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TIME–FREQUENCY ANALYSIS OF PSEUDODIFFERENTIAL OPERATORS

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Abstract. In this paper we apply a time–frequency approach to the study of pseudodifferential operators. Both the Weyl and the Kohn–Nirenberg correspondences are considered. In order to quantify the time–frequency content of a function or distribution, we use certain function spaces called modulation spaces. We deduce a time–frequency characterization of the twisted product $\sigma\sharp\tau$ of two symbols σ and τ, and we show that modulation spaces provide the natural setting to exactly control the time–frequency content of $\sigma\sharp\tau$ from the time–frequency content of σ and τ. As a consequence, we discuss some boundedness and spectral properties of the corresponding operator with symbol $\sigma\sharp\tau$.

1. Introduction

A pseudodifferential operator can be defined through the Weyl or the Kohn–Nirenberg correspondence by bijectively assigning to any distributional symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$ a linear operator $T_\sigma\colon \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$, so that the properties of the operator are in an appropriate way reflected in the properties of the symbol.

One way to construct a pseudodifferential operator is as a superposition of time–frequency shifts. Even though this is a classical idea, going back to H. Weyl [17], this interpretation has reblossomed in recent years in the context of the study of the harmonic analysis in the Heisenberg group ([11], [6]). As a consequence, a number of methods from the time–frequency analysis have been employed to the study of pseudodifferential operators (for instance: [16], [10], [14], [9], [18]).

In this paper, we are interested in pseudodifferential operators whose symbols satisfy certain integrability conditions in the time–frequency plane and are not necessarily smooth. The interest of these classes of operators stems partly from electrical engineering applications, in particular signal processing and time–varying filtering theory, where operators arising from the Weyl correspondence are used as models for time–frequency or time–varying filters (cf., for instance, [5], [12], [15]). In this context, the symbol is interpreted as

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the mask of the filter, since it weights selectively the different time–frequency components of the signal.

In our approach, the pseudodifferential operator T_σ is realized as a superposition of elementary rank–one operators through the time–frequency decomposition of the associated symbol σ. In order to exactly quantify the time–frequency content of the symbol, certain function spaces, called modulation spaces, are natural. These spaces are defined by prescribing the decay properties of the Short–Time Fourier Transform (STFT) of a given function or distribution, and they contain a large class of objects, including some classical function spaces (e.g, L^2, Sobolev spaces). The precise definition of modulation spaces, together with the basic time–frequency tools employed in this paper, are reviewed in Section 2.

In Section 3 we apply this time–frequency approach to study the composition of pseudodifferential operators. Let T_σ and T_τ be pseudodifferential operators having symbols σ and τ respectively. Then $T_{\sigma\sharp\tau} = T_\sigma T_\tau$ is a pseudodifferential operator with symbol $\sigma\sharp\tau$, where $\sigma\sharp\tau$ is called the *twisted product* of σ and τ. The direct computation of the twisted product (cf. [6, Section 2.3]) leads to a complicated integral formula, which is usually asymptotically expanded into a power series and approximated. However, such an expansion requires σ and τ to be arbitrarily smooth. Our time–frequency approach yields a characterization of the twisted product which does not require smoothness assumptions on the symbols. The STFT of $\sigma\sharp\tau$ behaves essentially as a matrix multiplication of the STFTs of σ and τ, from which the modulation space norm of $\sigma\sharp\tau$ can be controlled by the modulation space norms of σ and τ. In particular, the twisted product turns out to be closed on the modulation spaces $M_w^{p,p}$ with $w(x,y) = (1+|x|+|y|)^s$, for $1 \le p \le 2$.

Notation.

Let X, X_0, Y, Y_0 be Banach spaces. $\mathcal{L}(X,Y)$ is the space of all bounded linear operators from X to Y, and $\mathcal{L}(X) = \mathcal{L}(X,X)$. The norm of X is $\|\cdot\|_X$, or simply $\|\cdot\|$ if the context is clear. The dual space of X is X'. We write $\langle f, g\rangle$ for the action of $g \in X'$ on $f \in X$. We write T^* for the adjoint operator of T. A linear subspace $\mathcal{A}(X,Y)$ of $\mathcal{L}(X,Y)$ is an *operator ideal* if $UTV \in \mathcal{A}(X,Y)$ whenever $U \in \mathcal{L}(Y,Y_0)$, $T \in \mathcal{A}(X,Y)$, and $V \in \mathcal{L}(X_0,X)$. The Schatten class $\mathcal{I}_p \subset \mathcal{L}(\mathcal{H})$ consists of the compact operators on a Hilbert space $\mathcal{H}$ whose singular values lie in ℓ^p. $\mathcal{I}_p$ is an operator ideal and coincides with the class of Hilbert–Schmidt operators when $p = 2$ and with the trace–class operators when $p = 1$.

