathbb{R})$. If there is an *admissible* function $\psi \in L^2(\mathbb{R}^n)$ such that any $f \in L^2(\mathbb{R}^n)$ can be recovered via the reproducing formula

$$f = \int_{\mathbb{R}^n} \int_G \langle f, \psi_{M,t} \rangle\, \psi_{M,t}\, d\lambda(M)\, dt,$$

where λ is a measure on G, then such ψ is a *continuous wavelet* associated with Λ and the mapping

$$f \to \mathscr{W}_\psi f(M,t) = \langle f, \psi_{M,t} \rangle, \quad (M,t) \in \Lambda,$$

is the *continuous wavelet transform* with respect to Λ. Depending on the choice of G and ψ, there is a variety of continuous wavelet transforms [21, 24]. The simplest case is $G = \{aI : a > 0\}$, where I is the identity matrix. In this situation, we obtain the classical continuous wavelet transform

$$\mathscr{W}_\psi f(a,t) = a^{-n/2} \int_{\mathbb{R}^n} f(x)\, a^{-1}\, \overline{\psi(a^{-1}(x-t))}\, dx,$$

which was used in the Section 1 for $n = 1,2$. Note that, in this case, the dilation group G is isotropic since the dilation factor a acts in the same way for each coordinate direction. It is reasonable to expect that, by choosing more general dilation groups G, one obtains wavelet transforms with more interesting geometric properties.

2.2 The shearlet transform

The continuous shearlet transform is the continuous wavelet transform associated with a special subgroup $\mathscr{A}_S$ of $\mathscr{A}$ called the *shearlet group* (cf. [6, 7, 19, 20]). For a fixed $\beta = (\beta_1, \ldots, \beta_{n-1})$, where $0 < \beta_i < 1$, $1 \le i < n-1$, $\mathscr{A}_S$ consists of the elements (M_{as}, t), where

$$M_{as} = \begin{pmatrix} a & -a^{\beta_1} s_1 & \ldots & -a^{\beta_{n-1}} s_{n-1} \\ 0 & a^{\beta_1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a^{\beta_{n-1}} \end{pmatrix},$$

$a > 0$, $s = (s_1, \ldots, s_{n-1}) \in \mathbb{R}^{n-1}$, and $t \in \mathbb{R}^n$. Note that each matrix M_{as} is the product of the matrices $B_s A_a$, where

$$A_a = \begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & a^{\beta_1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & a^{\beta_{n-1}} \end{pmatrix}, \quad B_s = \begin{pmatrix} 1 & -s_1 & \ldots & -s_{n-1} \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

where A_a is an anisotropic dilation matrix and B_s is non-expanding matrix called a *shear matrix*. Hence, for an appropriate admissible function $\psi \in L^2(\mathbb{R}^n)$ and $\beta = (\beta_1, \ldots, \beta_{n-1})$, where $0 < \beta_i < 1$, the continuous shearlet transform is the mapping

$$f \to \langle f, \psi_{M_{as},t} \rangle, \quad (M_{as}, t) \in \mathscr{A}_S.$$

The analyzing elements $\psi_{M_{as},t}$ are called *shearlets* and are the affine functions

$$\psi_{M_{as},t}(x) = |\det M_{as}|^{-\frac{1}{2}} \psi(M_{as}^{-1}(x-t)).$$

In the following we will show that, thanks the geometric and analytic properties of shearlets, the continuous shearlet transform enables a very precise description of jump discontinuities of functions of several variables. For example, if $f = \chi_S$, where $S \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded region with piecewise smooth boundary, the continuous shearlet transform provides a characterization of the location and orientation of the boundary set through its asymptotic decay at fine scales.

2.3 *The shearlet transform* $(n = 2)$

Before applying the shearlet framework in dimensions $n = 2$, we need to specify the definition of the continuous shearlet transform that will be needed for our analysis.

For appropriate admissible functions $\psi^{(h)}, \psi^{(v)} \in L^2(\mathbb{R}^2)$, a fixed $0 < \beta < 1$ and matrices

$$M_{as} = \begin{pmatrix} a & -a^{\beta} s \\ 0 & a^{\beta} \end{pmatrix}, \quad N_{as} = \begin{pmatrix} a^{\beta} & 0 \\ -a^{\beta} s & a \end{pmatrix},$$

we define the *horizontal* and *vertical (continuous) shearlets* by

$$\psi^{(h)}_{a,s,t}(x) = |\det M_{as}|^{-\frac{1}{2}} \psi^{(h)}(M_{as}^{-1}(x-t)), \quad a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2,$$

and

$$\psi^{(v)}_{a,s,t}(x) = |\det N_{as}|^{-\frac{1}{2}}\,\psi^{(v)}(N_{as}^{-1}(x-t)), \quad a>0, s\in\mathbb{R}, t\in\mathbb{R}^2,$$

respectively. To ensure a more uniform covering of the range of directions through the shearing variable s, rather than using a single shearlet system where s range over $\mathbb{R}$, it will be convenient to use the two systems of shearlets defined above and let s range over a bounded interval.

To define our admissible functions $\psi^{(h)}, \psi^{(v)}$, for $\xi = (\xi_1,\xi_2)\in\mathbb{R}^2$ let

$$\hat{\psi}^{(h)}(\xi_1,\xi_2) = \hat{\psi}_1(\xi_1)\,\hat{\psi}_2(\tfrac{\xi_2}{\xi_1}), \quad \hat{\psi}^{(v)}(\xi_1,\xi_2) = \hat{\psi}_1(\xi_2)\,\hat{\psi}_2(\tfrac{\xi_1}{\xi_2}), \tag{1}$$

where

$$\int_0^\infty |\hat{\psi}_1(a\omega)|^2 \frac{da}{a} = 1, \text{ for a.e. } \omega\in\mathbb{R}, \text{ and } \operatorname{supp}\hat{\psi}_1 \subset [-2,-\tfrac{1}{2}]\cup[\tfrac{1}{2},2]; \tag{2}$$

$$\|\psi_2\|_2 = 1 \text{ and } \operatorname{supp}\hat{\psi}_2 \subset [-\tfrac{\sqrt{2}}{4},\tfrac{\sqrt{2}}{4}]. \tag{3}$$

Observe that, in the frequency domain, a shearlet $\psi^{(h)}_{a,s,t}$ has the form:

$$\hat{\psi}^{(h)}_{a,s,t}(\xi_1,\xi_2) = a^{\frac{1+\beta}{2}}\,\hat{\psi}_1(a\,\xi_1)\,\hat{\psi}_2(a^{\beta-1}(\tfrac{\xi_2}{\xi_1}-s))\,e^{-2\pi i\xi\cdot t}.$$

This shows each function $\hat{\psi}^{(h)}_{a,s,t}$ has support:

$$\operatorname{supp}\hat{\psi}^{(h)}_{a,s,t} \subset \{(\xi_1,\xi_2) : \xi_1\in[-\tfrac{2}{a},-\tfrac{1}{2a}]\cup[\tfrac{1}{2a},\tfrac{2}{a}], |\tfrac{\xi_2}{\xi_1}-s| \le a^{1-\beta}\}.$$

That is, its frequency support is a pair of trapezoids, symmetric with respect to the origin, oriented along a line of slope s. The support becomes increasingly thin as $a\to 0$. This is illustrated in Figure 1. The shearlets $\psi^{(v)}_{a,s,t}$ have similar properties, with frequency supports orientated along lines of slopes $\frac{1}{s}$.

For $0<a<\frac{1}{4}$ and $|s|\le\frac{3}{2}$, each system of continuous shearlets spans a subspace of $L^2(\mathbb{R}^2)$ consisting of functions having frequency supports in one of the horizontal or vertical cones defined in the frequency domain by

$$\mathscr{P}^{(h)} = \{(\xi_1,\xi_2)\in\mathbb{R}^2 : |\xi_1|\ge 2 \text{ and } |\tfrac{\xi_2}{\xi_1}|\le 1\}$$
$$\mathscr{P}^{(v)} = \{(\xi_1,\xi_2)\in\mathbb{R}^2 : |\xi_1|\ge 2 \text{ and } |\tfrac{\xi_2}{\xi_1}|> 1\}.$$

More precisely, the following proposition, which is a generalization of a result in [19], shows that the horizontal and vertical shearlets form a continuous reproducing system for the spaces of L^2 functions whose frequency support is contained in $\mathscr{P}^{(h)}$ and $\mathscr{P}^{(h)}$, respectively.

