Connectivity in the Set of Gabor frames

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Abstract

In this paper we present a constructive proof that the set of Gabor frames is path-connected in the $L^2(\mathbb{R}^n)$-norm. In particular, this result holds for the set of Gabor Parseval frames as well as for the set of Gabor orthonormal bases. In order to prove this result, we introduce a construction which shows exactly how to modify a Gabor frame or Parseval frame to obtain a new one with the same property. Our technique is a modification of a method used in [6] to study the connectivity of affine Parseval frames.

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1 Introduction

The study of the topological properties of Gabor and affine systems is an important topic in the wavelet and Gabor theory. Some special prominence has been given in the literature to the problem of connectivity. This question was originally raised in [1], where the significance of the problem is also emphasized. Despite several important contributions to the study of this problem (for example in [16], [15], [5], [6]) there are still a number of open questions.

In this paper, we are concerned with the problem of the connectivity for Gabor systems. In [5], [4], Gabardo, Han and Larson used an abstract result from the theory of von Neumann algebras to prove that the sets of Gabor Parseval frames and Gabor frames are path connected in the norm topology of L^2. Unfortunately, their proof is not constructive and this is a limitation since, in many situations, one would like to explicitly construct the path connecting any Gabor Parseval frame or frame to a fixed element in the same set. The main contribution of this paper is to present a *constructive* proof of these results. Unlike the abstract Hilbert space method in [5], [4], our approach involves an explicit deformation g_t, $t \in [0, 1]$, of an arbitrary frame generator g_0, connecting g_0 to a fixed band-limited generator g_1. If $g_0 \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p < \infty$, such deformation is continuous in L^p, $1 \le p < \infty$. Furthermore, we obtain an explicitly controlled deformation of the Gabor frame coefficients by uniformly continuous functions. The techniques developed in this paper are relevant to different problems, including the study of the stability of frames under perturbation.

To illustrate our technique, consider the one-dimensional Gabor systems

$$\mathcal{G}_b(g) = \{e^{2\pi i bmx}\, g(x-k) :\, k, m \in \mathbb{Z}\},$$

where $g \in L^2(\mathbb{R})$, $0 < b < 1$. For $E = [0, b)$, let $g_1 = (\chi_E)^\vee$. Then the system $\mathcal{G}_b(g_1)$ is a Parseval frame for $L^2(\mathbb{R})$. In our approach, we connect any g_0 such that $\mathcal{G}_b(g_0)$ is a Parseval frame for $L^2(\mathbb{R})$ to g_1 in the following way. For $0 \le t \le 1$, let $E_t = [0, bt)$ and define a deformation g_t by

$$\hat{g}_t(\xi) = \begin{cases} \hat{g}_0(\xi), & \xi \in \widehat{\mathbb{R}} \setminus \tau_b(E_t) \\ 1, & \xi \in E_t, \end{cases}$$

where $\tau_b(E_t) = \cup_{k\in\mathbb{Z}}(E_t + bk)$. As we mentioned, we can show that this deformation is continuous in L^p, $1 \le p < \infty$.

Before describing our approach in details, it will be useful to establish some notation and definitions.

1.1 Preliminaries

In this paper, $GL_n(\mathbb{R})$ denotes the $n \times n$ invertible matrices with real coefficients and, similarly, $GL_n(\mathbb{Q})$ denotes the $n \times n$ invertible matrices with rational coefficients. The Fourier transform is defined as $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi \cdot x}\, dx$ and the inverse Fourier transform is $\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi$. Throughout the paper, the space $\mathbb{T}^n$ will be identified with $[0,1)^n$. The Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$ is denoted by $\mu(\Omega)$.

Λ is a **lattice** in $\mathbb{R}^n$ if $\Lambda = A\mathbb{Z}^n$, where $A \in GL_n(\mathbb{R})$. Given a measurable set $\Omega \subseteq \mathbb{R}^n$ and a lattice Λ in $\mathbb{R}^n$, we say that Ω **tiles** $\mathbb{R}^n$ by Λ, or Ω is a **fundamental domain** of Λ, if the following two properties hold:

(i) $\bigcup_{\ell \in \Lambda} (\Omega + \ell) = \mathbb{R}^n$ a.e.;

(ii) $\mu\Big((\Omega + \ell) \cap (\Omega + \ell')\Big) = 0$ for any $\ell \neq \ell'$ in Λ.

We say that Ω **packs** $\mathbb{R}^n$ by Λ if only (ii) holds. Equivalently, Ω tiles $\mathbb{R}^n$ by Λ if and only if

$$\sum_{\ell \in \Lambda} \chi_\Omega(x - \ell) = 1 \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.1}$$

and Ω packs $\mathbb{R}^n$ by Λ if and only if

$$\sum_{\ell \in \Lambda} \chi_\Omega(x - \ell) \leq 1 \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.2}$$

Clearly, $\mu(\Omega) = |\det A|$ if Ω tiles by Λ, and $\mu(\Omega) \leq |\det A|$ if Ω packs by Λ. Furthermore, if Ω packs $\mathbb{R}^n$ by Λ and $\mu(\Omega) = |\det A|$, then Ω necessarily tiles $\mathbb{R}^n$ by Λ.

We need the following facts from the theory of frames. A countable sequence $\{g_i\}_{i \in \mathcal{I}}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **Bessel sequence** if there exists a constant $\beta > 0$ so that

$$\sum_{i \in \mathcal{I}} |\langle f, g_i \rangle|^2 \;\leq\; \beta\, \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If, in addition, there is a constant $0 < \alpha \leq \beta$ so that

$$\alpha\, \|f\|^2 \;\leq\; \sum_{i \in \mathcal{I}} |\langle f, g_i \rangle|^2 \;\leq\; \beta\, \|f\|^2 \qquad \text{for all } f \in \mathcal{H},$$

then $\{g_i\}_{i \in \mathcal{I}}$ is a **frame** for $\mathcal{H}$. The numbers α, β are called the **lower** and **upper frame bounds**, respectively. The frame is **tight** if α and β can be chosen so that $\alpha = \beta$, and is a **Parseval frame** if $\alpha = \beta = 1$. Given a frame $\{g_i\}_{i \in \mathcal{I}}$ of $\mathcal{H}$ with frame bounds α and β, the **frame operator** S, defined by $S\, f \;=\; \sum_{i \in \mathcal{I}} \langle f, g_i \rangle\, g_i$, is a bounded, invertible and positive mapping of $\mathcal{H}$ onto itself. This provides the **frame decomposition**:

$$f \;=\; \sum_{i \in \mathcal{I}} \langle f, S^{-1} g_i \rangle\, g_i \;=\; \sum_{i \in \mathcal{I}} \langle f, g_i \rangle\, S^{-1} g_i, \qquad \text{for all } f \in \mathcal{H}, \tag{1.3}$$