$L_w^{p,q}(\mathbf{R}^{2n})$ is the weighted mixed–normed space of functions f on $\mathbf{R}^{2n}$ with norm $\|f\|_{L_w^{p,q}} = (\int_{\mathbf{R}^n}(\int_{\mathbf{R}^n}|f(x,y)|^p\,w(x,y)^p\,dx)^{q/p}\,dy)^{1/q}$. If $w \equiv 1$ we write $L^{p,q}(\mathbf{R}^{2n})$. If $p = q$, we have the classical space $L^p(\mathbf{R}^{2n}) = L^{p,p}(\mathbf{R}^{2n})$. $\ell_w^{p,q}(\mathbf{Z}^{2n})$ is the space of sequences $a = (a_{km})_{k,m\in\mathbf{Z}^n}$ with norm $\|a\|_{\ell_w^{p,q}} = (\sum_m(\sum_k|a_{km}|^p\,w(k,m)^p)^{q/p})^{1/q}$. If $w \equiv 1$

we write $\ell^{p,q}(\mathbf{Z}^{2n})$. If $p = q$, we have the classical sequence space $\ell^p(\mathbf{Z}^{2n}) = \ell^{p,p}(\mathbf{Z}^{2n})$. $C(\mathbf{R}^n)$ is the space of continuous functions on $\mathbf{R}^n$, and $C_0(\mathbf{R}^n)$ is the space of continuous functions on $\mathbf{R}^n$ vanishing at infinity. $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space of all infinitely differentiable functions on $\mathbf{R}^n$ decaying rapidly at infinity, and $\mathcal{S}'(\mathbf{R}^n)$ is its topological dual, the space of tempered distributions. $H^s(\mathbf{R}^n)$ is the Sobolev space of functions defined by the norm $\|f\|_{H^s}^2 = \int_{\mathbf{R}^n} |\hat{f}(\gamma)|^2 (1 + |\gamma|^2)^s \, d\gamma$. The usual dot product of x, $y \in \mathbf{R}^n$ is denoted by juxtaposition, i.e., $xy = x_1 y_1 + \cdots + x_n y_n$. The symplectic form on $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbf{R}^n \times \mathbf{R}^n$ is $[\alpha, \beta] = \alpha_1 \beta_2 - \alpha_2 \beta_1$. The composition of f and g is $(f \circ g)(t) = f(g(t))$. The inner product of f, $g \in L^2(\mathbf{R}^n)$ is $\langle f, g \rangle = \int_{\mathbf{R}^n} f(t) \, \overline{g(t)} \, dt$; the same notation is used for the extension of the inner product to $\mathcal{S}(\mathbf{R}^n) \times \mathcal{S}'(\mathbf{R}^n)$. The Fourier transform is $\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int f(t) \, e^{-2\pi i \gamma t} \, dt$; the inverse Fourier transform is $\check{f}(\gamma) = \hat{f}(-\gamma)$. The Fourier transform maps $\mathcal{S}(\mathbf{R}^n)$ onto itself, and extends to $\mathcal{S}'(\mathbf{R}^n)$ by duality. The convolution of f and g is $(f * g)(x) = \int_{\mathbf{R}^n} f(x - t) g(t) \, dt$.

2. Background: Time–Frequency Analysis

We briefly review the Schrödinger representation of the Heisenberg group as a tool for constructing and analyzing pseudodifferential operators. We adopt most of the notation and conventions of Folland's book [6].

2.1. The Schrödinger representation. The Schrödinger representation of the Heisenberg group $\mathbf{H}^n = \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ is the map ρ from $\mathbf{H}^n$ to the group of unitary operators on $L^2(\mathbf{R}^n)$ defined by $\rho(a, b, t) f(x) = e^{2\pi i t} \, e^{\pi i a b} \, e^{2\pi i b x} \, f(x + a)$. In many considerations the t-variable is unimportant, so for $(a, b) \in \mathbf{R}^{2n}$ we define $\rho(a, b) f(x) = e^{\pi i a b} \, e^{2\pi i b x} \, f(x + a)$. We refer to $\rho(a, b) f$ as a *time-frequency shift* of f. We recall the following useful facts.