Proposition 1. *Let $\psi^{(h)}$ and $\psi^{(v)}$ be given by* (1) *with $\hat{\psi}_1$ and $\hat{\psi}_2$ satisfying* (2) *and* (3)*, respectively. Let*

$$L^2(\mathscr{P}^{(h)})^\vee = \{f\in L^2(\mathbb{R}^2) : \operatorname{supp}\hat{f}\subset\mathscr{P}^{(h)}\},$$

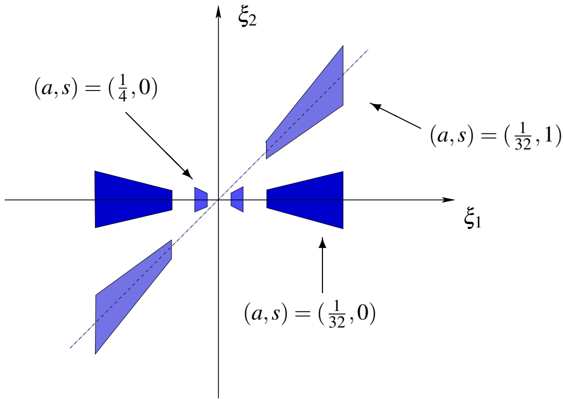

Fig. 1 Supports of the shearlets $\hat{\psi}^{(h)}_{ast}$ (in the frequency domain) for different values of a and s.

with a similar definition for $L^2(\mathscr{P}^{(v)})^\vee$. We have the following.

(i) For all $f \in L^2(\mathscr{P}^{(h)})^\vee$,

$$f = \int_{\mathbb{R}^2}\int_{-2}^{2}\int_{0}^{\frac{1}{4}} \langle f, \psi^{(h)}_{a,s,t}\rangle\, \psi^{(h)}_{a,s,t}\, \frac{da}{a^3}\, ds\, dt.$$

(ii) For all $f \in L^2(\mathscr{P}^{(v)})^\vee$,

$$f = \int_{\mathbb{R}^2}\int_{-2}^{2}\int_{0}^{\frac{1}{4}} \langle f, \psi^{(v)}_{a,s,t}\rangle\, \psi^{(v)}_{a,s,t}\, \frac{da}{a^3}\, ds\, dt.$$

The equalities are understood in the L^2 sense.

Note that $\frac{da}{a^3}\, ds\, dt$ is the left Haar measure of the shearlet group $\mathscr{A}_S$.

Using the horizontal and vertical shearlets, we define the *(fine-scale) continuous shearlet transform* on $L^2(\mathbb{R}^2)$ as the mapping

$$f \in L^2(\mathbb{R}^2 \setminus [-2,2]^2)^\vee \to \mathscr{SH}_\psi f(a,s,t), \quad a \in (0,\tfrac{1}{4}], s \in [-\infty,\infty], t \in \mathbb{R}^2,$$

given by

$$\mathscr{SH}_\psi f(a,s,t) = \begin{cases} \mathscr{SH}^{(h)}_\psi f(a,s,t) = \langle f, \psi^{(h)}_{a,s,t}\rangle, & \text{if } |s| \le 1 \\ \mathscr{SH}^{(v)}_\psi f(a,\frac{1}{s},t) = \langle f, \psi^{(v)}_{a,s,t}\rangle, & \text{if } |s| > 1. \end{cases}$$

In this expression, it is understood that the limit value $s = \pm\infty$ is defined and that $\mathscr{SH}_\psi f(a,\pm\infty,t) = \mathscr{SH}^{(v)}_\psi f(a,0,t)$.

The term *fine-scale* refers to the fact that this shearlet transform is only defined for the scale variable $a \in (0,1/4]$, corresponding to “fine scales”. In fact, as it is clear from Proposition 1, the shearlet transform $\mathscr{SH}_\psi f$ defines an isometry on

$L^2(\mathbb{R}^2 \setminus [-2,2]^2)^\vee$, the subspace of $L^2(\mathbb{R}^2)$ of functions with frequency support away from $[-2,2]^2$, but not on $L^2(\mathbb{R}^2)$. This is not a limitation since our method for the geometric characterization of singularities will require to derive asymptotic estimates as a approaches 0.

3 Shearlet analysis of jump discontinuities in dimension $n = 2$

To introduce the main ideas associated with the shearlet-based analysis of singularities, let us examine first the two-dimensional Heaviside function which was considered in Section 1. Using Plancherel theorem and denoting $t = (t_1, t_2) \in \mathbb{R}^2$, when $|s| < 1$ we have

$$\begin{aligned}
\mathscr{S}\mathscr{H}_\psi H(a,s,t) &= \langle H, \psi^{(h)}_{a,s,t} \rangle \\
&= \int_{\mathbb{R}^2} \hat{H}(\xi_1,\xi_2)\, \overline{\hat{\psi}^{(h)}_{a,s,t}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\
&= \int_{\mathbb{R}^2} \frac{\delta_2(\xi_1,\xi_2)}{2\pi i \xi_1}\, \overline{\hat{\psi}^{(h)}_{a,s,t}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\
&= \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1}\, \overline{\hat{\psi}_{a,s,t}}(\xi_1, 0)\, d\xi_1 \\
&= a^{\frac{1+\beta}{2}} \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1}\, \overline{\hat{\psi}_1}(a\,\xi_1)\, \overline{\hat{\psi}_2}(a^{\beta-1}s)\, e^{2\pi i \xi_1 t_1}\, d\xi_1 \\
&= a^{\frac{1+\beta}{2}}\, \overline{\hat{\psi}_2}(a^{\beta-1}s) \int_{\mathbb{R}} \hat{\gamma}(\eta)\, e^{2\pi i \eta \frac{t_1}{a}}\, d\eta,
\end{aligned}$$

where $\hat{\gamma}(\eta) = \frac{1}{2\pi i \eta}\overline{\hat{\psi}_1}(\eta)$. Hence, using the same argument from the introduction, under the assumption that $\hat{\psi}_1 \in C_c^\infty(\mathbb{R})$ we have that $\mathscr{S}\mathscr{H}_\psi H(a,s,t)$ exhibits rapid asymptotic decay, as $a \to 0$, for all $(t_1,t_2) \in \mathbb{R}^2$ when $t_1 \neq 0$. If $t_1 = 0$ and $s \neq 0$, the term $\overline{\hat{\psi}_2}(a^{\beta-1}s)$ will vanish as $a \to 0$, due to the support assumptions on $\hat{\psi}_2$. Finally, if $t_1 = 0$ and $s = 0$, we have that

$$\mathscr{S}\mathscr{H}_\psi H(a,0,(0,t_2)) = a^{\frac{1+\beta}{2}}\, \overline{\hat{\psi}_2}(0) \int_{\mathbb{R}} \hat{\gamma}(\eta)\, d\eta.$$

Hence, provided that $\hat{\psi}_2(0) \neq 0$ and $\int_{\mathbb{R}} \hat{\gamma}(\eta)\, d\eta \neq 0$, we have the estimate

$$\mathscr{S}\mathscr{H}_\psi H(a,0,(0,t_2)) = O(a^{\frac{1+\beta}{2}}).$$

A similar computation shows that $\mathscr{S}\mathscr{H}_\psi H(a,s,t)$ exhibits rapid asymptotic decay, as $a \to 0$, for all $|s| > 1$. In summary, under appropriate assumptions on ψ_1 and ψ_2, the continuous shearlet transform of H decays rapidly, asymptotically for $a \to 0$, for all t and s, unless t is on the discontinuous line and s corresponds to the normal direction of the discontinuous line at t.

The same properties of the continuous shearlet transform observed on the two-dimensional Heaviside function can be extended to any function of the form $f = \chi_S$ where $S \subset \mathbb{R}^2$ is a compact region whose boundary, denoted by ∂S, is a simple piecewise smooth curve, of finite length L. To define the normal orientation to the boundary curve ∂S, let $\alpha(t)$, $0 \le t \le L$ be a parametrization of ∂S. Let $p_0 = \alpha(t_0)$ and let $s_0 = \tan(\theta_0)$ with $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We say that s_0 *corresponds to the normal direction of* ∂S *at* p_0 if $(\cos\theta_0, \sin\theta_0) = \pm\mathbf{n}(t_0)$.

The following theorem generalizes a result proved originally in [10] for the special case $\beta = \frac{1}{2}$.

