TEST #1

No books or notes allowed. Please, write clearly and justify all your steps, to get proper credit for your work.

(1)[3 Pts] (i) State the definition of orthogonal complement of an inner product space $V$.

(ii) Let $V = \mathbb{R}^3$ and consider the subspace of $V$ given by

$$V_0 = \text{span}\{(1, 0, 2), (-1, -1, 1)\}.$$ 

Find the orthogonal complement of $V_0$ in $V$.

(2)[3 Pts] Consider the inner product space $V = L^2([0, 1])$. Compute the orthogonal projection of the function $f(x) = x^2$, for $x \in [0, 1]$, onto the subspace $V_0 = \text{span}\{\phi, \psi\}$, where

$$\phi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(3)[3 Pts] Consider the sequence of functions $(f_n)$ defined by

$$f_n(x) = \begin{cases} nx & 0 \leq x < \frac{1}{n} \\ 1 & \frac{1}{n} \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Draw the graph of $f_n(x)$ for two values of $n$ (e.g., $n = 2, 4$). Show that $(f_n)$ converges to the function $f(x) = 1$, $x \in [0, 1]$, in the $L^2$ norm.
TEST #2

Please, write clearly and justify all your steps, to get proper credit for your work.

(1)[5 Pts] Let \( f(x) = \cos^2(x) \).
(a) Sketch a graph of \( f \) over the interval \([-\pi, \pi]\).
(b) Expand the function \( f(x) = \cos^2(x) \) in a Fourier series valid on the interval \(-\pi \leq x \leq \pi\).
(b) Does the Fourier series of \( f \) converge uniformly to \( f \)? Justify your answer.

(2)[6 Pts] Consider the function \( f(x) = \begin{cases} -x^2 & -1 \leq x < 0 \\ x^2 & 0 \leq x \leq 1 \end{cases} \).
(a) Sketch a graph of \( f \) over the interval \([-1, 1]\).
(b) Expand the function \( f \) in a Fourier series valid on the interval \(-1 \leq x \leq 1\).
[HINT: You can take advantage of the symmetry of \( f \)]

(3)[4 Pts] (a) Show that if \( f \) is continuous on the interval \( 0 \leq x \leq a \), then its even periodic extension is continuous everywhere. Justify your answer.
(b) What about the odd periodic extension? What conditions are necessary to ensure that the odd periodic of \( f \) is continuous everywhere? Justify your answer.
[HINT: In both cases, it suffices to check the behavior near \( x = 0 \)]
TEST #3

Please, write clearly and justify all your steps, to get proper credit for your work. Open book test

(1)[5 Pts] Let
\[ f(t) = \begin{cases} t & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

(a) Compute \( \hat{f} \), the Fourier transform of \( f \).
(b) Express the real and imaginary part of \( \hat{f} \).
[HINT: You can take advantage of the symmetry of \( f \)]

(2)[5 Pts] Let
\[ \phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \]
and
\[ g(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \]

(a) Compute \( h(x) = (\phi * g)(x) \).
(b) Sketch the graphs of \( \phi \), \( f \) and \( h \) over the interval \([-1, 3]\).

(3)[3 Pts] Consider the filter
\[ f \rightarrow f * h_d, \]
where
\[ h_d(t) = \begin{cases} 1/d & 0 \leq t < d \\ 0 & \text{otherwise} \end{cases} \]

Let \[ f(t) = e^{-t} (\cos 2t + \sin 3t + \cos 18t + \sin 90t) , \quad t \in [0, 2\pi]. \]

Which value(s) of the parameter \( d \) for the filter \( h_d \) will ensure that the components of the signal \( f \) with frequencies above 50 are removed and the frequencies in the range 0 to 18 are retained? Justify your answer.
The orthogonal complement of \( V_0 \) in \( V \) is the set of all vectors in \( V \) which are orthogonal to \( V_0 \):

\[
V_0^\perp = \{ v \in V : \langle v, w \rangle = 0 \ \forall w \in V_0 \} \tag{1PT}
\]