Proposition 2.1. *Let $f \in L^2(\mathbf{R}^n)$ and let a, b, a', $b' \in \mathbf{R}^n$. Then:*

(a) $\|\rho(a, b) f\|_{L^2} = \|f\|_{L^2}$,

(b) $(\rho(a, b) f)^\wedge = \rho(-b, a) \hat{f}$,

(c) $(\rho(a, b))^{-1} = (\rho(a, b))^* = \rho(-a, -b)$,

(d) $\rho(a, b) \, \rho(a', b') f = e^{\pi i (a b' - a' b)} \rho(a + a', \, b + b') f$.

The (cross-)*ambiguity function*, or *Fourier-Wigner transform*, of f, $g \in L^2(\mathbf{R}^n)$ is:

$$A(f, g)(a, b) \;=\; \langle \rho(a, b) f, g \rangle \;=\; \int_{\mathbf{R}^n} e^{\pi i a b} \, e^{2\pi i b x} \, f(x + a) \, \overline{g(x)} \, dx.$$

If $f = g$, we write $A(f,f) = A(f)$. The ambiguity function extends to a map from $\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^{2n})$ and from $\mathcal{S}'(\mathbf{R}^n)\times\mathcal{S}'(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^{2n})$. A slight change in the definition yields the *Short–Time Fourier Transform* (STFT) of a distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ with respect to a window $g \in \mathcal{S}(\mathbf{R}^n)$:

$$S_g f(a,b) \;=\; \int_{\mathbf{R}^n} f(x)\,\overline{g(x-a)}\,e^{-2\pi i x b}\,dx \;=\; e^{-\pi i a b}\langle \rho(a,-b)f, g\rangle \;=\; e^{-\pi i a b} A(f,g)(a,-b).$$

The *Wigner transform* of f, $g \in L^2(\mathbf{R}^n)$ is the Fourier transform of the ambiguity function of f and g:

$$W(f,g)(\xi,x) \;=\; A(f,g)^\wedge(\xi,x) \;=\; \int_{\mathbf{R}^n} e^{-2\pi i p\xi}\, f(x+\tfrac{p}{2})\,\overline{g(x-\tfrac{p}{2})}\,dp. \tag{2.1}$$

We set $W(f) = W(f,f)$. Similarly to the ambiguity function, also the Wigner transform extends to a map from $\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^{2n})$ and from $\mathcal{S}'(\mathbf{R}^n)\times\mathcal{S}'(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^{2n})$.

The following facts will be useful (cf. [6, Sec. 1.4 and 1.8]).

Proposition 2.2. *Let f, $g \in L^2(\mathbf{R}^n)$ and let a, b, u_1, u_2, v_1, $v_2 \in \mathbf{R}^n$. Then:*

(a) $A(f,g) \in L^2(\mathbf{R}^{2n})$, *with* $\|A(f,g)\|_{L^2} = \|f\|_{L^2}\,\|g\|_{L^2}$.

(b) $A(f,g) \in C_0(\mathbf{R}^{2n})$, *and* $\|A(f,g)\|_{L^\infty} \le \|f\|_{L^2}\,\|g\|_{L^2}$.

(c) $A(f,g)(a,b) = \overline{A(g,f)(-a,-b)}$.

(d) $A(\rho(u_1,u_2)f, \rho(v_1,v_2)g)(a,b) \;=\; e^{\pi i(u_1v_2-u_2v_1)}\, e^{\pi i((u_2+v_2)a-(u_1+v_1)b)}$
$\times\, A(f,g)(a+u_1-v_1, b+u_2-v_2)$.

(e) $W(\rho(u_1,u_2)f, \rho(v_1,v_2)g)(a,b) \;=\; e^{\pi i(u_2v_1-u_1v_2)}\, e^{\pi i((u_1-v_1)a+(u_2-v_2)b)}$
$\times\, W(f,g)(a-\frac{u_2+v_2}{2}, b+\frac{u_1+v_1}{2})$.

(f) (Moyal's Identity) $\big\langle W(f_1,g_1),\, W(f_2,g_2)\big\rangle = \langle f_1,f_2\rangle\,\langle g_2,g_1\rangle$.