Theorem 1. *Let* ψ_1, ψ_2 *be chosen such that*

- $\hat{\psi}_1 \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\hat{\psi}_1 \subset [-2,-\frac{1}{2}] \cup [\frac{1}{2}, 2]$, *is odd, nonnegative on* $[\frac{1}{2}, 2]$ *and it satisfies* $\int_0^\infty |\hat{\psi}_1(a\xi)|^2 \frac{da}{a} = 1$, *for a.e.* $\xi \in \mathbb{R}$; (4)
- $\hat{\psi}_2 \in C_c^\infty(\mathbb{R})$, $\operatorname{supp}\hat{\psi}_2 \subset [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]$, *is even, nonnegative, decreasing in* $[0, \frac{\sqrt{2}}{4})$, *and* $\|\psi_2\|_2 = 1$. (5)

Let $\frac{1}{3} < \beta < 1$. *For* $B = \chi_S$, *where* $S \subset \mathbb{R}^2$ *is a compact set whose boundary* ∂S *is a simple piecewise smooth curve, the following holds.*

(i) If $p \notin \partial S$ *then, for all* $s \in \mathbb{R}$,

$$\lim_{a \to 0^+} a^{-N}\, \mathscr{SH}_\psi B(a,s,p) = 0, \quad \textit{for all } N > 0.$$

(ii) If $p_0 \in \partial S$ *is a regular point,* s_0 *corresponds to the normal direction of* ∂S *at* p_0 *and* $s \neq s_0$, *then*

$$\lim_{a \to 0^+} a^{-N}\, \mathscr{SH}_\psi B(a,s,p_0) = 0, \quad \textit{for all } N > 0.$$

(iii) If $p_0 \in \partial S$ *is a regular point,* s_0 *corresponds to the normal direction of* ∂S *at* p_0 *and* $s = s_0$, *then*

$$\infty > \lim_{a \to 0^+} a^{-\frac{1+\beta}{2}}\, \mathscr{SH}_\psi B(a,s_0,p_0) \neq 0.$$

That is, if $p_0 \in \partial S$, the continuous shearlet transform decays rapidly, asymptotically for $a \to 0$, unless $s = s_0$ corresponds to the normal direction of ∂S at p_0, in which case

$$\mathscr{SH}_\psi B(a,s_0,p_0) = O(a^{\frac{1+\beta}{2}}), \text{ as } a \to 0.$$

Theorem 1 generalizes to the case of functions of the form $f = \chi_S$ where $S \subset \mathbb{R}^2$ and the boundary curve ∂S contains corner points. In this case, if p_0 is a corner point and s corresponds to one of the normal directions of ∂S at p_0, then the continuous shearlet transform has a decay rate of order $O(a^{\frac{1+\beta}{2}})$, as $a \to 0$, similar to the situation of regular points. For other values of s, however, the asymptotic decay rate depends both on the tangent and the curvature at p_0 (cf. [10]).

Theorems 1 was originally proved in [10] for the case $\beta = 1/2$ and its proof was successively simplified ans streamlined in [13]. In the following section, we sketch the main ideas of the proof, highlighting how to extend the proof from [13] to the case $\beta \neq 1/2$.

3.1 Proof of Theorems 1 (sketch)

The argument used for the two-dimensional Heaviside function cannot be extended to this case directly since this would require an explicit expression of the Fourier transform of the function $B = \chi_S$. Instead, we can apply the divergence theorem which allows us to express the Fourier transform of B as a line integral over ∂S:

$$\begin{aligned} \hat{B}(\xi) = \widehat{\chi}_S(\xi) &= \int_S e^{-2\pi i \xi \cdot x}\, dx \\ &= -\frac{1}{2\pi i \|\xi\|^2} \int_{\partial S} e^{-2\pi i \langle \xi, x\rangle}\, \xi \cdot \mathbf{n}(x)\, d\sigma(x), \end{aligned} \tag{6}$$

for all $\xi \neq 0$, where ∂S is the boundary of S, $\mathbf{n}$ is the unit outward normal to S, and σ is 1-dimensional Hausdorff measure on $\mathbb{R}^2$.

Hence, using (6), we have that

$$\begin{aligned} \mathscr{SH}_\psi B(a,s,p) &= \langle B, \psi^{(d)}_{a,s,p} \rangle \\ &= \langle \hat{B}, \hat{\psi}^{(d)}_{a,s,p} \rangle \\ &= \int_{\mathbb{R}^2} \hat{B}(\xi)\, \overline{\hat{\psi}^{(d)}_{a,s,p}(\xi)}\, d\xi \\ &= -\frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{\overline{\hat{\psi}^{(d)}_{a,s,p}(\xi)}}{\|\xi\|^2} \int_{\partial S} e^{-2\pi i \xi \cdot x}\, \xi \cdot \mathbf{n}(x)\, d\sigma(x)\, d\xi, \end{aligned} \tag{7}$$

where the upper-script in $\psi^{(d)}_{a,s,p}$ is either $d = h$, when $|s| \leq 1$, or $d = v$, when $|s| > 1$.

One can observe that the asymptotic decay of the shearlet transform $\mathscr{SH}_\psi B(a,s,p)$, as $a \to 0$, is only determined by the values of the boundary ∂S which are "close" to p. Hence, for $\varepsilon > 0$, let $D(\varepsilon, p)$ be the ball in $\mathbb{R}^2$ of radius ε and center p, and $D^c(\varepsilon, p) = \mathbb{R}^2 \setminus D(\varepsilon, p)$. Using (7), we can write the shearlet transform of B as

$$\mathscr{SH}_\psi B(a,s,p) = I_1(a,s,p) + I_2(a,s,p),$$

where

$$I_1(a,s,p) = -\frac{1}{2\pi i}\int_{\mathbb{R}^2} \frac{\overline{\hat{\psi}^{(d)}_{a,s,p}(\xi)}}{\|\xi\|^2}\int_{\partial S\cap D(\varepsilon,p)} e^{-2\pi i\xi\cdot x}\,\xi\cdot\mathbf{n}(x)\,d\sigma(x)\,d\xi, \tag{8}$$

$$I_2(a,s,p) = -\frac{1}{2\pi i}\int_{\mathbb{R}^2} \frac{\overline{\hat{\psi}^{(d)}_{a,s,p}(\xi)}}{\|\xi\|^2}\int_{\partial S\cap D^c(\varepsilon,p)} e^{-2\pi i\xi\cdot x}\,\xi\cdot\mathbf{n}(x)\,d\sigma(x)\,d\xi. \tag{9}$$

The Localization Lemma below (whose assumptions are satisfied by the shearlet generator function in Theorem 1) shows that I_2 has rapid asymptotic decay at fine scales. For its proof, we need the following "repeated integration by parts" lemma whose proof follows easily from induction and the standard integration by parts result. Note that this version of the Localization Lemma is more general than the one appeared in [10, 13], since it does not assume a special form of the function ψ.

Lemma 1. *Let $N\in\mathbb{Z}^+=\{1,2,3,\dots\}$ and let $f,g\in C^N(\mathbb{R})$ be such that $f^{(n)}g^{(N-1-n)}$ vanishes at ∞, for all $n=0,\dots,N-1$, and $f^{(n)}g^{(N-n)}\in L^1(\mathbb{R})$, for all $n=0,\dots,N$. Then,*

$$\int_{\mathbb{R}} f(x)g^{(N)}(x)\,dx = (-1)^N\int_{\mathbb{R}} f^{(N)}(x)g(x)\,dx.$$

Lemma 2 (Localization Lemma). *Fix $p\in\mathbb{R}^2$ and let $N\in\mathbb{Z}^+$. Suppose that*

(i) $\hat{\psi}^{(d)}\in C^N(\mathbb{R}^2)$, *for $d=h,v$;*
(ii) $\partial^\omega\hat{\psi}^{(d)}\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, *for all $0\le|\omega|\le N-1$ and $d=h,v$;*
(iii) $\partial^\omega\hat{\psi}^{(d)}/r_d^{N+1-|\omega|}\in L^1(\mathbb{R}^2)$, *for all $0\le|\omega|\le N$ and $d=h,v$, where*

$$r_d(\xi) = \begin{cases} |\xi_1|, & \text{if } d=h\\ |\xi_2|, & \text{if } d=v.\end{cases}$$

Then, there exists a constant $0<C<\infty$ such that

$$|I_2(a,s,p)| \le C\,a^{N\beta+(1-\beta)/2},$$

for all a and s.