with convergence in $\mathcal{H}$. The sequence $\{S^{-1}g_i\}_{i\in\mathcal{I}}$ is also a frame for $\mathcal{H}$, called the **canonical dual frame** of $\{g_i\}_{i\in\mathcal{I}}$, and has upper and lower frame bounds β^{-1} and α^{-1}, respectively. If the frame is tight, then $S^{-1} = \alpha^{-1} I$, where I is the identity operator, and the frame decomposition becomes:

$$f = \frac{1}{\alpha} \sum_{i\in\mathcal{I}} \langle f, g_i\rangle\, g_i \qquad \text{for all } f \in \mathcal{H}, \tag{1.4}$$

with convergence in $\mathcal{H}$. Equations (1.3) and (1.4) show that a frame provides a basis-like representation. In general, however, a frame need not be a basis, and the elements $\{g_i\}$ need not to be linearly independent. In particular, if $\{g_i\}_{i\in\mathcal{I}}$ is a Parseval frame, then $\|g_i\| \le 1$ for all $i \in \mathcal{I}$, and the frame is an orthonormal basis for $\mathcal{H}$ if and only if $\|g_i\| = 1$ for all $i \in \mathcal{I}$ (cf. [10, Ch.8]). We refer to [3, 10] for additional information about frames.

The **Gabor systems** generated by $g \in L^2(\mathbb{R}^n)$ and associated with the matrices $B, C \in GL_n(\mathbb{R})$ are the collections

$$\mathcal{G}_{B,C}(g) = \{M_{Bm}\, T_{Ck}\, g : m, k \in \mathbb{Z}^n\}, \tag{1.5}$$

where T_y, M_z, $y, z \in \mathbb{R}^n$, are, respectively, the **translation** and **modulation** operators, which are defined by

$$T_y f(x) = f(x - y), \qquad M_z f(x) = e^{2\pi i z\cdot x} f(x).$$

If we change the order in which the translations and modulations are applied, we obtain the systems $\widetilde{\mathcal{G}}_{B,C}(g) = \{T_{Ck}\, M_{Bm}\, g : m, k \in \mathbb{Z}^n\}$. Simple calculations show that $T_{Ck}\, M_{Bm}\, g = e^{-2\pi i Bm\cdot Ck}\, M_{Bm}\, T_{Ck}\, g$, for every $m, k \in \mathbb{Z}^n$, and that $(T_{Ck}\, M_{Bm}\, g)^\wedge = M_{-Ck}\, T_{Bm}\, \hat{g}$, for every $m, k \in \mathbb{Z}^n$. These observations immediately imply that

Lemma 1.1. *(a) $\mathcal{G}_{B,C}(g)$ is a frame for $L^2(\mathbb{R}^n)$ if and only if $\widetilde{\mathcal{G}}_{B,C}(g)$ is a frame for $L^2(\mathbb{R}^n)$; furthermore, the frame constants α and β can be taken to be the same.*

(b) $\mathcal{G}_{B,C}(g)$ is a frame for $L^2(\mathbb{R}^n)$ if and only if $\mathcal{G}_{C,B}(\hat{g})$ is a frame for $L^2(\mathbb{R}^n)$; furthermore, the frame constants α and β can be taken to be the same.

Even though (1.5) is the customary definition of a Gabor system, it is easy to see that, by re-scaling the functions g, one can represent the same system using only one matrix (instead of the two matrices B and C). To show that this is the case, observe that

$$M_{Bm}\, T_{Ck}\, g(x) = e^{2\pi i Bm\cdot x}\, g(x - Ck) = e^{2\pi i Bm\cdot x}\, |\det C|^{-1/2}\, |\det C|^{1/2}\, g(C(C^{-1}x - k)).$$

Using the change of variables $y = C^{-1}x$ and the new function $g'(x) = g(C\,x)$, we have that $M_{Bm}\, T_{Ck}\, g(x) = M_{C^T Bm}\, T_k\, g'(y)$. Thus, we have:

$$\mathcal{G}_{B,C}(g) = \mathcal{G}_{C^T B}(g') = \{M_{C^T Bm}\, T_k\, g' : m, k \in \mathbb{Z}^n\}.$$

Therefore, without loss of generality, any Gabor system can be represented as

$$\mathcal{G}_B(g) = \{M_{Bm}\, T_k\, g : m, k \in \mathbb{Z}^n\}, \tag{1.6}$$

where $B \in GL_n(\mathbb{R})$. In order to simplify the notation, we will adopt this definition of Gabor systems in the following.

The **Gabor frame operator** $S_B(g)$, associated with the system $\mathcal{G}_B(g)$, $g \in L^2(\mathbb{R}^n)$, is given by

$$S_B(g)\, f = \sum_{m\in\mathbb{Z}^n} \sum_{k\in\mathbb{Z}^n} \langle f, M_{Bm}\, T_k\, g\rangle\, M_{Bm}\, T_k\, g(x).$$

We can consider the operator $S_B(g)$ as a sesquilinear form $\langle f, f\rangle \mapsto \langle S_B(g)\, f, f\rangle$ on $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. For $i, j \in \mathbb{Z}^n$, let

$$\gamma_{i,j}(\xi, \hat{g}) = \sum_{k\in\mathbb{Z}^n} \hat{g}(\xi + i - Bk)\, \overline{\hat{g}(\xi + j - Bk)},$$

and let $\Gamma(\xi)$ be the matrix whose (i, j)-entries are the functions $\gamma_{i,j}(\xi, \hat{g})$. Observe that these functions satisfy $\gamma_{i,j}(\xi, \hat{g}) = \gamma_{0,j-i}(\xi + i, \hat{g})$. We, thus, have the "matrix representation" for $S_B(g)$ that was obtained by Ron and Shen in [14] (cf. also [7, Sec. 6.3]):

$$\big\langle S_B(g)\, f,\, f\big\rangle = \sum_{j\in\mathbb{Z}^n} \sum_{i\in\mathbb{Z}^n} \int_{[0,1)^n} \gamma_{i,j}(\xi, \hat{g})\, \overline{\hat{f}(\xi + i)}\, \hat{f}(\xi + j)\, d\xi. \tag{1.7}$$

Observe that this representation is well-defined for all $f \in \mathcal{D}$, where

$$\mathcal{D} = \big\{f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat{f} \text{ is compact}\big\} \tag{1.8}$$

is dense in $L^2(\mathbb{R}^n)$. This representation of the Gabor frame operator is used to derive the following complete characterization of the functions $g \in L^2(\mathbb{R}^n)$ such that $S_B(g)$ is a Parseval frame or, more generally, a frame for $L^2(\mathbb{R}^n)$.