(g) If $f \in L^1(\mathbf{R}^n)$, then $\int A(f)(a,b)\,da = \hat f(-\frac{b}{2})\overline{\hat f(\frac{b}{2})}$, $\int A(f)(a,b)\,db = f(\frac{a}{2})\overline{f(-\frac{a}{2})}$.

(h) $S_g f(a,b) \;=\; e^{2\pi i a b}\, S_{\hat g}\hat f(b,-a)$.

The *Weyl correspondence* is the 1-1 correspondence between a distributional symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$ and the pseudodifferential operator $L_\sigma = \sigma(D,X)\colon \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ defined implicitly by:

$$\langle L_\sigma f, g\rangle \;=\; \langle \hat\sigma, A(g,f)\rangle \;=\; \langle \sigma, W(g,f)\rangle,$$

where $f, g \in \mathcal{S}(\mathbf{R}^n)$. L_σ is the *Weyl transform* of σ. The *Kohn–Nirenberg correspondence* assigns to a symbol τ the operator $K_\tau = \sigma(D, X)_{KN}$ defined implicitly by:

$$\langle K_\tau f, g\rangle \;=\; \langle \hat{\tau}, e^{\pi i \xi x} A(g, f)\rangle. \tag{2.2}$$

K_τ is the *Kohn–Nirenberg transform* of τ. Equation (2.2) shows that the operators L_σ in the Weyl correspondence and K_τ in the Kohn–Nirenberg correspondence are equal if and only if their symbols are related by $\hat{\sigma}(\xi, x) = \hat{\tau}(\xi, x)\, e^{-\pi i \xi x}$. Therefore, statements invariant under multiplication by $e^{-\pi i \xi x}$ will be valid for one correspondence if and only if they are valid for the other.

2.2 Modulation Spaces. The modulation spaces measure the joint time-frequency distribution of $f \in \mathcal{S}'(\mathbf{R}^d)$. For background and detailed information on their properties we refer to [1], [2], [3], [8].

Let w be a subadditive positive weight function on $\mathbf{R}^{2n}$, i.e., $1 \le w(\alpha) < \infty$, and $w(\alpha + \beta) \le w(\alpha)\, w(\beta)$ for all α, $\beta \in \mathbf{R}^{2n}$. We assume that w has at most polynomial growth, i.e., for all $\alpha \in \mathbf{R}^{2n}$ we have $w(\alpha) \le C|\alpha|^N$ for some C, $N \ge 0$. Let $1 \le p$, $q \le \infty$. Given a window function $g \in \mathcal{S}(\mathbf{R}^n)$, denote by $M^{p,q}_w(\mathbf{R}^n)$ the space of all distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ for which the norm

$$\|f\|_{M^{p,q}_w(\mathbf{R}^n)} \;=\; \|S_g f\|_{L^{p,q}_w(\mathbf{R}^{2n})} \;=\; \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |S_g f(x, y)|^p\, w(x, y)^p \, dx\right)^{q/p} dy\right)^{1/q}$$

is finite, with obvious modifications if p or $q = \infty$. If $w \equiv 1$ then we write $M^{p,q}(\mathbf{R}^n)$. The space $M^{p,q}_w(\mathbf{R}^n)$ is a Banach space whose definition is independent of the choice of window g, i.e., different choices of windows g yield equivalent norms. The assumptions on the weight w guarantees that the modulation spaces are defined in the realm of tempered distributions, and that $\mathcal{S}$ is dense in all modulation spaces $M^{p,q}_w$ for all $1 \le p, q \le \infty$ (cf. [1], [8, Section 11.1]). For $1 \le p$, $q < \infty$, the dual space of $M^{p,q}(\mathbf{R}^n)$ is $(M^{p,q}(\mathbf{R}^n))' = M^{p',q'}(\mathbf{R}^n)$, where p', q' satisfy $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

We recall the following invariance properties of the modulation spaces. If $w(\pm x, \pm y) = w(y, x)$, it follows from Proposition 2.2(h) that $M^{p,p}_w(\mathbf{R}^n)$ is invariant under the Fourier transform. Moreover, the modulation spaces are invariant under the metaplectic representation (cf. [3, Theorem 29]). In particular, multiplication by $e^{-\pi i x y}$ leaves the space $M^{p,q}_w(\mathbf{R}^n)$ invariant for each $1 \le p, q \le \infty$, i.e., $\|f\|_{M^{p,q}_w} = \|e^{-\pi i x y} f\|_{M^{p,q}_w}$. We will use this property in Section 3 to transfer statements between Weyl and Kohn–Nirenberg correspondences.

