A Comparison of Fractal Dimension and Spectrum Coefficient Characterization of $1/f^{\alpha}$ Noise

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Letter to the Editor

A Comparison of Fractal Dimension and Spectrum Coefficient Characterization of $1/f^\alpha$ Noise

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Although the advantages of a spectral characterization of noise are many and well known, it may be interesting to consider studying $1/f^\alpha$ noise from the point of view of fractal analysis in the time domain, in order to assess if this can be a useful tool or even a complementary perspective to the usual spectral analysis.

Fractal geometry in fact is a natural framework for the description of complex systems, and it has been widely applied to the study of noise [1]. As in a spectrum, fractals are also characterized by parameters that are intimately related to the physics of the phenomena. The most important parameter of a fractal object is its *fractal dimension* D [1, 2] which is a quantitative measure of the "jaggedness" of a curve or, which is the same, of its space-filling properties. A fractal object has a fractal dimension $1 < D < 2$, as opposed to a standard (Euclidean) curve, for which $D = 1$.

The existence of a relationship between the fractal dimension of the time variation of a noise signal and the logarithmic slope of its spectral density (i.e. the spectrum index α) has been discussed by several authors. A heuristic approach proposing an analytical relationship was attempted in [3] and [4]. The linear dependence $\alpha = 5 - 2D$ was found for the special class of *self-affine* profiles (functions $y(x)$ which are scale-invariant on rescaling of both y and x axes). This hypothesis, however, is quite restrictive [4].

The approach used in this work is to abandon the search for an analytical relationship, but not restrict the analysis to any special class of functions. Our goal is to compare the efficiency of the characterization of $1/f^\alpha$ noise by determination of the fractal dimension in the time domain with that of spectral characterization by determination of the spectrum index α.

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The first attempt in this direction was made by Gagnepain et al. [5]. In their own judgement, they were limited by the lack of precise evaluation methods for the fractal dimension. Since then, methods for the evaluation of D have progressed considerably and the present analysis takes advantage of such improvements.

In this work, random time series with $1/f^\alpha$ spectra were generated by means of the algorithm suggested in [3]. This method simply consists in constraining the power spectrum $S(f)$ of the Fourier series

$$X(t) = \sum_{k=0}^{N-1} a_k \, e^{2 i \pi k t}$$

to be $S(f) \propto 1/f^\alpha$, that is by imposing the condition that the coefficients a_k are subject to

$$E\{|a_k|^2\} \propto \frac{1}{k^\alpha},$$

where $E\{\ \}$ represents the expected value, and the phase of the coefficients a_k can be chosen randomly. A time series generated in this way nicely simulates a very general $1/f^\alpha$ noise.

The fractal dimension of such simulated $1/f^\alpha$ noise was then evaluated using the "variation" algorithm suggested in [2], which has been proven to be reliable and robust. The philosophy of this method can be summarized as follows: if $f(x)$ is a fractal curve, there exists at least one part of the considered domain of x in which $f(x)$ is nowhere or almost nowhere differentiable. It follows that the slope of the line passing through two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the curve tends to infinity when x_1 tends to x_2. The fractal dimension is estimated from the rate at which this slope increases when the two points collapse. A least squares fit on a suitable estimator of the latter determines both D, as the regression coefficient, and its uncertainty δD as the 90 % confidence limit.

A brief outline of our analysis follows.

(a) $1/f^\alpha$ noise series with α varying in steps of 0,2 between 0 and 4 (which is the range of practical interest) were numerically generated.

(b) The fractal dimension of such series was evaluated.

(c) This process was repeated for twenty statistically independent noise series, for each value of α. The values of D obtained in this way were then averaged.

Results are shown in Figure 1 where the fractal dimension is shown as a function of the spectrum index α. A comparison with the results of Gagnepain [5] and with the linear relationship $D = (5-\alpha)/2$ is also provided. All curves cross for $\alpha = 2$, where $D = 1,5$. However, the slope of the approximately linear region of our results (solid line), for α varying between 1 and 3, is smaller than the slope $-1/2$ of the analytical approach (see [3, 4]), and appears smoother than Gagnepain's results. Furthermore, our curve also has a smoother transition to the asymptotic values $D \to 2$ and $D \to 1$, when $\alpha \to 0$ or $\alpha > 3$, respectively.