Theorem 1.2 ([14]). (*a*) *The system $\mathcal{G}_B(g)$ is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if, for all $m \in \mathbb{Z}^n$,*

$$\gamma_{0,m}(\xi, \hat{g}) = \sum_{k\in\mathbb{Z}^n} \hat{g}(\xi - Bk)\, \overline{\hat{g}(\xi - Bk + m)} = \delta_{0,m} \tag{1.9}$$

for a.e. $\xi \in \mathbb{R}^n$.

(*b*) *The system $\mathcal{G}_B(g)$ is a frame for $L^2(\mathbb{R}^n)$ if and only if $\alpha = \operatorname{ess\,inf}_{\xi\in\mathbb{R}^n} \|\Gamma(\xi, \hat{g})\|_{op} > 0$ and $\beta = \operatorname{ess\,sup}_{\xi\in\mathbb{R}^n} \|\Gamma(\xi, \hat{g})\|_{op} < \infty$, where α and β are the frame bounds for $\mathcal{G}_B(g)$.*

Theorem 1.2 implies the following:

Corollary 1.3. *If $\mathcal{G}_B(g)$ is a Parseval frame for $L^2(\mathbb{R}^n)$ then $|\det B| \leq 1$, and $\mathcal{G}_B(g)$ is an orthonormal basis if and only if $|\det B| = 1$.*

Proof. Let $\mathcal{G}_B(g)$ be a Parseval frame and Ω be a fundamental domain for $B\,\mathbb{Z}^n$. Then, by Theorem 1.2, part (a),

$$\|g\| = \sum_{k\in\mathbb{Z}^n} \int_{\Omega+B\,k} |\hat{g}(\xi)|^2\,d\xi = \int_{\Omega} \sum_{k\in\mathbb{Z}^n} |\hat{g}(\xi+B\,k)|^2\,d\xi = \mu(\Omega) = |\det B|.$$

The proof now easily follows since, as we mentioned before, if $\mathcal{G}_B(g)$ is a Parseval frame, then $\|T_k\,M_{Bm}\,g\| = \|g\| \le 1$ for all $m,k \in \mathbb{Z}^n$, and is an orthonormal basis if and only if $\|g\| = 1$. $\square$

More generally one can show that, if $|\det B| > 1$, then the Gabor system $\mathcal{G}_B(g)$ cannot be a frame. This result is known as **Rieffel's theorem** (cf. [13], [11]).

2 Connectivity in the set of Gabor frames

In this section, we show that the set of Gabor frames is pathwise connected under the L^2–norm. In particular, the set of Gabor Parseval frames is also pathwise connected.

Let $\mathbf{F}_B$ (resp., $\mathbf{PF}_B$) be the sets of all functions $g \in L^2(\mathbb{R}^n)$ such that $\mathcal{G}_B(g)$, given by (1.6), is a frame (resp., Parseval frame) for $L^2(\mathbb{R}^n)$. This is the main result of this paper:

Theorem 2.1. *Let $B \in GL_n(\mathbb{Q})$ and suppose that $|\det B| \le 1$. Then the following is true:*
(a) *the set $\mathbf{F}_B$ is path connected in the norm topology of $L^2(\mathbb{R}^n)$;*
(b) *the set $\mathbf{PF}_B$ is path connected in the norm topology of $L^2(\mathbb{R}^n)$.*

Before presenting the proof of this theorem, let us make a few remarks. As we explained before, this theorem was originally proved in [5] as an application from the theory of von Neumann algebras. Unlike the original argument, however, we will present a *constructive* proof of this result. Observe that the assumption $B \in GL_n(\mathbb{Q})$ is needed in the construction that we use in our proof. We will make additional comments about this in Section 2.3. Finally, recall that, by Rieffel's theorem, the assumption $|\det B| \le 1$ is a necessary condition for the Gabor systems $\mathcal{G}_B(g)$ to form a frame.

Our proof of Theorem 2.1 is based on a new general result, which shows how to modify an element $g \in \mathbf{F}_B$ in such a way that the modified function g' is in the same set; this result is discussed in Section 2.1. The proof of Theorem 2.1 is presented in Section 2.2.

2.1 Basic concepts

It is easy to see from equations (1.9) that if J is a fundamental domain of $B\mathbb{Z}^n$, then the Gabor system $\mathcal{G}_B(g_J)$, where $\hat{g}_J = \chi_J$ is a Parseval frame (and, thus, a frame) for $L^2(\mathbb{R}^n)$. In order to prove the connectivity in the set $\mathbf{F}_B$, we have to join any given $g \in \mathbf{F}_B$ to g_J

through a continuous path inside the set $\mathbf{F}_B$. The idea that we will use is to move points from the support of $\hat{g}$ into J in such a way that the conditions of Theorem 1.2 are preserved. A similar idea was used in [15] and [6] to prove a number of connectivity results about affine systems.

We will need the following definition. Given $B \in GL_n(\mathbb{R})$, we denote the **orbit space with respect to** B of a measurable set $E \subset \mathbb{R}^n$ as the set

$$\tau_B(E) = \bigcup_{k\in\mathbb{Z}^n} (E + B\ k).$$

If $B = I$, we simply write $\tau(E) = \tau_I(E)$.

The following properties of the orbit space of a set will be useful.

Lemma 2.2. *Let $\{E_k : k \geq 1\} \subset \mathbb{R}^n$ be measurable sets. We have the following:*

(a) $\tau_B(\bigcup_{k\geq 1} E_k) = \bigcup_{k\geq 1} \tau_B(E_k)$.

(b) $\tau_B(E_1) \setminus \tau_B(E_2) \subseteq \tau_B(E_1 \setminus E_2)$.

Proof. (a) If $\xi \in \tau_B(\bigcup_{k\geq 1} E_k)$, then $\xi = \eta + Bm$, for some $\eta \in \bigcup_{k\geq 1} E_k$, $m \in \mathbb{Z}^n$. Thus $\eta \in E_{k_0}$, for some $k_0 \geq 1$, and, as a consequence, $\xi = \eta + Bm \in \tau_B(E_{k_0})$. It follows that $\xi = \eta + Bm \in \bigcup_{k\geq 1} \tau_B(E_k)$. Conversely, if $\xi \in \bigcup_{k\geq 1} \tau_B(E_k)$, then $\xi \in \tau_B(E_{k_0})$ for some $k_0 \geq 1$. Then $\xi = \eta + Bm$, for some $\eta \in E_{k_0}$, $m \in \mathbb{Z}^n$. As a consequence, $\eta \in \bigcup_{k\geq 1} E_k$ and, hence, $\xi = \eta + Bm \in \tau_B(\bigcup_{k\geq 1} E_k)$.