Among the modulation spaces the following well-known function spaces occur.

(a) $M^{2,2}(\mathbf{R}^n) = L^2(\mathbf{R}^n)$.

(b) (Weighted L^2-spaces) If $w(x,y) = (1+|x|)^s$, then $M_w^{2,2}(\mathbf{R}^n) \;=\; L_w^2(\mathbf{R}^n)$.

(c) (Sobolev spaces) If $w(x,y) = (1+|y|)^s$, then $M_w^{2,2}(\mathbf{R}^n) \;=\; H^s(\mathbf{R}^n)$.

(d) If $w(x,y) = (1+|x|+|y|)^s$, then $M_w^{2,2}(\mathbf{R}^n) \;=\; L_s^2(\mathbf{R}^n) \cap H^s(\mathbf{R}^n)$.

(e) (Feichtinger's algebra) $M^{1,1}(\mathbf{R}^n) = S_0(\mathbf{R}^n)$.

We recall that the space S_0 is contained in L^2, and that it is an algebra under both convolution and pointwise multiplication. The space S_0 plays an important role in abstract harmonic analysis (cf. [4]).

2.3 Time–Frequency Expansion of the Weyl Operator.

By realizing the symbol of the Weyl operator L_σ as a superposition of time–frequency shifts, it is possible to express L_σ in terms of elementary rank–one operators. The fundamental result needed for the time–frequency analysis is the following inversion formula, which is proved in an abstract context in [1]:

THEOREM 2.3. *If $\Phi \in \mathcal{S}(\mathbf{R}^{2n})$ with $\|\Phi\|_{L^2} = 1$, and $\sigma \in M_w^{p,q}(\mathbf{R}^{2n})$ with $1 \le p, q < \infty$, then:*

$$\sigma \;=\; \iint_{\mathbf{R}^{4n}} \langle \sigma, \rho(\alpha,\beta)\Phi\rangle \, \rho(\alpha,\beta)\Phi \, d\alpha \, d\beta, \tag{2.3}$$

where the integral converges in the norm of $M_w^{p,q}(\mathbf{R}^{2n})$. If $p = \infty$ or $q = \infty$ or if $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$, then (2.3) holds with weak convergence of the integral.

The following consequence of Theorem 2.3 is proved in [9, Lemma 3.2].

THEOREM 2.4. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ with $\|\phi\|_{L^2} = 1$, and let $\Phi = W(\phi,\phi)$. Let $\sigma \in M_w^{p,q}(\mathbf{R}^{2n})$, with $1 \le p, q \le \infty$. Denote by $N\colon \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}^{4n}$ the linear transformation*

$$N(\xi,\eta) \;=\; N(\xi_1,\xi_2,\eta_1,\eta_2) \;=\; \big(\tfrac{\xi_2+\eta_2}{2}, -\tfrac{\xi_1+\eta_1}{2}, \xi_1-\eta_1, \xi_2-\eta_2\big), \tag{2.4}$$

where $\xi = (\xi_1,\xi_2)$, $\eta = (\eta_1,\eta_2) \in \mathbf{R}^n \times \mathbf{R}^n$. Then, for $f \in \mathcal{S}(\mathbf{R}^n)$ we have:

$$L_\sigma f \;=\; \iint_{\mathbf{R}^{4n}} S_\Phi\sigma(N(\xi,\eta)) \, e^{\pi i[\xi,\eta]} \, \langle f, \rho(\eta)\phi\rangle \, \rho(\xi)\phi \, d\xi \, d\eta. \tag{2.5}$$

This integral converges as in Theorem 2.3.

From now on we will let ϕ denote an arbitrary but fixed function in $\mathcal{S}(\mathbf{R}^n)$ such that $\phi(t) = \phi(-t)$ and $\|\phi\|_{L^2} = 1$. For example, we could take $\phi(x) = 2^{n/4} \, e^{-\pi x^2}$. We set $\Phi = W(\phi)$. We will let N denote the linear transformation defined in (2.4), and $\tilde{N}$ the linear transformation $\tilde{N}(\xi,\eta) = N(\eta,\xi)$.

3. Composition of Pseudodifferential Operators

Suppose that L_σ, L_τ are Weyl operators that map $\mathcal{S}(\mathbf{R}^n)$ into itself. Then $L_{\sigma\sharp\tau} = L_\sigma L_\tau$ maps $\mathcal{S}(\mathbf{R}^n)$ into itself, and its symbol $\sigma\sharp\tau \in \mathcal{S}'(\mathbf{R}^{2n})$ is called the *twisted product* of σ and τ.