Proof. Fix $0<a\le 1/4$ and $s\in\mathbb{R}$. We may assume that $s\le 1$ and $d=h$. Substituting for $\hat{\psi}^{(h)}_{a,s,p}$ and using (9), the change of variable $\eta_1=a\xi_1$ and $\eta_2=a^\beta\xi_2-a^\beta s\xi_1$, and some algebraic manipulation, we have

$$\begin{aligned}
&I_2(a,s,p)\\
&= \frac{-a^{(1+\beta)/2}}{2\pi i}\int_{\mathbb{R}^2}\frac{\overline{\hat{\psi}^{(h)}(a\xi_1,a^\beta\xi_2-a^\beta s\xi_1)}}{\|\xi\|^2}\int_{\partial S\cap D^c(\varepsilon,p)} e^{-2\pi i\xi\cdot(x-p)}\,\xi\cdot\mathbf{n}(x)\,d\sigma(x)\,d\xi\\
&= \frac{-a^{-(1+\beta)/2}}{2\pi i}\int_{\mathbb{R}^2}\frac{\overline{\hat{\psi}^{(h)}(\eta)}}{a^{-2}\eta_1^2+(a^{-\beta}\eta_2+a^{-1}s\eta_1)^2}\int_{\partial S\cap D^c(\varepsilon,p)} e^{-2\pi i(a^{-1}\eta_1,a^{-\beta}\eta_2+a^{-1}s\eta_1)\cdot(x-p)}\\
&\quad\times(a^{-1}\eta_1,a^{-\beta}\eta_2+a^{-1}s\eta_1)\cdot\mathbf{n}(x)\,d\sigma(x)\,d\eta\\
&= \frac{-a^{(1-\beta)/2}}{2\pi i}\int_{\mathbb{R}^2}\frac{\overline{\hat{\psi}^{(h)}(\eta)}}{\eta_1^2+(a^{1-\beta}\eta_2+s\eta_1)^2}\int_{\partial S\cap D^c(\varepsilon,p)} e^{-2\pi i a^{-1}\eta_1[(x_1-p_1)+s(x_2-p_2)]}
\end{aligned}$$

$$\times (\eta_1, a^{1-\beta}\eta_2 + s\eta_1)\cdot \mathbf{n}(x) e^{-2\pi i a^{-\beta}\eta_2(x_2-p_2)}\, d\sigma(x)\, d\eta. \tag{10}$$

Note also that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left| \frac{\overline{\hat{\psi}^{(h)}(\eta)}}{\eta_1^2 + (a^{1-\beta}\eta_2 + s\eta_1)^2} \right| \int_{\partial S\cap D^c(\varepsilon,p)} \left| (\eta_1, a^{1-\beta}\eta_2 + s\eta_1)\cdot \mathbf{n}(x) \right| \\
&\quad \times \left| e^{-2\pi i a^{-1}\eta_1[(x_1-p_1)+s(x_2-p_2)]} e^{-2\pi i a^{-\beta}\eta_2(x_2-p_2)} \right| d\sigma(x)\, d\eta \\
&\quad \le \int_{\mathbb{R}^2} \frac{|\hat{\psi}^{(h)}(\eta)|}{\eta_1^2 + (a^{1-\beta}\eta_2 + s\eta_1)^2} \int_{\partial S\cap D^c(\varepsilon,p)} \|(\eta_1, a^{1-\beta}\eta_2 + s\eta_1)\| \|\mathbf{n}(x)\|\, d\sigma(x)\, d\eta \\
&\quad \le \sigma(\partial S) \int_{\mathbb{R}^2} \frac{|\hat{\psi}^{(h)}(\eta)|}{r_h(\eta)}\, d\eta < \infty,
\end{aligned} \tag{11}$$

where, in the last inequality, we have used properties (ii) and (iii) in the statement of this lemma.

Choose $\delta > 0$ not depending on s and disjoint Borel measurable subsets $E_q \subset \mathbb{R}^2$, for $q = 1,2$, satisfying

$$E_q \subset \{x \in \mathbb{R}^2 : |(x_q - p_q) + s_q(x_2 - p_2)| \ge \delta\} \text{ and } E_1 \cup E_2 = \partial S \cap D^c(\varepsilon, p), \tag{12}$$

where $s_1 = s$ and $s_2 = 0$. Then, using (10), (11), and the Fubini-Tonelli theorem it follows that

$$\begin{aligned}
I_2(a,s,p) = \frac{-a^{(1-\beta)/2}}{2\pi i} \sum_{q=1,2} \int_{E_q} \int_{\mathbb{R}^2} f_a(x,\eta) \\
\times e^{-2\pi i a^{-1}\eta_1[(x_1-p_1)+s(x_2-p_2)]} e^{-2\pi i a^{-\beta}\eta_2(x_2-p_2)}\, d\eta\, d\sigma(x),
\end{aligned} \tag{13}$$

where $f_a : \partial S \times \mathbb{R}^2 \to \mathbb{C}$ is defined by

$$f_a(x,\eta) = \frac{(\eta_1, a^{1-\beta}\eta_2 + s\eta_1)\cdot \mathbf{n}(x)}{\eta_1^2 + (a^{1-\beta}\eta_2 + s\eta_1)^2} \overline{\hat{\psi}^{(h)}(\eta)}$$

for a.e. (x,η). We require the following claim, whose proof is a straightforward application of induction and the quotient rule.

For each $q \in \{1,2\}$ and $n \in \{0,\ldots,N\}$, there exists $L_n^q \in \mathbb{Z}^+$ and, for each $l = 1,\ldots,L_n^q$, there exist $\gamma_l^{qn} \ge 0$, $c_l^{qn} \in L^\infty(S,\sigma)$ not depending on a, η, or s, a monomial $m_l^{qn} : \mathbb{R}^2 \to \mathbb{R}$, and a multi-index ω_l^{qn} with $|\omega_l^{qn}| \le n$ and $|\omega_l^{qn}| = \deg(m_l^{qn}) - 2^{n+1} + n + 1$ such that

$$\frac{\partial^n}{\partial \eta_q^n} f_a(x,\eta) = \sum_{l=1}^{L_n^q} \frac{a^{\gamma_l^{qn}} c_l^{qn}(x) m_l^{qn}(\eta_1, a^{1-\beta}\eta_2 + s\eta_1)}{(\eta_1^2 + (a^{1-\beta}\eta_2 + s\eta_1)^2)^{2^n}} \overline{\partial^{\omega_l^{qk}} \hat{\psi}^{(h)}(\eta)},$$

for a.e. (x,η). We are using monomial in the strict sense (i.e., $\eta_1\eta_2$ is a monomial but $-\eta_1\eta_2$ and $2\eta_1\eta_2$ are not).

If $q \in \{1,2\}$, choose r such that $\{q,r\} = \{1,2\}$. If $m : \mathbb{R}^2 \to \mathbb{R}$ is a monomial and $\gamma \in \mathbb{R}$, then, by switching to spherical coordinates, it is clear that $|m(\eta)|/\|\eta\|^\gamma \leq 1/\|\eta\|^{\gamma-\deg(m)}$, for all $\eta \neq 0$. Using this and the claim, if $n \in \{0,\dots,N\}$, we have

$$\begin{aligned}\left|\frac{\partial^n}{\partial \eta_q^n} f_a(x,\eta)\right| &\leq \sum_{l=1}^{L_n^q} \|c_l^{qn}\|_\infty \left|\frac{m_l^{qn}(\eta_1, a^{1-\beta}\eta_2 + s\eta_1)}{(\eta_1^2 + (a^{1-\beta}\eta_2 + s\eta_1)^2)^{2^n}}\right| |\partial^{\omega_l^{qn}} \hat{\psi}^{(h)}(\eta)| \\ &\leq \sum_{l=1}^{L_n^q} \|c_l^{qn}\|_\infty \frac{|\partial^{\omega_l^{qn}} \hat{\psi}^{(h)}(\eta)|}{\|(\eta_1, a^{1-\beta}\eta_2 + s\eta_1)\|^{k+1-|\omega_l^{qn}|}} \\ &\leq \sum_{l=1}^{L_n^q} \|c_l^{qn}\|_\infty \frac{|\partial^{\omega_l^{qn}} \hat{\psi}^{(h)}(\eta)|}{r_h(\eta)^{n+1-|\omega_l^{qn}|}},\end{aligned} \tag{14}$$

for a.e. (x,η). The second inequality, together with the claim and property (ii) of $\psi^{(h)}$, implies that $\frac{\partial^n}{\partial \eta_q^n} f_a(x,\cdot)$ vanishes at ∞, for $n = 0,\dots,N-1$ and σ-a.e. x. The third inequality, together with the claim and properties (ii) and (iii) of $\psi^{(h)}$ implies that $\frac{\partial^n}{\partial \eta_q^n} f_a(x,\cdot) \in L^1(\mathbb{R}^2)$, for $n = 0,\dots,N$ and σ-a.e. x.