Since \( \mathbb{R}^3 \) is 3-dimensional and \( V_0 \) is 2-dimensional, \( V_0^\perp \) is the span of the vectors which are orthogonal to \((1,0,2)\) and \((-1,-1,1)\).

\[
\begin{pmatrix}
1 \\
0 \\
2
\end{pmatrix} \times
\begin{pmatrix}
-1 \\
-1 \\
1
\end{pmatrix} = \begin{pmatrix}
2 \\
-1 \\
-1
\end{pmatrix} = (2, -1, -1) \tag{1PT}
\]

Hence \( V_0^\perp = \{ v \in V : v = \alpha (2, -1, -1), \ \alpha \in \mathbb{R}^2 \} \tag{1PT} \)

\[\text{1) } (i) \tag{1PT}\]

\( p_0 \) and \( q \) are an orthonormal basis of \( V_0 \). Hence, \( p_0 \), the orthogonal projection of \( f \) onto \( V_0 \) is given by

\[p_0(x) = \langle f, p_0 \rangle \psi(x) + \langle f, q \rangle \psi(x) \tag{1PT}\]

\[<f, p_0> = \int_0^1 x^2 \psi(x) \, dx = \frac{1}{3} \tag{1PT}\]

\[<f, q> = \int_0^1 x^2 \psi(x) \, dx = \int_0^{1/2} x^2 \, dx - \int_0^{1/2} x^2 \, dx = \frac{x^3}{3} \bigg|_0^{1/2} = \frac{1}{8} - \frac{1}{3} = \frac{1}{24} \tag{1PT}\]

\[= -\frac{1}{4} \tag{1PT}\]

\[p_0(x) = \frac{1}{3} \psi(x) - \frac{1}{4} \psi(x) \tag{1PT}\]

Need to show that \( f_n \to f \) in \( L^2 \) norm.

That is \( \| f_n - f \| \to 0 \).

That is \( \int_0^1 (f_n(x) - f(x))^2 \, dx = \int_0^1 (mx - 1)^2 \, dx = \frac{1}{3} \left( \frac{1}{n} \right) \to 0 \) as \( n \to \infty \).

This shows that \( \lim_{n \to \infty} \| f_n - f \| = \lim_{n \to \infty} \frac{1}{3n} = 0 \).
\( f(x) = \cos^2 x \)

\[ \text{[1 Pt] Draw graph} \]

(a) 

\[ \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \quad (\cos x \text{ is } \pi\text{-periodic}) \]

(b) By the uniqueness of the Fourier series, it follows that

The Fourier series of \( f \), over the interval \([-\pi, \pi]\) is

\[ f(x) = \frac{1}{2} + \frac{1}{2} \cos 2x \]

(c) Since \( f \) is continuous and piecewise smooth, then \( f(x) \) converges uniformly to \( f \) on \([-\pi, \pi]\)

\[ \text{[1 Pt]} \]

\[ \text{[1 Pt]} \]

\( f(x) \)

Neko Hit F is an odd function

\[ \text{[1 Pt]} \]

(b) Since \( f \) is an odd function, its Fourier series only contain

\( \cos \) terms. Hence, it can be computed as

\[ F(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \quad \text{where} \quad b_k = 2 \int_{0}^{1} f(x) \sin(k\pi x) \, dx \]

\[ b_k = 2 \int_{0}^{1} \cos^2 x \sin(k\pi x) \, dx = 2 \left[ \frac{-x^2 \cos(k\pi x)}{k\pi} \right]_{0}^{1} + \frac{1}{k\pi} \left[ x \sin(k\pi x) \right]_{0}^{1} \]

\[ = 2 \left[ \frac{-1}{k\pi} \cos(k\pi) + \frac{1}{k\pi} \sin(k\pi) \right] \]

\[ = 2 \left[ \frac{-1}{k\pi} \cos(k\pi) + \frac{1}{k\pi} \sin(k\pi) \right] \]

\[ = 2 \left[ \frac{1}{k\pi} \left( \frac{1}{k\pi} - 1 \right) \right] \]

\[ = 2 \left( \frac{(-1)^k}{k\pi} + \frac{2}{(k\pi)^3}(-1)^k - 1 \right) \]

\[ \text{[5 Pts]} \]

\( \text{This is all you get} \)

\( \text{FULL} \)
(a) If \( f \) is continuous on \([0, a]\), then its periodic extension satisfies \( f(x) = f(x + na) \) \( \forall n \in \mathbb{Z} \) and \( f(x) = f(-x) \).