(b) If $\xi \in \tau_B(E) \setminus \tau_B(F)$, then $\xi = e + Bm$, with $e \in E$, $m \in \mathbb{Z}^n$, but $e \notin F$, otherwise it would be $\xi = e + Bm \in \tau_B(F)$. Thus $e \in E \setminus F$ and, as a consequence, $\xi = e + Bm \in \tau_B(E \setminus F)$. □

We now deduce a proposition which shows how to modify a function $g \in \mathbf{F}_B$ to obtain a new one. For a measurable set $E \subset \mathbb{R}^n$, let g_E be defined by

$$\hat{g}_E(\xi) = \begin{cases} \hat{g}(\xi), & \xi \in \mathbb{R}^n \setminus \tau_B(E) \\ \chi_E(\xi), & \xi \in \tau_B(E). \end{cases} \tag{2.1}$$

In other words, we have $\hat{g}_E = \chi_E + \hat{g}\,\chi_{\mathbb{R}^n\setminus\tau_B(E)}$. We have the following result:

Proposition 2.3. *Let $g \in \mathbf{F}_B$, with lower and upper frame bounds α and β, respectively, and $E \subset \mathbb{R}^n$ be a measurable set with $\mu(E) < \infty$. Assume that*

(i) *E packs $\mathbb{R}^n$ by $\mathbb{Z}^n$,*

(ii) *E packs $\mathbb{R}^n$ by $B\mathbb{Z}^n$,*

(iii) $\tau\big(\tau_B(E)\big) = \tau_B(E)$.

Then g_E, given by (2.1), is also an element of $\mathbf{F}_B$ with lower and upper frame bounds α_E and β_E, respectively, where

$$\alpha_E = \alpha,\ \beta_E = \beta, \qquad \textit{if } \mu(E) = 0, \tag{2.2}$$

$$\alpha_E = \beta_E = 1, \qquad \textit{if } \mu(\tau_B(E)) = 0, \tag{2.3}$$

$$\alpha_E = \min(1,\alpha),\ \beta_E = \max(1,\beta), \qquad \textit{otherwise.} \tag{2.4}$$

Proof. It follows immediately from (iii) that both $\tau_B(E)$ and $\mathbb{R}^n \setminus \tau_B(E)$ are unions of orbits for translations by $\mathbb{Z}^n + B\,\mathbb{Z}^n$. Thus, if $\xi \in \tau_B(E)$, then $\xi + \ell - Bk \in \tau_B(E)$ for any $\ell, k \in \mathbb{Z}^n$. Similarly, if $\xi \in \mathbb{R}^n \setminus \tau_B(E)$, then $\xi + \ell - Bk \in \mathbb{R}^n \setminus \tau_B(E)$ for any $\ell, k \in \mathbb{Z}^n$.

It follows that, for any $i, j \in \mathbb{Z}^n$, if $\xi \in \mathbb{R}^n \setminus \tau_B(E)$, then

$$\gamma_{i,j}(\xi, \hat{g}_E) = \sum_{k\in\mathbb{Z}^n} \hat{g}_E(\xi + i - Bk)\, \overline{\hat{g}_E(\xi + j - Bk)} = \gamma_{i,j}(\xi, \hat{g}),$$

and, if $\xi \in \tau_B(E)$, then

$$\gamma_{i,j}(\xi, \hat{g}_E) = \sum_{k\in\mathbb{Z}^n} \chi_E(\xi + i - Bk)\, \overline{\chi_E(\xi + j - Bk)}.$$

Because of assumptions (i) and (ii), this last equation is zero, unless $i = j$. Thus, combining these observations, we have:

$$\gamma_{i,j}(\xi, \hat{g}_E) = \begin{cases} \gamma_{i,j}(\xi, \hat{g}) & \text{if } \xi \in \mathbb{R}^n \setminus \tau_B(E) \\ \delta_{i,j} & \text{if } \xi \in \tau_B(E). \end{cases} \tag{2.5}$$

Let $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (1.8). Using the matrix representation of the Gabor frame operator, given by (1.7), and equation (2.5) we have:

$$\begin{aligned} \big\langle S_B(g_E)\, f,\, f\big\rangle = &\sum_{i\in\mathbb{Z}^n} \int_{[0,1)^n \bigcap \tau_B(E)} \overline{\hat{f}(\xi + i)}\, \hat{f}(\xi + i)\, d\xi + \\ &+ \sum_{i,j\in\mathbb{Z}^n} \int_{[0,1)^n \setminus \tau_B(E)} \gamma_{i,j}(\xi, \hat{g})\, \overline{\hat{f}(\xi + i)}\, \hat{f}(\xi + j)\, d\xi. \end{aligned} \tag{2.6}$$

By a simple periodization argument (cf. [11, Sec. 4]), we have that $\sum_{i\in\mathbb{Z}^n} \int_{[0,1)^n} \overline{\hat{h}(\xi + i)}\, \hat{h}(\xi + i)\, d\xi = \|h\|^2$, for all $h \in L^2(\mathbb{R}^n)$, and, similarly, if $E \subset [0,1]^n$,

$$\sum_{i\in\mathbb{Z}^n} \int_{[0,1)^n \cap E} \overline{\hat{h}(\xi + i)}\, \hat{h}(\xi + i)\, d\xi = \int_{\tau(E)} |\hat{h}(\xi)|^2\, d\,\xi = \|\hat{h}\|^2_{L^2(\tau(E))}, \tag{2.7}$$

for all $h \in L^2(\mathbb{R}^n)$. Using the assumption (iii), we have that $\tau\big([0,1]^n \cap \tau_B(E)\big) = \tau_B(E)$ and $\tau\big([0,1]^n \setminus \tau_B(E)\big) = \widehat{\mathbb{R}}^n \setminus \tau_B(E)$. Thus, using these observations and equation (2.7) in (2.6), we have

$$\big\langle S_B(g_E)\, f,\, f\big\rangle = \|\hat{f}\|_{L^2(\tau(E))} + \big\langle S_B(g)\, f_S,\, f_S\big\rangle, \tag{2.8}$$

where $\hat{f}_S = \chi_S \hat{f}$ and $S = \widehat{\mathbb{R}}^n \setminus \tau_B(E)$. Since $g \in \mathbf{F}_B$, then, for all $\hat{f}_S = \chi_S \hat{f}$ where $f \in \mathcal{D}$ we have:

$$\big\langle S_B(g)\, f_S,\, f_S\big\rangle \geq \alpha\, \|f_S\|^2 \quad \text{and} \quad \big\langle S_B(g)\, f_S,\, f_S\big\rangle \leq \beta\, \|f_S\|^2.$$

The extension of these inequalities to all $f \in L^2(\mathbb{R}^n)$ follows by a standard density argument. These observations, together with (2.8), show that $g_E \in \mathbf{F}_B$ and that the frame bounds α_E and β_E are given by (2.2), (2.3) or (2.4), depending on the set E. $\square$

In the special case when one starts from a function $g \in \mathbf{PF}_B$, then Proposition 2.3 shows that also $g_E \in \mathbf{PF}_B$, since, in this situation, $\alpha_E = \alpha = 1$ and $\beta_E = \beta = 1$. Furthermore, in the case of Parseval frames, we can deduce an even stronger result which gives necessary and sufficient condition so that, if $g \in \mathbf{PF}_B$, then also $g_E \in \mathbf{PF}_B$. The following proposition is inspired by [6, Prop. 2.1], where a similar result is proved in the case of affine systems.