3.1 Formula for the twisted product. Using the time–frequency approach of Section 2.3, we deduce a useful characterization of the twisted product. We begin by assuming that σ, $\tau \in \mathcal{S}(\mathbf{R}^{2n})$. In this case the twisted product $\sigma\sharp\tau$ can be expressed [6, Sec.2.3] as:

$$(\sigma\sharp\tau)(\omega) \;=\; 2^{2n} \iint_{\mathbf{R}^{4n}} \sigma(u)\,\tau(v)\,e^{4\pi i[\omega-v,\omega-u]}\,du\,dv.$$

Using Theorem 2.3 to expand σ and τ, we obtain:

$$(\sigma\sharp\tau)(\omega) \;=\; 2^{2n} \iiiint_{\mathbf{R}^{8n}} \langle\sigma,\rho(\psi,\zeta)\Phi\rangle\,\langle\tau,\rho(\gamma,\delta)\Phi\rangle\,((\rho(\psi,\zeta)\Phi)\sharp(\rho(\gamma,\delta)\Phi))(\omega)\,d\psi\,d\zeta\,d\gamma\,d\delta \tag{3.1}$$

Recall that, if $F \in \mathcal{S}(\mathbf{R}^{2n})$, then $S_\Phi F(a,b) = e^{-\pi iab}\,\langle F,\rho(-a,b)\Phi\rangle$. By Proposition 2.2(e), $\rho(M(x,y))\Phi = W(\rho(x)\phi,\rho(y)\phi)$, where $M\colon \mathbf{R}^{2n}\times\mathbf{R}^{2n}\to\mathbf{R}^{4n}$ is the linear transformation for which $(-a,b) = M(x,y)$ iff $(a,b) = N(x,y)$. It follows that the change of variables $(a,b) = N(x,y)$ yields:

$$S_\Phi F(N(x,y)) \;=\; e^{-\pi iab}\,\langle F,\rho(-a,b)\Phi\rangle \;=\; e^{-\pi i[x,y]}\,\langle F, W(\rho(x)\phi,\rho(y)\phi)\rangle \tag{3.2}$$

Therefore, the change of variables $(-\psi,\zeta) = N(\xi,\eta)$, $(-\gamma,\delta) = N(\mu,\nu)$ in (3.1) yields:

$$\begin{aligned}(\sigma\sharp\tau)(\omega) \;=\; 2^{2n} \iiiint_{\mathbf{R}^{8n}} & S_\Phi\sigma(N(\xi,\eta))\,S_\Phi\tau(N(\mu,\nu))\,e^{\pi iP} \\ & \times \big(W(\rho(\xi)\phi,\rho(\eta)\phi)\sharp W(\rho(\mu)\phi,\rho(\nu)\phi)\big)(\omega)\,d\xi\,d\eta\,d\mu\,d\nu\end{aligned} \tag{3.3}$$

where $P = [\mu,\nu] + [\xi,\eta]$. By direct calculations, we have:

$$W(\rho(\xi)\phi,\rho(\eta)\phi)\sharp W(\rho(\mu)\phi,\rho(\nu)\phi) \;=\; \langle\rho(\mu)\phi,\rho(\eta)\phi\rangle\,W(\rho(\xi)\phi,\rho(\nu)\phi). \tag{3.4}$$

Consequently, using (3.2) and (3.4) into (3.3), we obtain:

$$\begin{aligned}S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta)) = 2^{2n}\,e^{-\pi i[\alpha,\beta]} \iiiint_{\mathbf{R}^{8n}} & S_\Phi\sigma(N(\xi,\eta))\,S_\Phi\tau(N(\mu,\nu))\,\langle\rho(\mu)\phi,\rho(\eta)\phi\rangle \\ & \times e^{\pi iP}\,\big\langle W(\rho(\xi)\phi,\rho(\nu)\phi),\,W(\rho(\alpha)\phi,\rho(\beta)\phi)\big\rangle\,d\xi\,d\eta\,d\mu\,d\nu.\end{aligned} \tag{3.5}$$