In particular, \( f(0^+) = \lim_{h \to 0^+} f(h) = \lim_{h \to 0^-} f(-h) = f(0^-) \).

Thus, the periodic extension is continuous at \( x = 0 \).

(b) The odd extension satisfies \( f(x) = -f(-x) \).

Hence, \( \lim_{h \to 0^+} f(h) \neq \lim_{h \to 0^-} f(-h) = \lim_{h \to 0^+} f(h) \).

In general, \( \lim_{h \to 0^+} f(h) \neq \lim_{h \to 0^-} f(h) \).

The two sides are the same \( \iff \ f(h = f(h), \)

\[ \lim_{h \to 0^+} f(h) = -\lim_{h \to 0^-} f(h) = 0 \]

That is, the odd extension of \( f \) is continuous \( \iff \ f(0) = 0 \).
1. (a) \[ \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) e^{-iwt} \, dt \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\frac{1}{2}} t e^{-iwt} \, dt \]

Notice:
\[ t e^{-iwt} = t \cos wt - i t \sin wt \]

Since \( t \cos wt \) is odd, it gives no contribution to \( \hat{f}(w) \)

\[ = \frac{2i}{\sqrt{2\pi}} \int_{0}^{\frac{1}{2}} t (-\sin wt) \, dt \]

\[ = \frac{2i}{\sqrt{2\pi}} \left[ \frac{t}{w} \cos wt \right]_{0}^{\frac{1}{2}} - \frac{1}{w} \int_{0}^{\frac{1}{2}} \cos wt \, dt \]

\[ = \frac{2i}{\sqrt{2\pi}} \left[ \frac{1}{2w} \cos (\frac{1}{2}w) - \frac{1}{2w} \sin (\frac{1}{2}w) \right] \]

(b) \( \hat{f}(w) \) is purely imaginary

\[ \Re \hat{f}(w) = 0 \]

2. The convolution acts as a local averaging operator on windows of size \( d \).

To remove frequencies above 50, need

\[ d \geq \frac{2\pi}{50} \approx 0.12 \]

To preserve frequencies below 18, need

\[ d \ll \frac{2\pi}{18} \approx 0.33 \]

For example, we can choose \( d = 0.15 \).
\[ h(x) = (\varphi * g)(x) = \int_{\mathbb{R}} \varphi(t) g(x-t) \, dt \]

\[ = \int_{0}^{1} g(x-t) \, dt \]

It is clear from the graph of \( g(x-t) \) that \( h(x) = 0 \) if \( x \leq 0 \)

and \( h(x) = 0 \) if \( x-1 > 1 \Rightarrow x > 2 \)

Hence, we only need to examine \( 0 \leq x \leq 2 \)

If \( 0 \leq x \leq 1 \), then

\[ h(x) = \int_{0}^{x} (x-t) \, dt = \left[ xt - \frac{t^2}{2} \right]_{0}^{x} = x^2 - \frac{x^2}{2} = \frac{x^2}{2} \]

If \( 1 < x \leq 2 \), then

\[ h(x) = \int_{x-1}^{x} (x-t) \, dt = \left[ xt - \frac{t^2}{2} \right]_{x-1}^{x} = x - \frac{x}{2} - (x-1)x + \frac{(x-1)^2}{2} \]

\[ = x \left( 1 - \frac{x}{2} \right) \]

Conclusion

\[ h(x) = \begin{cases} 
0 & \text{if } x < 0 \\
0 & \text{if } x > 2 \\
\frac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\
x \left( 1 - \frac{x}{2} \right) & \text{if } 1 < x \leq 2 
\end{cases} \]