Proposition 2.4. *If $g \in \mathbf{PF}_B$ and $E \subset \mathbb{R}^n$ is a measurable set, then g_E, given by (2.1), is also an element of $\mathbf{PF}_B$ if and only if:*

(i) *E packs $\mathbb{R}^n$ by $\mathbb{Z}^n$,*

(ii) *E packs $\mathbb{R}^n$ by $B\mathbb{Z}^n$*

(iii) $\mu\Big(supp\, \hat{g} \bigcap \big(\tau(E) \setminus \tau_B(E)\big)\Big) = 0.$

In addition, we have that $\|g\| = \|g_E\|$.

Remark 2.5. Observe that, if the hypotheses of Proposition 2.3 are satisfied, then also conditions (i), (ii) and (iii) in Proposition 2.4 are satisfied.

In addition, observe that, since a Gabor Parseval frame $\mathcal{G}(g)$ is an orthonormal basis if and only if $\|g\| = 1$, then Proposition 2.4 also implies that $\mathcal{G}(g_E)$ is an orthonormal basis whenever $\mathcal{G}(g)$ is an orthonormal basis.

Proof. For $g \in L^2(\mathbb{R}^n)$ and $m \in \mathbb{Z}^n$, define

$$t_m(\xi, \hat{g}) = \sum_{k \in \mathbb{Z}^n} \hat{g}(\xi - Bk)\, \overline{\hat{g}(\xi - Bk + m)}. \tag{2.9}$$

Observe that $t_m(\xi, \hat{g})$ is constant on each orbit $\tau_B(\xi) = \{\xi' \in \mathbb{R}^n : \xi' = \xi + B\, k,\, k \in \mathbb{Z}^n\}$, and that $t_m(\xi, \hat{g}) = t_{-m}(\xi + m, \hat{g})$. Recall that, by Theorem 1.2, $t_m(\xi, \hat{g}) = \delta_{m,0}$ for a.e.

$\xi \in \mathbb{R}^n$. By the same theorem we also have that $g_E \in \mathbf{PF}_B$ if and only if $t_m(\xi, \hat{g}_E) = \delta_{m,0}$ for a.e. $\xi \in \mathbb{R}^n$.

We consider first the case $m = 0$. If $\xi \in \mathbb{R}^n \setminus \tau_B(E)$, then $\xi - Bk \in \mathbb{R}^n \setminus \tau_B(E)$ for each $k \in \mathbb{Z}^n$ and

$$t_0(\xi, \hat{g}_E) = t_0(\xi, \hat{g}) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n \setminus \tau_B(E).$$

On the other hand, if $\xi \in \tau_B(E)$, then $\xi - Bk \in \tau_B(E)$ for each $k \in \mathbb{Z}^n$. Since $\hat{g}_E(\xi) = \chi_E(\xi)$, when $\xi \in \tau_B(E)$, then

$$t_0(\xi, \hat{g}_E) = \chi_E + \sum_{k \neq 0} \chi_E(\xi + B\,k).$$

It follows that $t_0(\xi, \hat{g}_E) = 1$ for $\xi \in \tau_B(E)$ if and only if E is a packing set by $B\mathbb{Z}^n$. We have thus proved that $t_0(\xi, \hat{g}_E) = 1$ for a.e. $\xi \in \mathbb{R}^n$ if and only if E is a packing set by $B\mathbb{Z}^n$.

Let us consider now the function $t_m(\xi, \hat{g}_E)$ for $m \neq 0$. By the argument in the case $m = 0$, we can now assume that E is a packing set by $B\mathbb{Z}^n$. Fix $m \neq 0$. If $(\xi - Bk) \in \mathbb{R}^n \setminus \tau_B(E)$ and $(\xi - Bk + m) \in \mathbb{R}^n \setminus \tau_B(E)$, for some $k \in \mathbb{Z}^n$, then

$$t_m(\xi, \hat{g}_E) = t_m(\xi, \hat{g}) = 0.$$

Suppose now that $\xi - Bk \in \tau_B(E)$. Then $\xi \in \tau_B(E)$ and we can write $\xi = \eta - Bk_0$ for $\eta \in E$, $k_0 \in \mathbb{Z}^n$. Thus, with the change of indices $k' = k + k_0$, since E packs $\mathbb{R}^n$ by $B\mathbb{Z}^n$, we have:

$$\begin{aligned}
t_m(\xi, \hat{g}_E) &= \sum_{k \in \mathbb{Z}^n} \hat{g}_E(\xi - Bk)\, \overline{\hat{g}_E(\xi - Bk + m)} \\
&= \sum_{k \in \mathbb{Z}^n} \hat{g}_E(\eta - B(k + k_0))\, \overline{\hat{g}_E(\eta - B(k + k_0) + m)} \\
&= \sum_{k' \in \mathbb{Z}^n} \hat{g}_E(\eta - Bk')\, \overline{\hat{g}_E(\eta - Bk' + m)} \\
&= \chi_E(\eta)\, \overline{\hat{g}_E(\eta + m)}.
\end{aligned}$$

Now there are two cases to consider: either $\eta + m \in \tau_B(E)$ or $\eta + m \notin \tau_B(E)$. If $\eta + m \in \tau_B(E)$, then

$$t_m(\xi, \hat{g}_E) = \chi_E(\eta)\, \overline{\hat{g}_E(\eta + m)} = \chi_E(\eta)\, \chi_E(\eta + m),$$

and this quantity is zero if and only if E packs $\mathbb{R}^n$ by $\mathbb{Z}^n$. On the other hand, if $\eta + m \notin \tau_B(E)$, then $\eta + m$ must be in $(E + \mathbb{Z}^n) \setminus \tau_B(E)$ and so $t_m(\xi, \hat{g}_E) = 0$ if and only if condition (iii) holds. Finally, consider the case $(\xi - Bk + m) \in \tau_B(E)$. In this situation, we have $\xi + m \in \tau_B(E)$ and, since $t_m(\xi, \hat{g}_E) = t_{-m}(\xi + m, \hat{g}_E)$, the analysis of this case is exactly the same as the previous case.

We have thus proved that, for $m \neq 0$, $t_m(\xi, \hat{g}_E) = 0$ for a.e. $\xi \in \mathbb{R}^n$ if and only if conditions (ii) and (iii) hold.