Finally, from (3.5), Moyal's identity (Proposition 2.2(f)) implies:

$$S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta)) = 2^{2n}\, e^{-\pi i[\alpha,\beta]} \iiiint_{\mathbf{R}^{8n}} S_\Phi\sigma(N(\xi,\eta))\, S_\Phi\tau(N(\mu,\nu))\, \langle\rho(\mu)\phi, \rho(\eta)\phi\rangle \\ \times\; e^{\pi i P}\, \langle\rho(\xi)\phi, \rho(\alpha)\phi\rangle\, \langle\rho(\beta)\phi, \rho(\nu)\phi\rangle\, d\xi\, d\eta\, d\mu\, d\nu. \tag{3.6}$$

Set

$$\begin{aligned} S'_\sigma(\alpha,\eta) &= \int_{\mathbf{R}^{2n}} |S_\Phi\sigma(N(\xi,\eta))|\, |A(\phi)(\alpha-\xi)|\, d\xi, \\ S'_\tau(\mu,\beta) &= \int_{\mathbf{R}^{2n}} |S_\Phi\tau(N(\mu,\nu))|\, |A(\phi)(\beta-\nu)|\, d\nu. \end{aligned}$$

These functions are smoothed versions of the STFTs of σ and τ. The following result shows that the STFT of $\sigma\sharp\tau$ is controlled by the (continuous) matrix multiplication of S'_σ and S'_τ.

Proposition 3.1. *Let $\sigma,\, \tau \in \mathcal{S}(\mathbf{R}^{2n})$. Then:*

$$|S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta))| \;\le\; 2^{2n}\, \big\langle S'_\sigma(\alpha,\cdot) * |A(\phi)|,\; S'_\tau(\cdot,\beta)\big\rangle.$$

Proof. By Proposition 2.1, $\langle\rho(a)\phi, \rho(b)\phi\rangle\rangle = e^{\pi i[a,b]}A(\phi)(a-b)$, so (3.6) implies:

$$\begin{aligned} |S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta))| \;&\le\; 2^{2n} \iiiint_{\mathbf{R}^{8n}} |S_\Phi\sigma(N(\xi,\eta))|\, |S_\Phi\tau(N(\mu,\nu))|\, |A(\phi)(\mu-\eta)| \\ &\qquad\times\; |A(\phi)(\alpha-\xi)|\, |A(\phi)(\nu-\beta)|\, d\xi\, d\eta\, d\mu\, d\nu \\ &=\; 2^{2n} \iint_{\mathbf{R}^{4n}} S'_\sigma(\alpha,\eta)\, |A(\phi)(\mu-\eta)|\, S'_\tau(\mu,\beta)\, d\mu\, d\eta \\ &=\; 2^{2n}\, \big\langle S'_\sigma(\alpha,\cdot) * |A(\phi)|,\; S'_\tau(\cdot,\beta)\big\rangle. \quad \square \end{aligned}$$

3.2 Main theorem.

Proposition 3.1 shows that the time–frequency content of $\sigma\sharp\tau$ is controlled by the time–frequency content of σ and τ. The modulation spaces provide the natural setting to exactly characterize this relationship. Our main results in this direction are collected in the following theorem and corollary, and will be proved in Section 3.3.

THEOREM 3.2. *Let* $w(\alpha,\beta) = (1+|\alpha|^2+|\beta|^2)^{s/2}$, $w1(\alpha,\beta) = (1+|\alpha|^2)^{s/2}$, $w2(\alpha,\beta) = (1+|\beta|^2)^{s/2}$, *with* $s \geq 0$.

(a) *If* $\sigma,\, \tau \in M_w^{p,p}(\mathbf{R}^{2n})$, *with* $1 \leq p \leq 2$, *then* $\sigma\sharp\tau \in M_w^{p,p}(\mathbf{R}^{2n})$, *with*

$$\|\sigma\sharp\tau\|_{M_w^p} \leq C\, \|\sigma\|_{M_w^p}\, \|\tau\|_{M_w^p},$$

where $C = 2^{\frac{3}{2}s}\, 2^{3n}\, \|A(\phi)\|_{L^1_{w1}}\, \|A(\phi)\|_{L^1_{w2}}$. *In particular, if* $s = 0$, *then* $C = 2^{5n}$.