Using the observations of the previous paragraph, the Fubini-Tonelli theorem, Lemma 1, (12), (14), the claim, and property (i) of $\psi^{(h)}$, we obtain

$$\begin{aligned}&\left|\int_{E_q}\int_{\mathbb{R}^2} f_a(x,\eta) e^{-2\pi i a^{-1}\eta_1[(x_1-p_1)+s(x_2-p_2)]} e^{-2\pi i a^{-\beta}\eta_2(x_2-p_2)}\, d\eta\, d\sigma(x)\right| \\ &\leq \int_{E_q}\int_{\mathbb{R}} \left|\int_{\mathbb{R}} f_a(x,\eta) e^{-2\pi i a^{-\beta_q}\eta_q[(x_q-p_q)+s_q(x_2-p_2)]}\, d\eta_q\right| d\eta_r\, d\sigma(x) \\ &= \int_{E_q}\int_{\mathbb{R}} \left|\int_{\mathbb{R}} f_a(x,\eta) \frac{\partial^N}{\partial \eta_q^N}\left(\frac{e^{-2\pi i a^{-\beta_q}\eta_q[(x_q-p_q)+s_q(x_2-p_2)]}}{\left(-2\pi i a^{-\beta_q}[(x_q-p_q)+s_q(x_2-p_2)]\right)^N}\right) d\eta_q\right| d\eta_r\, d\sigma(x) \\ &= \int_{E_q}\int_{\mathbb{R}} \left|\int_{\mathbb{R}} \frac{\partial^N}{\partial \eta_q^N} f_a(x,\eta) \frac{e^{-2\pi i a^{-\beta_q}\eta_q[(x_q-p_q)+s_q(x_2-p_2)]}}{\left(-2\pi i a^{-\beta_q}[(x_q-p_q)+s_q(x_2-p_2)]\right)^N} d\eta_q\right| d\eta_r\, d\sigma(x) \\ &\leq \frac{a^{N\beta_q}}{(2\pi\delta)^N} \int_{E_q}\int_{\mathbb{R}^2} \left|\frac{\partial^N}{\partial \eta_q^N} f_a(x,\eta)\right| d\eta\, d\sigma(x) \\ &\leq \frac{\sigma(\partial S) a^{N\beta_q}}{(2\pi\delta)^N} \sum_{l=1}^{L_N^q} \|c_l^{qN}\|_\infty \|\partial^{\omega_l^{qN}} \hat{\psi}^{(h)}/r_h^{N+1-|\omega_l^{qN}|}\|_1,\end{aligned}$$

where $\beta_1 = 1$ and $\beta_2 = \beta$. The lemma follows from the claim, property (iii) of $\psi^{(h)}$, the above inequality, and (13). □

For the analysis of the term I_1, we will use a local approximation of the curve ∂S.

Let $\alpha(t)$ be the boundary curve ∂S, with $0 \leq t \leq L$, and $p \in \partial S$. Without loss of generality, we may assume that $L > 1$ and $p = (0,0) = \alpha(1)$. We can write the boundary curve near p as $\mathscr{C} = \partial S \cap D(\varepsilon,(0,0))$, where

$$\mathscr{C} = \{\alpha(t) : 1-\varepsilon \le t \le 1+\varepsilon\}.$$

Rather than using the arclength representation of $\mathscr{C}$, we can also write $\mathscr{C} = \{(G(u),u),\ -\varepsilon \le u \le \varepsilon\}$, where $G(u)$ is a smooth function. Since $p=(0,0)$, then $G(0)=0$. Hence we define the quadratic approximation of ∂S near $p=(0,0)$ by $\partial S_0 = (G_0(u),u)$, where G_0 is the Taylor polynomial of degree 2 of G centered at the origin, given by $G_0(u) = G'(0)u + \frac{1}{2}G''(0)u^2$. Accordingly, we define $B_0 = \chi_{S_0}$, where S_0 is obtained by replacing the curve ∂S in $B = \chi_S$ with the quadratic curve ∂S_0 near the point $p=(0,0)$.

The following lemma, which is a generalization from [13], shows that to derive the estimates of Theorem 1 it is sufficient to replace the set B with set B_0, since this produces a "low-order" error. Note that this approximation result only holds for $\frac{1}{3} < \beta < 1$, that is, when the anisotropic scaling factor of the dilation matrices is not too high. The argument provided below does not extend to smaller values of β. Possibly this restriction could be removed by considering a higher order polynomial approximation for the boundary curve ∂S, but this would make the rest of the proof of Theorem 1 significantly more involved.

Lemma 3. *Let $\frac{1}{3} < \beta < 1$. For any $|s| \le \frac{3}{2}$, we have*

$$\lim_{a\to 0^+} a^{-\frac{1+\beta}{2}} \left|\mathscr{SH}_\psi B(a,s,0)) - \mathscr{SH}_\psi B_0(a,s,0)\right| = 0.$$

Proof. Let $p=(0,0)\in\partial S$. Since we assume $|s|\le\frac{3}{2}$, we need to use the system of 'horizontal' shearlets only.

Let γ be chosen such that $\frac{1+\beta}{4} < \gamma < \beta$ (this can be satisfied for $\frac{1}{3} < \beta < 1$) and assume that a is sufficiently small, so that $a^\gamma \ll 1$. A direct calculation shows that

$$\begin{aligned}\left|\mathscr{SH}_\psi B(a,s,0) - \mathscr{SH}_\psi B_0(a,s,0)\right| &\le \int_{\mathbb{R}^2} |\psi^{(h)}_{a,s,0}(x)|\,|\chi_S(x) - \chi_{S_0}(x)|\,dx \\ &= T_1(a) + T_2(a),\end{aligned}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$ and

$$\begin{aligned}T_1(a) &= a^{-\frac{1+\beta}{2}} \int_{D(a^\gamma,(0,0))} |\psi^{(h)}(M_{as}^{-1}x)|\,|\chi_S(x)-\chi_{S_0}(x)|\,dx,\\ T_2(a) &= a^{-\frac{1+\beta}{2}} \int_{D^c(a^\gamma,(0,0))} |\psi^{(h)}(M_{as}^{-1}x)|\,|\chi_S(x)-\chi_{S_0}(x)|\,dx.\end{aligned}$$

Observe that:

$$T_1(a) \le C a^{-\frac{1+\beta}{2}} \int_{D(a^\gamma,(0,0))} |\chi_S(x)-\chi_{S_0}(x)|\,dx.$$

To estimate the above quantity, it is enough to compute the area between the regions S and S_0. Since G_0 is the Taylor polynomial of G of degree 2, we have

$$T_1(a) \leq C\, a^{-\frac{1+\beta}{2}} \int_{|x|<a^\gamma} |x|^3 dx \leq C\, a^{4\gamma-\frac{1+\beta}{2}}.$$

Since $\gamma > \frac{1+\beta}{4}$, the above estimate shows that $T_1(a) = o(a^{\frac{1+\beta}{2}})$.