To show that $\|g\|^2 = \|g_E\|^2$, observe that

$$\|g\|^2 - \|g_E\|^2 = \int_{\tau_B(E)} |\hat{g}(\xi)|^2 \, d\xi - \int_{\tau_B(E)} \chi_E(\xi) \, d\xi = \int_E \sum_{k \in \mathbb{Z}^n} |\hat{g}(\xi + Bk)|^2 \, d\xi - \mu(E).$$

Since $\mathcal{G}(g)$ is a Parseval frame, then, by Theorem 1.2, $\sum_{k\in\mathbb{Z}^n} |\hat{g}(\xi + Bk)|^2 = 1$ a.e., and, thus, $\|g\|^2 - \|g_E\|^2 = 0$. $\square$

2.2 Proof of the main theorem

We will now construct a family of sets E_S satisfying the assumptions of Proposition 2.3. The following basic result, which is due to Han and Wang [9], will be useful.

Theorem 2.6. *Let $\Lambda_1 = A_1\mathbb{Z}^n$ and $\Lambda_2 = A_2\mathbb{Z}^n$ be two lattices in $\mathbb{R}^n$ such that $|\det A_1| \leq |\det A_2|$. Then there exists a measurable set $J \subseteq \mathbb{R}^n$ such that J tiles $\mathbb{R}^n$ by Λ_1 and packs $\mathbb{R}^n$ by Λ_2. Furthermore, if A_1 and A_2 are in $GL_n(\mathbb{Q})$, then J can be chosen to be a finite union of congruent hyper-rectangles in $\mathbb{R}^n$.*

Let $B \in GL_n(\mathbb{R})$ with $|\det B| \leq 1$, and let $J \subset \mathbb{R}^n$ be a measurable set such that J tiles $\mathbb{R}^n$ by $B\mathbb{Z}^n$ and packs $\mathbb{R}^n$ by $\mathbb{Z}^n$. By Theorem 2.6, such a set exists. For a measurable set $S \subseteq J$, define

$$E_S = \tau_B\big(\tau(S)\big) \bigcap J. \tag{2.10}$$

Since $E_S \subset J$, the assumptions (i) and (ii) of Proposition 2.3 are satisfied. We need to show that condition (iii) of Proposition 2.3 is also satisfied. Since J is a fundamental domain for the $B\,\mathbb{Z}^n$ translations, then each point $\eta \in \tau_B\big(\tau(S)\big)$ is of the form $\eta = \xi + B\,k$, for some $k \in \mathbb{Z}^n$ and $\xi \in E_S$. This implies that $\tau_B(E_S) = \tau_B\big(\tau(S)\big)$. Therefore, we have that

$$\tau\big(\tau_B(E_S)\big) = \tau\Big(\tau_B\big(\tau(S)\big)\Big) = \tau_B\Big(\tau\big(\tau(S)\big)\Big) = \tau_B\big(\tau(S)\big) = \tau_B(E_S),$$

and, as a consequence, the set E_S satisfies condition (iii) of Proposition 2.3. For E_S constructed as above, let g_{E_S} be defined by

$$\hat{g}_{E_S}(\xi) = \begin{cases} \hat{g}(\xi), & \xi \in \mathbb{R}^n \setminus \tau_B(E_S) \\ \chi_{E_S}(\xi), & \xi \in \tau_B(E_S). \end{cases} \tag{2.11}$$

Thus, using Propositions 2.3 and 2.4, we immediately deduce:

Proposition 2.7. *Let $g \in L^2(\mathbb{R}^n)$, $B \in GL_n(\mathbb{R})$ with $|\det B| \leq 1$, and g_{E_S} be given by (2.11). Then:*

(a) if $g \in \mathbf{F}_B$, then also $g_{E_S} \in \mathbf{F}_B$;

(b) if $g \in \mathbf{PF}_B$, then also $g_{E_S} \in \mathbf{PF}_B$.

We are now ready to prove Theorem 2.1

Proof of Theorem 2.1

(a) We will prove that, for a given $g \in \mathbf{F}_B$, there is a continuous path $\{g_t : 0 \le t \le 1\}$ connecting g to $g_1 = (\chi_J)^\vee$, where $J = B[0,1)^n$ (this is a fundamental domain by $B\mathbb{Z}^n$). This implies that the set $\mathbf{F}_B$ is path–connected. We will achieve this in a few steps.

Step 1. We will start by constructing a continuous set-valued function $S(t)$, where $0 \le t \le 1$, such that $S(0)$ is a set of measure zero, $S(1) = J$, and $S(t) \subseteq S(t')$ for each $t' > t$.

Since $B \in GL_n(\mathbb{Q})$, using the Smith canonical form (cf. [12]) we can find unimodular integral matrices U and V such that UBV is diagonal, that is

$$UBV = \begin{pmatrix} \frac{p_1}{q_1} & 0 & \cdots & 0 \\ 0 & \frac{p_2}{q_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{p_n}{q_n} \end{pmatrix}, \tag{2.12}$$

where p_i and q_i are relatively prime for each i. Thus, without loss of generality, we can assume that B is diagonal as in (2.12) (this is equivalent to making a change of variables that maps that lattice $B\mathbb{Z}^n$ into $UBV\mathbb{Z}^n$ and the lattice $\mathbb{Z}^n$ into itself). Now, by Theorem 2.6, the set J can be chosen to be a finite union of congruent hyper-rectangles R_k in $\mathbb{R}^n$ with sides parallel to the coordinate axes, that is $J = \bigcup_{k=1}^M R_k$. For each hyper-rectangle R_k, let ℓ be the side length and $x_k = (x_{k1}, \cdots, x_{kn})$ be the coordinate of the lower left vertex. For each k, $1 \le k \le M$, let $R_k(t)$ be the hyper-rectangle in $\mathbb{R}^n$ whose lower left vertex is the same as the lower left vertex of R_k, and whose side length is $t\ell$, with $0 \le t \le 1$. Hence, for each $1 \le k \le M$, $R_k(0) = \{x_k\}$, $R_k(1) = R_k$, and $R(t) \subseteq R(t')$ for each $t' > t$. Now let

$$S(t) = \bigcup_{k=1}^{M} R_k(t).$$

It immediately follows from this construction that $S(0) = \{x_1, \ldots, x_M\}$, $S(1) = J$, and $S(t) \subseteq S(t')$ for each $t' > t$. To show that $S(t)$ is continuous, let $0 \le t < t' \le 1$, then, for each k, $1 \le k \le M$, we have

$$\begin{aligned} \mu\big(R_k(t') \setminus R_k(t)\big) &= (t')^n - t^n \\ &= (t' - t)((t')^{n-1} + (t')^{n-2}t + \cdots + t't^{n-2} + t^{n-1}) \\ &\le n\,(t' - t) \to 0 \end{aligned}$$

as $t' - t \to 0$, and, thus, also $\mu\big(S(t') - S(t)\big) \to 0$.