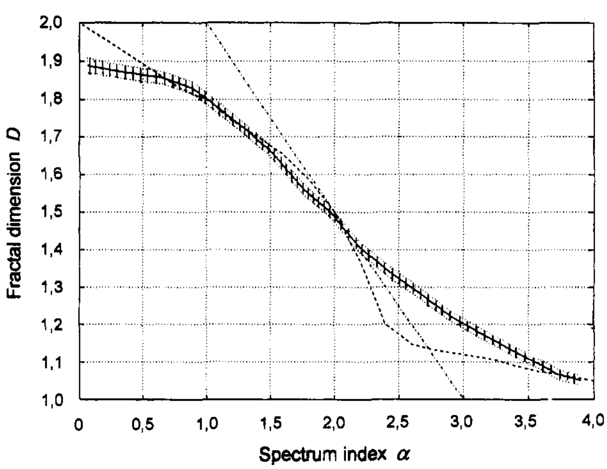

Figure 1. Relationship between fractal dimension D and spectrum index α. Our results (solid line), together with the appropriate uncertainty range (hatched area), are compared with the results of [5] (dashed line) and with the linear relationship $D = (5-\alpha)/2$ [3, 4] (dot-dashed line).

In order to compare the characterization of noise by fractal dimension and by spectrum index from the point of view of the efficiency, the uncertainties associated with each were also evaluated. Results are shown in Figure 2 for the particular case of $\alpha = 2$.

It is well known that in order to obtain a good estimate of a power spectrum, a large amount of data is necessary because many data segments must be processed and the results averaged [6]. The evaluation of the fractal dimension requires fewer data. For each chosen value of α we proceeded as follows:

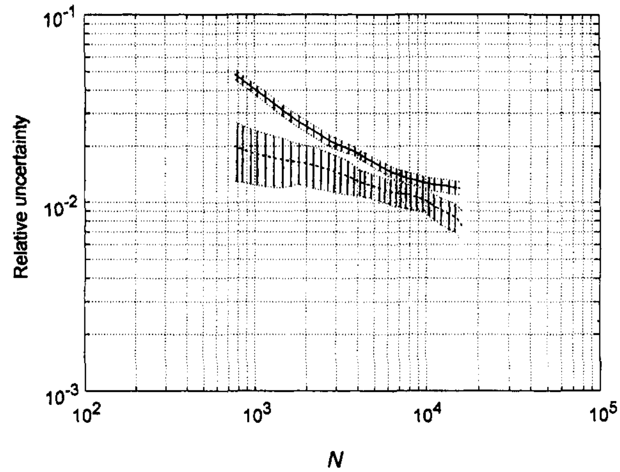

Figure 2. Relative uncertainties of fractal dimension $\delta D/D$ (dotted line) and of spectrum index $\delta\alpha/\alpha$ (solid line) versus the number of samples N. Hatched area indicates the confidence range of each curve.

(a) A sequence of N data was generated, with N varying between 7×10^2 and 2×10^4 in different trials.

(b) Both α and its 90 % uncertainty $\delta\alpha$ were evaluated by least squares fitting over the power spectrum.

(c) This process was repeated for twenty statistically independent noise series. The resulting averaged relative uncertainty $\delta\alpha/\alpha$ is shown in Figure 2, versus the number of samples N.

(d) The fractal dimension D was calculated for each series and then averaged, and its uncertainty δD was estimated by the method already described. The relative uncertainty $\delta D/D$, averaged over twenty statistically independent noise series, is the quantity shown in Figure 2, versus the number of samples N.

A comparison of the two uncertainty curves shows that in a log-log plot the relative uncertainty of the α estimation as a function of N decreases approximately as $\sqrt{N}$. The slope of the uncertainty of D appears to be slightly smaller. The uncertainty associated with the evaluation of the fractal dimension is also slightly smaller than the uncertainty obtained for the estimate of the spectrum index with the data length examined.

As a conclusion, there seems to be an advantage in using the fractal dimension description instead of the spectrum index, particularly when a small amount of data is available. The fractal approach appears attractive because the algorithms for the calculation of D are reliable, accurate and simple to apply, and require less data than the usual spectrum analysis in order to obtain a good estimate. On the other hand, it appears from Figure 1 that characterization of noise processes by their spectrum index α has higher discrimination capabilities for values of α approaching zero or higher than 3.

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