The assumptions on the generator function $\psi^{(h)}$ of the shearlet system $\psi^{(h)}$ imply that, for each $N > 0$, there is a constant $C_N > 0$ such that $|\psi(x)| \leq C_N\,(1+|x|^2)^{-N}$. Also note that $(M_{as})^{-1} = A_a^{-1} B_s^{-1}$, where $B_s^{-1} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $A_a^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-\beta} \end{pmatrix}$. It is easy to verify that, for all $|s| \leq \frac{3}{2}$, there is a constant $C_0 > 0$ such that $\|B_s^{-1}x\|^2 \geq C_0\|x\|^2$, or $(x_1+sx_2)^2 + x_2^2 \geq C_0(x_1^2+x_2^2)$ for all $x \in \mathbb{R}^2$. Thus, for $a < 1$, we can estimate $T_2(a)$ as:

$$\begin{aligned}
T_2(a) &\leq C\, a^{-\frac{1+\beta}{2}} \int_{D^c(a^\gamma,(0,0))} |\psi^{(h)}(M_{as}x)|\, dx \\
&\leq C_N\, a^{-\frac{1+\beta}{2}} \int_{D^c(a^\gamma,(0,0))} \Big(1+(a^{-1}(x_1+sx_2))^2+(a^{-\beta}x_2)^2\Big)^{-N} dx \\
&\leq C_N\, a^{-\frac{1+\beta}{2}} \int_{D^c(a^\gamma,(0,0))} \Big((a^{-\beta}(x_1+sx_2))^2+(a^{-\beta}x_2)^2\Big)^{-N} dx \\
&\leq C_N\, a^{2\beta N-\frac{1+\beta}{2}} \int_{D^c(a^\gamma,(0,0))} (x_1^2+x_2^2)^{-N}\, dx \\
&= C_N\, a^{2\beta N-\frac{1+\beta}{2}} \int_{a^\gamma}^{\infty} r^{1-2N}\, dr \\
&= C_N\, a^{2N(\beta-\gamma)}\, a^{2\gamma-\frac{1+\beta}{2}},
\end{aligned}$$

where the constant C_0 was absorbed in C_N. Since $\gamma < \beta$ and N can be chosen arbitrarily large, it follows that $T_2(a) = o(a^{\frac{1+\beta}{2}})$. □

The proof of Theorem 1 can now be completed using Lemmata 2 and 3, following the arguments from [13].

4 Shearlet analysis of general singularities

The shearlet analysis of singularities extends beyond the case of functions of the form χ_S considered in the previous sections. The results presented below illustrate the shearlet analysis of singularities of rather general functions.

As a first case, we will examine the case of 'general' functions of two variables containing jump discontinuities. Let S be a bounded open subset of $\mathbb{R}^2$ and assume that its boundary ∂S is generated by a C^3 curve that can be parametrized as $(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta)$ where $\rho(\theta) : [0,2\pi) \to [0,1]$ is a radius function. We will consider functions of the form $f\,\chi_S$, where f is a smooth functions. Note that this model is a special case of the class of cartoon-like images, where the set ∂S describes the edge of an object. Similar image models are commonly used, for example, in the variational approach to image processing (cf. [5, Ch.3]).

We have the following result, which is a refinement from an observation in [14].

Theorem 2. *Let ψ_1, ψ_2, β be chosen as in Theorem 1. Let $B = f\chi_S$, where $S \subset \mathbb{R}^2$ is a bounded region whose boundary ∂S is a simple C^3 curve and $f \in C^\infty(\mathbb{R}^2)$. Then we have the following results.*

(i) If $p \notin \partial S$ then, for all $s \in \mathbb{R}$,

$$\lim_{a\to 0^+} a^{-N}\,\mathscr{SH}_\psi B(a,s,p) = 0, \quad \textit{for all } N > 0.$$

(ii) If $p_0 \in \partial S$ is a regular point, s_0 corresponds to the normal direction of ∂S at p_0 and $s \neq s_0$, then

$$\lim_{a\to 0^+} a^{-N}\,\mathscr{SH}_\psi B(a,s,p_0) = 0, \quad \textit{for all } N > 0.$$

(iii) If $p_0 \in \partial S$ is a regular point, s_0 corresponds to the normal direction of ∂S at p_0, $s = s_0$ and $f(p) \neq 0$, then

$$\lim_{a\to 0^+} a^{-\frac{1+\beta}{2}}\,\mathscr{SH}_\psi B(a,s_0,p_0) \neq 0.$$

For simplicity of notation, we will prove Theorem 2 in the special case where $\beta = \frac{1}{2}$. The case of general $\frac{1}{3} < \beta < 1$ can be easily derived from here. For the proof, we need first the following lemma (where we assume $\beta = \frac{1}{2}$).

Lemma 4. *Let $S \subset \mathbb{R}^2$ be a bounded region whose boundary ∂S is a simple C^3 curve. Assume that $p_0 \in \partial S$ is a regular point and P_S is a polynomial with $P_S(p_0) = 0$. For any $N > 0$, we have*

(i) $\lim_{a\to 0} a^{-N}\langle P_S\,\chi_S, \psi^{(h)}_{a,s,p_0}\rangle = 0, \quad s \neq \pm\mathbf{n}(p_0)$,
(ii) $\lim_{a\to 0} a^{-\frac{5}{4}}\langle P_S\,\chi_S, \psi^{(h)}_{a,s,p_0}\rangle = C, \quad s = \pm\mathbf{n}(p_0)$,

where C is a finite real number.

Proof. We only prove the lemma when P_S is a polynomial of degree 2, since the same argument works for a polynomial of degree > 2. Without loss of generality, we may assume $p_0 = (0,0)$ and that near p_0, we have that $\partial S = \{(g(u),u),\ -\varepsilon < u < \varepsilon\}$, where $g(u) = Au^2 + Bu$. Also we may write $s = \tan\theta_0$ with $\theta_0 = 0$.

Recall that, by the divergence theorem,

$$\begin{aligned}\widehat{\chi_S}(\rho,\theta) &= -\frac{1}{2\pi i\rho}\int_{\partial S} e^{-2\pi i\rho\,\Theta(\theta)\cdot x}\,\Theta(\theta)\cdot\mathbf{n}(x)\,d\sigma(x)\\ &= -\frac{1}{2\pi i\rho}\int_0^L e^{-2\pi i\rho\,\Theta(\theta)\cdot\alpha(t)}\,\Theta(\theta)\cdot\mathbf{n}(t)\,dt.\end{aligned}$$

Since $P_S(0)=0$, we can write $P_S(x)$ as $A_1x_1+A_2x_2+A_3x_1^2+A_4x_1x_2+A_5x_2^2$. Let $P_S(\frac{i}{2\pi}D)$ be the differential operator obtained from the polynomial $P_S(x)$ by replacing x_1 with $\frac{i}{2\pi}\frac{\partial}{\partial\xi_1}$ and x_2 with $\frac{i}{2\pi}\frac{\partial}{\partial\xi_2}$.

A direct computation gives that

$$\begin{aligned}
&\frac{\partial}{\partial\xi_1}\left(\hat\psi_1(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))\right)\\
&=a\hat\psi_1{}'(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))-\frac{\xi_2}{\xi_1^2}a^{-\frac12}\hat\psi_1(a\xi_1)\hat\psi_2{}'(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s)),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial\xi_1^2}\left(\hat\psi_1(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))\right)\\
&=a^2\hat\psi_1{}''(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))-a^{\frac12}\frac{\xi_2}{\xi_1^2}\hat\psi_1{}'(a\xi_1)\hat\psi_2{}'(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))\\
&+a^{-\frac12}\frac{2\xi_2}{\xi_1^3}\hat\psi_1(a\xi_1)\hat\psi_2{}'(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))-a^{\frac12}\frac{\xi_2}{\xi_1^2}\hat\psi_1{}'(a\xi_1)\hat\psi_2{}'(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s))\\
&+(a^{-\frac12}\frac{\xi_2}{\xi_1^2})^2\hat\psi_1(a\xi_1)\hat\psi_2{}''(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s)).
\end{aligned}$$

Using these expressions, we obtain that

$$\begin{aligned}
\langle P_S\,\chi_S,\psi^{(h)}_{a,s,p}\rangle &= \langle\chi_S,P_S\,\psi^{(h)}_{a,s,p}\rangle\\
&=\langle\widehat{\chi_S},\widehat{P_S\,\psi^{(h)}_{a,s,p}}\rangle\\
&=\langle\widehat{\chi_S},P_S(\frac{i}{2\pi}D)(\widehat{\psi^{(h)}_{a,s,p}})\rangle\\
&=\sum_{m=1}^{5}J_m(a,s,p),
\end{aligned}$$

where, using $p=(0,0)$,

$$\begin{aligned}
J_1(a,s,0) &= \frac{A_1 i}{2\pi}\langle\widehat{\chi_S},\frac{\partial}{\partial\xi_1}(\hat\psi_1(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s)))\rangle\\
J_2(a,s,0) &= \frac{A_2 i}{2\pi}\langle\widehat{\chi_S},\frac{\partial}{\partial\xi_2}(\hat\psi_1(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s)))\rangle\\
J_3(a,s,0) &= -\frac{A_3}{(2\pi)^2}\langle\widehat{\chi_S},\frac{\partial^2}{\partial\xi_1^2}(\hat\psi_1(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s)))\rangle\\
J_4(a,s,0) &= -\frac{A_4}{(2\pi)^2}\langle\widehat{\chi_S},\frac{\partial^2}{\partial\xi_1\partial\xi_2}(\hat\psi_1(a\xi_1)\hat\psi_2(a^{-\frac12}(\frac{\xi_2}{\xi_1}-s)))\rangle
\end{aligned}$$

$$J_5(a,s,0) = -\frac{A_5}{(2\pi)^2}\langle \widehat{\chi_S}, \frac{\partial^2}{\partial \xi_2^2}(\hat{\psi}_1(a\xi_1)\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_1}-s)))\rangle$$

Since $s \neq \pm\mathbf{n}(p_0)$, by integration by parts, it is easy to see that for each $N>0$, we have $\widehat{\chi_S}(a^{-1}\rho,\theta) = O(a^N)$, as $a \to 0$, uniformly for all ρ and θ. For each J_m, let $\xi = \rho\,\Theta(\theta)$ and $a\rho = \rho'$, by the Localization Lemma (Lemma 2) we see that $J_m = O(a^N)$ for $m = 1,2,3,4,5$ and this proves part (i).