Now define $E_t = E_{S(t)}$, where the set $E_{S(t)}$ is given by (2.10), with $S = S(t)$ and, finally, set $g_t = g_{E_t}$, where g_{E_t} is given by (2.1), with $E = E_t$.

By Proposition 2.7, $g_t \in \mathbf{F}_B$ for each $t \in [0,1]$. In particular, for $t = 0$, $g_0 = g$ a.e., since $E_0 = \{x_1, \ldots, x_k\}$ and, for $t = 1$, $g_1 = (\chi_J)^\vee$, since $E_1 = J$ and $\tau_B(E_1) = \tau_B(J) = \mathbb{R}^n$.

Step 2. We now need to show the path $\{g_t : 0 \le t \le 1\}$ is continuous in the L^2–norm. That is, we have to show that:

$$\lim_{t \to t'} \|g_t - g_{t'}\|_2 = 0, \tag{2.13}$$

for $0 \le t < t' \le 1$.

Observe that, since $S(t) \subseteq S(t')$ for $t' > t$, then, for all $0 \le t < t' \le 1$, we have

$$E_t = E_{S(t)} = \tau_B\big(\tau(S(t))\big) \cap J \ \subseteq\ \tau_B\big(\tau(S(t'))\big) \cap J = E_{t'}. \tag{2.14}$$

For $0 \le t < t' \le 1$, we can show that $\hat{g}_t = \hat{g}_{t'}$ on the set $(\mathbb{R}^n \setminus \tau_B(E_{t'})) \cup \tau_B(E_t)$. In fact, by (2.14), $\hat{g}_t = \hat{g} = \hat{g}_{t'}$ on $\mathbb{R}^n \setminus \tau_B(E_{t'})$ and $\hat{g}_t = \chi_{E_t} = \hat{g}_{t'}$ on $\tau_B(E_t)$. Thus, we have:

$$\|g_t - g_{t'}\|_2^2 = \|\hat{g}_t - \hat{g}_{t'}\|_2^2 = \int_{\tau_B(E_{t'}) \setminus \tau_B(E_t)} |\hat{g}_t(\xi) - \chi_{E_{t'}}(\xi)|^2 \, d\xi. \tag{2.15}$$

Since $|\hat{g}_t(\xi) - \chi_{E_{t'}}(\xi)|^2 \le 2\,(|\hat{g}_t(\xi)|^2 + \chi_{E_{t'}}(\xi)^2)$, from (2.15) we obtain:

$$\|g_t - g_{t'}\|_2^2 \le \int_{\tau_B(E_{t'}) \setminus \tau_B(E_t)} 2\left(|\hat{g}_t(\xi)|^2 + \chi_{E_{t'}}(\xi)\right) d\xi. \tag{2.16}$$

Observe that

$$\begin{aligned}
\int_{\tau_B(E_{t'}) \setminus \tau_B(E_t)} \chi_{E_{t'}}(\xi)\, d\xi &\le \sum_{k \in \mathbb{Z}^n} \int_{E_{t'} \setminus E_t} \chi_{E_{t'}}(\xi - Bk)\, d\xi \\
&= \int_{E_{t'} \setminus E_t} \chi_{E_{t'}}(\xi)\, d\xi \\
&= \mu\big(E_{t'} \setminus E_t\big).
\end{aligned} \tag{2.17}$$

Also observe that, since $g \in \mathbf{F}_B$, then, by Theorem 1.2, $\sum_{k \in \mathbb{Z}^n} |\hat{g}(\xi - Bk)|^2 \le \beta$ for a.e. $\xi \in \mathbb{R}^n$, where $\beta < \infty$ is the upper frame bound for the Gabor frame $\mathcal{G}_B(g)$. Thus:

$$\begin{aligned}
\int_{\tau_B(E_{t'}) \setminus \tau_B(E_t)} |\hat{g}_t(\xi)|^2\, d\xi &\le \sum_{k \in \mathbb{Z}^n} \int_{E_{t'} \setminus E_t} |\hat{g}_t(\xi - Bk)|^2\, d\xi \\
&= \int_{E_{t'} \setminus E_t} \sum_{k \in \mathbb{Z}^n} |\hat{g}(\xi - Bk)|^2\, d\xi \\
&\le \beta\, \mu\big(E_{t'} \setminus E_t\big).
\end{aligned} \tag{2.18}$$

Therefore, from equation (2.16), using (2.17) and (2.18), we obtain:

$$\|g_t - g_{t'}\|_2^2 \le 2\,(\beta + 1)\, \mu\big(E_{t'} \setminus E_t\big). \tag{2.19}$$

Thus, to complete the proof, we have to show that $\mu(E_{t'} \setminus E_t)$ approaches zero as $|t-t'| \to 0$.

Since $S(t) = \bigcup_{k=1}^{M} R_k(t)$, then

$$E_t = \tau_B\Big(\tau(S(t))\Big) \bigcap J = (\bigcup_{k=1}^{M} R_k(t) + \mathbb{Z}^n + B\mathbb{Z}^n) \bigcap J.$$

For each hyper-rectangle $R_k(t)$, $1 \le k \le M$, we have

$$R_k(t) = I_{x_{k1}}(t) \times \cdots \times I_{x_{kn}}(t),$$

where $I_{x_{ki}}(t)$ is the interval $[x_{ki}, x_{ki} + t\ell)$. Thus, using the fact that $\mathbb{Z} + p_i/q_i\mathbb{Z} = 1/q_i\,(q_i\mathbb{Z} + p_i\mathbb{Z}) = 1/q_i\,\mathbb{Z}$, we have:

$$\begin{aligned}
E_t &= \Big(\bigcup_{k=1}^{M} \big(I^k_{x_1}(t) \times \cdots \times I^k_{x_n}(t)\big) + \mathbb{Z}^n + B\mathbb{Z}^n\Big) \bigcap J \\
&= \Big(\bigcup_{k=1}^{M} \big(I^k_{x_1}(t) + \mathbb{Z} + p_1/q_1\mathbb{Z}\big) \times \cdots \times \big(I^k_{x_n}(t) + \mathbb{Z} + p_n/q_n\mathbb{Z}\big)\Big) \bigcap J \\
&= \Big(\bigcup_{k=1}^{M} \big(I^k_{x_1}(t) + 1/q_1\mathbb{Z}\big) \times \cdots \times \big(I^k_{x_n}(t) + 1/q_n\mathbb{Z}\big)\Big) \bigcap J \\
&= \Big(\bigcup_{k=1}^{M} R_k(t) + Q^{-1}\mathbb{Z}^n\Big) \bigcap J, \\
&= \tau_{Q^{-1}}\Big(S(t)\Big) \bigcap J,
\end{aligned}$$

where $Q = \begin{pmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & q_n \end{pmatrix}$. Since $S(t)$ is continuous, this quantity tends to zero as $(t'-t) \to 0$. Thus, by (2.19), the path $\{g_t : 0 \le t \le 1\}$ is also continuous, and this completes the proof of part (a).