For $s = \pm\mathbf{n}(p_0)$, let us first examine the term J_1. By the Localization Lemma, we can assume that J_1 has the following expression.

$$\begin{aligned}
&J_1(a,s,0)\\
&= \frac{A_1 i}{2\pi}\int_{R^2}\widehat{\chi_S}(\xi)\frac{\partial}{\partial \xi_1}\left(\hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1}))\right)d\xi\\
&= -\frac{a^{\frac{3}{4}}A_1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{-2\pi i\rho\,\Theta(\theta)\cdot(g(u),u)}\,\Theta(\theta)\cdot\mathbf{n}(u)\,du\\
&\times\left(a\hat{\psi}_1{}'(a\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}\tan\theta) - \frac{a^{-\frac{1}{2}}\tan\theta}{\rho\cos\theta}\hat{\psi}_1(a\rho\cos\theta)\hat{\psi}_2{}'(a^{-\frac{1}{2}}\tan\theta)\right)d\theta\,d\rho\\
&= -\frac{a^{-\frac{1}{4}}A_1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{-2\pi i a^{-1}\rho\,\Theta(\theta)\cdot(g(u),u)}\,\Theta(\theta)\cdot\mathbf{n}(u)\,du\\
&\quad\times\left(a\hat{\psi}_1{}'(\rho\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}\tan\theta) - \frac{a^{\frac{1}{2}}\tan\theta}{\rho\cos\theta}\hat{\psi}_1(\rho\cos\theta)\hat{\psi}_2{}'(a^{-\frac{1}{2}}\tan\theta)\right)d\theta\,d\rho\\
&= J_{11}(a,s,0) + J_{12}(a,s,0),
\end{aligned}$$

where

$$\begin{aligned}
J_{11}(a,s,0) = -\frac{a^{-\frac{1}{4}}A_1}{(2\pi)^2}&\int_0^\infty\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{-2\pi i a^{-1}\rho\,\Theta(\theta)\cdot(g(u),u)}\,\Theta(\theta)\cdot\mathbf{n}(u)\,du\\
&\times a\,\hat{\psi}_1{}'(\rho\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}\tan\theta)\,d\theta\,d\rho,
\end{aligned}$$

$$\begin{aligned}
J_{12}(a,s,0) = \frac{a^{-\frac{1}{4}}A_1}{(2\pi)^2}&\int_0^\infty\int_0^{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{-2\pi i a^{-1}\rho\,\Theta(\theta)\cdot(g(u),u)}\,\Theta(\theta)\cdot\mathbf{n}(u)\,du\\
&\times\frac{a^{\frac{1}{2}}\tan\theta}{\rho\cos\theta}\,\hat{\psi}_1(\rho\cos\theta)\,\hat{\psi}_2{}'(a^{-\frac{1}{2}}\tan\theta)\,d\theta\,d\rho.
\end{aligned}$$

Then, similar to the part (iii) of the proof of Theorem 1, we examine the oscillatory integrals J_{11} and J_{12} depending on the behaviour of the phase $\Theta(\theta)\cdot(g(u),u)$. As in the proof of Theorem 1, part (iii), there are two cases to consider depending on $A=0$ or $A\neq 0$ (recall that $g(u) = Au^2+Bu$). In either case, we break up the interval $[0,2\pi]$ into $[-\frac{\pi}{2},\frac{\pi}{2}]\bigcup(\frac{\pi}{2},\frac{3\pi}{2}]$ and let $t = a^{-\frac{1}{2}}\tan\theta)$, $u' = a^{-\frac{1}{2}}u$. Thus, we have the following estimates.

Case 1: $A \neq 0$. We will only consider the case $A > 0$ since the case $A < 0$ is similar. Using the formulas of Fresnel integrals, we have

$$\begin{aligned}
&\lim_{a\to 0^+}(2\pi)^2 2\sqrt{A}a^{-\frac{7}{4}}J_{11}(a,s,0)\\
&= -A_1\sqrt{A}\int_0^\infty \hat{\psi}_1{}'(\rho)\int_{-1}^1 e^{\frac{\pi i\rho}{2A}t^2}\hat{\psi}_2(t)dt\int_{-\infty}^\infty e^{-2\pi i\rho A u^2}du\,d\rho\\
&+A_1\sqrt{A}\int_0^\infty \hat{\psi}_1{}'(\rho)\int_{-1}^1 e^{-\frac{\pi i\rho}{2A}t^2}\hat{\psi}_2(t)dt\int_{-\infty}^\infty e^{2\pi i\rho A u^2}du\,d\rho\\
&= A_1\int_0^\infty \frac{\hat{\psi}_1{}'(\rho)}{\sqrt{\rho}}\int_{-1}^1\Big(\cos(\frac{\pi\rho}{2A}t^2)-\sin(\frac{\pi\rho}{2A}t^2)\Big)\hat{\psi}_2(t)dt\,d\rho\\
&= C_{11},
\end{aligned}$$

where C_{11} is a finite real number.

A similar calculation gives that

$$\begin{aligned}
&\lim_{a\to 0^+}(2\pi)^2 2\sqrt{A}a^{-\frac{7}{4}}J_{12}(a,s,0)\\
&= A_1\sqrt{A}\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 e^{\frac{\pi i\rho}{2A}t^2}t\hat{\psi}_2{}'(t)dt\int_{-\infty}^\infty e^{-2\pi i\rho A u^2}du\,d\rho\\
&+A_1\sqrt{A}\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 e^{-\frac{\pi i\rho}{2A}t^2}t\hat{\psi}_2{}'(t)dt\int_{-\infty}^\infty e^{2\pi i\rho A u^2}du\,d\rho\\
&= A_1\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho^{\frac{3}{2}}}\int_{-1}^1\Big(\cos(\frac{\pi\rho}{2A}t^2)+\sin(\frac{\pi\rho}{2A}t^2)\Big)t\hat{\psi}_2{}'(t)dt\,d\rho\\
&= C_{12},
\end{aligned}$$

where C_{12} is a finite real number.

The same argument applied to the term J_2 gives that

$$\begin{aligned}
&\lim_{a\to 0^+}(2\pi)^2 2\sqrt{A}a^{-\frac{5}{4}}J_{2}(a,s,0)\\
&= A_2\sqrt{A}\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 e^{\frac{\pi i\rho}{2A}t^2}\hat{\psi}_2{}'(t)dt\int_{-\infty}^\infty e^{-2\pi i\rho A u^2}du\,d\rho\\
&+A_2\sqrt{A}\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 e^{-\frac{\pi i\rho}{2A}t^2}\hat{\psi}_2{}'(t)dt\int_{-\infty}^\infty e^{2\pi i\rho A u^2}du\,d\rho\\
&= A_2\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho^{\frac{3}{2}}}\int_{-1}^1\Big(\cos(\frac{\pi\rho}{2A}t^2)-\sin(\frac{\pi\rho}{2A}t^2)\Big)\hat{\psi}_2{}'(t)dt\,d\rho = C_2 = 0,
\end{aligned}$$

where C_2 is a finite real number and, similarly,

$$\lim_{a\to 0^+}a^{-\frac{11}{4}}J_3(a,s,0) = C_3,\ \ \lim_{a\to 0^+}a^{-\frac{9}{4}}J_4(a,s,0) = C_4,\ \ \lim_{a\to 0^+}a^{-\frac{7}{4}}J_5(a,s,0) = C_5,$$

where C_3, C_4, C_5 are finite real numbers.