(b) Let us choose any $g \in \mathbf{PF}_B$ and let $g_1 = (\chi_J)^\vee$, where $J = B[0,1)^n$, as in part (a). Observe that $g_1 \in \mathbf{PF}_B$. The proof that g is path–connected to g_1 then follows exactly as in part (a). $\square$

From Remark 2.5 we also deduce the following.

Corollary 2.8. *The set of Gabor orthonormal bases for $L^2(\mathbb{R}^n)$ is path connected in the norm topology of $L^2(\mathbb{R}^n)$.*

Remark 2.9. *A simple modification of the proof of Theorem 2.1 shows that, if we assume that $g \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then we can prove that the path g_t, $1 \le t \le 1$ is continuous not only in the L^2 norm, but also in L^p, $1 \le p < \infty$.*

2.3 Irrational Gabor frames

The construction that we have used in the previous section cannot be extended, in general, to the case of Gabor systems $\mathcal{G}_B(g)$ where $B \in GL_n(\mathbb{R})$ rather than $B \in GL_n(\mathbb{Q})$. We point out, however, that this is a limitation of the constructive proof, and that Theorem 2.1 holds for all $B \in GL_n(\mathbb{R})$, as shown in [4].

In order to show that the constructive proof breaks down, let us consider, for the moment, the one-dimensional case. As in the previous section, let $J = [0, b)$, $S = S_0 \subseteq J$, and let the sets $\{S_k : k \geq 0\}$ and E_S be defined by (2.10). The following Lemma shows that when $b \notin \mathbb{Q}$ and S has measure larger than zero, then the sequence of sets $\{S_k : k \geq 0\}$ converges very rapidly to J.

Lemma 2.10. *Let $J = [0, b)$, with $b \notin \mathbb{Q}$. If $\mu(S) > 0$ and S has nonempty interior, then $E_S = [0, b)$.*

Proof. Without loss of generality, let $S = [0, \tau)$, $0 < \tau < b$ (if $\tau = b$, then there is nothing to prove). Given $0 < \epsilon < \tau/2$, choose x_0 such that $\epsilon < x_0 < \tau - \epsilon$. Now take $y_0 \in (0, b)$. We can show that there is a point $y \in (y_0 - \epsilon, y_0 + \epsilon)$, where $y = x + k + lb$, for some $x \in (x_0 - \epsilon, x_0 + \epsilon)$, $k, l \in \mathbb{Z}$. In fact, given $y \in (y_0 - \epsilon, y_0 + \epsilon)$, since $\{k + lb : k, l \in \mathbb{Z}\}$ is a dense set in $\mathbb{R}$, we can find $k, l \in \mathbb{R}$ such that

$$|(y - x_0) - (k + l)| < \epsilon.$$

Hence, there is an $x \in (x_0 - \epsilon, x_0 + \epsilon) \in S$ so that $y = x + k + lb$, for those $k, l \in \mathbb{Z}$. Since $S_2 = S + \mathbb{Z} + b\mathbb{Z}$, this implies that $y \in S_2$. The same construction can be repeated for each $y \in (0, b)$, since $\epsilon > 0$ can be chosen arbitrarily small. $\square$

As a consequence of this lemma, if we define the path $S(t) = t[0, b)$, $E_t = E_{S(t)}$, as we did before, we will not be able to obtain a continuous path. In fact, if $b \notin \mathbb{Q}$, the set-valued function E jumps from $E_0 = \{0\}$ to $E_t = [0, b)$ when $t > 0$.

In the higher dimensions, we have different cases. Let W be the maximal subspace of $\mathbb{R}^n$ such that $(\mathbb{Z}^n + B\,\mathbb{Z}^n) \bigcap W$ is dense in W. If $B \in GL_n(\mathbb{Q})$, then W is a lattice, and we have the situation that we have considered in the Section 2.2. On the other hand, if $B \in GL_n(\mathbb{R})$ and $B \notin GL_n(\mathbb{Q})$, we can distinguish two cases: $\mathbb{Z}^n + B\,\mathbb{Z}^n$ is either dense in $\mathbb{R}^n$ or is not. In the first case, we have $W = \mathbb{R}^n$ and the argument of Lemma 2.10 shows that the construction of Section 2.2 fails. In the second case, we have $1 \leq \dim(W) \leq n$ and it is possible to extend the argument of Section 2.2 to this situation.

References

[1] X. Dai and D.R. Larson, Wandering vectors for unitary systems and orthogonal wavelets, Memoirs of Amer. Math. Soc. 134 (1998).

[2] I. Daubechies, Ten Lectures on Wavelets, CBMS–NSF Regional Conference Series in Applied mathematics, Vol.61, SIAM, Philadelphia, 1992.

[3] R.J. Duffin and A.C. Schaeffer, A Class of Nonharmonic Fourier Series, Trans. Amer. Math. Soc. 72 (1952) 341–366, .

[4] J.P. Gabardo and D. Han, Aspects of Gabor analysis and operator algebras. In: Advances in Gabor analysis, 129–152, Appl. Numer. Harmon. Anal., Birkhuser Boston, Boston, MA, 2003.

[5] J.P. Gabardo, D. Han and D. Larson, Gabor frames and operator algebras, Wavelet Applications in Signal and Image Processing, Proc. SPIE 4119 (2000) 337-345.

[6] G. Garrigós, E. Hernández, H. Šikić, F. Soria, G. Weiss and E. Wilson, Connectivity in the set of tight frame wavelets, Glasnik Mathematički 38(58) (2003) 75–98.

[7] K. Gröchenig, Foundations of Time–Frequency Analysis, Birkäuser, Boston, 2001.

[8] C. Heil and D. Walnut, Continuous and Discrete Wavelet Transforms, SIAM Rev. 31(4) (1989) 628-666.

[9] D. Han and Y. Wang, Lattice tiling and the Weyl-Heisenberg frames, Geometric and Functional Analysis (to appear) (2003).

[10] E. Hernández and G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton FL, 1996.

[11] D. Labate, A Unified Characterization of Reproducing Systems Generated by a Finite Family, J. Geometric Anal. 12(3) (2002) 469–491.

[12] M. Newman, Integral Matrices, Academic Press, New York, 1972.

[13] M.A. Rieffel, Von Neumann Algebras Associated with Pair of Lattices in Lie Groups, Math. Ann. 257 (1981) 403-418.

[14] A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in* $L^2(\mathbb{R}^d)$, Duke Math. J., **89**, (1997), 237-282.

[15] D.M. Speegle, The s-elementary wavelets are path-connected, Proc. Amer. Math. Soc. 127 (1999) 223–233.

[16] WUTAM Consortium, Basic Properties of Wavelets, Journal of Fourier Analysis and Applications 4(4) (1998) 575-594.