In general, for $m=(m_1,m_2)\in\mathbb{N}\times\mathbb{N}$, we have

$$\lim_{a\to 0^+} a^{-(\frac{3}{4}+m_1+\frac{m_2}{2})}\langle\widehat{\chi}_S, \frac{\partial^m}{\partial\xi^m}\left(\hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_1}-s))\right)\rangle = C_m,$$

where C_m is a finite real number for each fixed m. This shows that part (ii) holds for the case $A\neq 0$.

Case 2: $A=0$. Using an argument similar to the one used in the proof of part (iii) of Theorem 1, we have that

$$\begin{aligned}
&\lim_{a\to 0^+}(2\pi)^2 2a^{-\frac{7}{4}}\langle\widehat{\chi}_S, \frac{\partial}{\partial\xi_1}(\hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_1}-s)))\rangle\\
&=\int_0^\infty \hat{\psi}_1{}'(\rho)\int_{-1}^1 \hat{\psi}_2(t)e^{-2\pi i\rho tu}\,dt\,du\,d\rho-\int_0^\infty \hat{\psi}_1{}'(\rho)\int_{-1}^1 \hat{\psi}_2(t)e^{2\pi i\rho tu}\,dt\,du\,d\rho\\
&+\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 t\,\hat{\psi}_2{}'(t)e^{-2\pi i\rho tu}\,dt\,du\,d\rho-\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 t\,\hat{\psi}_2{}'(t)e^{2\pi i\rho tu}\,dt\,du\,d\rho\\
&=0,
\end{aligned}$$

where we have used the assumption that $\hat{\psi}_1$ is odd and $\hat{\psi}_2$ is even.
Similarly

$$\begin{aligned}
&\lim_{a\to 0^+}(2\pi)^2 2a^{-\frac{5}{4}}\langle\widehat{\chi}_S, \frac{\partial}{\partial\xi_2}(\hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_1}-s)))\rangle\\
&=\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 \hat{\psi}_2{}'(t)e^{-2\pi i\rho tu}\,dt\,du\,d\rho-\int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\int_{-1}^1 \hat{\psi}_2{}'(t)e^{2\pi i\rho tu}\,dt\,du\,d\rho\\
&=0.
\end{aligned}$$

Also in this case, in general, for $m=(m_1,m_2)\in\mathbb{N}\times\mathbb{N}$, we have

$$\lim_{a\to 0^+} a^{-(\frac{3}{4}+m_1+\frac{m_2}{2})}\langle\widehat{\chi}_S, \frac{\partial^m}{\partial\xi^m}\left(\hat{\psi}_1(a\xi_1)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\frac{\xi_2}{\xi_1}-s))\right)\rangle = C_m,$$

where C_m is a finite real number for each fixed m. □

We can now complete the proof of the theorem
Proof of Theorem 2. It will be sufficient to consider the horizontal shearlet system $\{\psi_{a,s,p}^{(h)}\}$ since the analysis of the vertical system is similar.
(i) For any $p\notin\partial S$ using the argument from the proof of Lemma 3, one can find the Taylor polynomial P_S of f at p of degree N' such that, for any $N\in\mathbb{N}$,

$$\lim_{a\to 0^+} a^{-N}|\langle P_S\,\chi_S, \psi_{a,s,p}^{(h)}\rangle-\langle f\,\chi_S, \psi_{a,s,p}^{(h)}\rangle|=0.$$

As in the proof of Lemma 4, we convert $P_S(x)$ into the differential operator $P_S(\frac{i}{2\pi}D)$. Then, by the Localization Lemma 2 it follows that

$$\lim_{a\to 0^+} a^{-N} |\langle P_S\, \chi_S, \psi^{(h)}_{a,s,p}\rangle| = 0.$$

This completes the proof of part (i).

(ii) As in the proof of part (i), we can replace $B = f\,\chi_S$ by the expression $P_s\,\chi_S$. Then part (ii) follows from the argument used in the proof of part (i) of Lemma 4.

(iii) Again we can replace $B = f\,\chi_S$ by $P_s\,\chi_S$. Then, using Lemma 3, we see that near p the boundary curve can be replaced by $(g(u),u)$ where, as in Lemma 4, g is a polynomial of degree 2. Since $P_s(p) = f(p) \neq 0$, Lemma 4 and part (iii) of Theorem 1 imply that

$$\lim_{a\to 0^+} a^{-\frac{3}{4}}\, \mathscr{SH}_\psi B(a,s_0,p_0) = f(0) \lim_{a\to 0^+} a^{-\frac{3}{4}}\, \mathscr{SH}_\psi \chi_S(a,s_0,p_0) \neq 0. \quad \square$$

As yet another class of two-dimensional singularities, let us consider the case of discontinuities in the derivative. As a prototype of such singularities, let as examine the two-dimensional ramp function $x_1 H(x_1,x_2)$, where H is the two-dimensional Heaviside function defined in Section 3. Using a calculation very similar to Section 3, we obtain:

$$\begin{aligned}
&\mathscr{SH}_\psi(x_1 H)(a,s,t) = \langle x_1 H, \psi_{a,s,t}\rangle \\
&= -\frac{1}{2\pi i}\int_{\mathbb{R}^2} \partial_1 \hat{H}(\xi_1,\xi_2)\, \overline{\hat{\psi}_{a,s,t}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\
&= \frac{1}{2\pi i}\int_{\mathbb{R}^2} \hat{H}(\xi_1,\xi_2)\, \partial_1 \overline{\hat{\psi}_{a,s,t}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\
&= \int_{\mathbb{R}^2} \frac{\delta_2(\xi_1,\xi_2)}{2\pi i \xi_1}\, \partial_1 \overline{\hat{\psi}_{a,s,t}}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \\
&= \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1}\, \partial_1 \overline{\hat{\psi}_{a,s,t}}(\xi_1,\xi_2)|_{\xi_2=0}\, d\xi_1 \\
&= a^{\frac{1+\beta}{2}}\, \overline{\hat{\psi}_2}(a^{\beta-1}s) \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1}\, \partial_1 \left(\overline{\hat{\psi}_1}(a\,\xi_1)\, e^{2\pi i \xi_1 t_1}\right) d\xi_1 \\
&= a^{\frac{1+\beta}{2}}\, \overline{\hat{\psi}_2}(a^{\beta-1}s) \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1}\, \left(a\partial_1 \left(\overline{\hat{\psi}_1}\right)(a\,\xi_1) + 2\pi i t_1\, \overline{\hat{\psi}_1}(a\,\xi_1)\right) e^{2\pi i \xi_1 t_1}\, d\xi_1.
\end{aligned}$$

As in the case of shearlet transform of H, under the assumption that $\hat{\psi}_1 \in C_c^\infty(\mathbb{R})$ it follows that $\mathscr{SH}_\psi(x_1 H)(a,s,t)$ decays rapidly, asymptotically for $a \to 0$, for all (t_1,t_2) when $t_1 \neq 0$, and for $t_1 = 0$, $s \neq 0$. On the other hand, if $t_1 = 0$ and $s = 0$ we have:

$$\mathscr{SH}_\psi(x_1 H)(a,s,t) = a^{\frac{3+\beta}{2}}\, \overline{\hat{\psi}_2}(0) \int_{\mathbb{R}} \frac{1}{2\pi i \xi_1}\, \partial_1 \left(\overline{\hat{\psi}_1}\right)(a\,\xi_1)\, d\xi_1.$$

Provided that $\hat{\psi}_2(0) \neq 0$ and that the integral on the right hand side of the equation above is non-zero, it follows that $\mathscr{SH}_\psi(x_1 H)(a,s,t) = O(a^{\frac{3+\beta}{2}})$.

This result suggests that, under appropriate assumptions on ψ_1 and ψ_2, the analysis of Section 3 extends to singularities that behave locally as the ramp function. The complete discussion of this problem is beyond the scope of this paper.

Finally, we remark that the analysis of singularities using the continuous shearlet transform extends 'naturally' to the 3D setting. In particular, one can derive a characterization result of jump discontinuities which follows rather closely the analysis we presented in the 2D setting even though not all arguments from the 2D case carry over to this case (cf. [11, 12]). However, the analysis of the irregular boundary points and other types of singularities is more involved and only partial results are currently available in the references cited above.

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