(b) *If* $\sigma \in M_w^{p,p}(\mathbf{R}^{2n})$ *and* $\tau \in M_w^{p',p'}(\mathbf{R}^{2n})$, *with* $p \geq 2$ *and* $\frac{1}{p}+\frac{1}{p'} = 1$, *then* $\sigma\sharp\tau \in M_w^{p,p}(\mathbf{R}^{2n})$, *with*

$$\|\sigma\sharp\tau\|_{M_w^p} \leq C\, \|\sigma\|_{M_w^p}\, \|\tau\|_{M_w^{p'}},$$

where $C = 2^{\frac{3}{2}s}\, 2^{3n}\, \|A(\phi)\|_{L^1_{w1}}\, \|A(\phi)\|_{L^1_{w2}}$. *In particular, if* $s = 0$, *then* $C = 2^{5n}$.

COROLLARY 3.3.

(a) *If* $\sigma,\, \tau \in S_0(\mathbf{R}^{2n})$, *then* $\sigma\sharp\tau \in S_0(\mathbf{R}^{2n})$, *and* $\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_1} \leq 2^{4n}\, \|\sigma\|_{S_0} \|\tau\|_{S_0}$.

(b) *If* $\sigma,\ \tau \in L^2_s(\mathbf{R}^{2n}) \cap H^s(\mathbf{R}^{2n})$, *where* $w(\alpha,\beta) = (1+|\alpha|^2+|\beta|^2)^{s/2}$ *with* $s \geq 0$, *then* $\sigma\sharp\tau \in L^2_s(\mathbf{R}^{2n}) \cap H^s(\mathbf{R}^{2n})$. *If, in addition,* $s > n$, *then* $\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_1} \leq C\, \|\sigma\|_{L^2_s\cap H^s} \|\tau\|_{L^2_s\cap H^s}$, *where* C *is a constant which does not depend on* σ *or* τ.

(c) *If* $\sigma,\, \tau \in L^2(\mathbf{R}^{2n})$, *then* $\sigma\sharp\tau \in L^2(\mathbf{R}^{2n})$, *and* $\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_2} \leq 2^{5n}\, \|\sigma\|_{L^2} \|\tau\|_{L^2}$.

REMARK 3.4. (i) Theorem 3.2 shows that the modulation spaces $M_w^{p,p}$ are algebras under twisted product when $1 \leq p \leq 2$. As special cases, Corollary 3.3 shows that the spaces S_0, L^2 and $L^2_s \cap H^s$ are also algebras under twisted product. In addition, in these cases we have the following property. Recall that if $\sigma \in S_0$ or $\sigma \in L^2_s \cap H^s$, then L_σ lies in a closed subspace of $\mathcal{I}_1$ (cf. [8, Theorem 3], [10, Proposition 5.4]). As a consequence, the composition of operators L_σ and L_τ with symbols in S_0 or $L^2_s \cap H^s$ is closed not only on $\mathcal{I}_1$ (as follows from the ideal property of trace–class operators, already), but also on the closed subspaces of $\mathcal{I}_1$ defined when the symbols lie in these modulation spaces.

(ii) Using the observations of Section 2.2, it is easy to transfer the results of Corollary 3.3 to the Kohn–Nirenberg correspondence. Indeed, by equation (2.2), $K_\omega = L_{T\omega}$, where $(T\omega)^\wedge = e^{-\pi i x\xi}\hat{\omega}$. Since $\|(T\omega)^\wedge\|_{M_w^{p,q}} = \|\hat{\omega}\|_{M_w^{p,q}}$ (cf. Section 2.2), it follows that Corollary 3.3 holds for the Kohn–Nirenberg correspondence as well without any changes.

(iii) Part (c) of Corollary 3.3 is known (cf. [6, Proposition 1.33]) but it is reported for completeness. Observe that the operator $L_{\sigma\sharp\tau}$ is not only in $\mathcal{I}_2$, but also in $\mathcal{I}_1$, being the composition of two Hilbert–Schmidt operators.

3.3 Proof of Theorem 3.2.

In order to prove Theorem 3.2, we require some technical lemmas. The first set of lemmas shows that the $L^{p,q}_w$–norm of $S_\Phi(\sigma\sharp\tau)$ can be controlled by the $L^{p,q}_w$–norms of $S_\Phi\sigma$ and $S_\Phi\tau$ (Lemmas 3.5–3.7). The next lemma examines the effect of the linear transformation N and, in particular, the relationship between the $L^{p,q}_w$-norm of $S_\Phi F\circ N$ and the modulation space norm of F (Lemma 3.8).

Lemma 3.5. *Let $1\le p,q,r\le\infty$, with $\frac{1}{r}+\frac{1}{r'}=1$, be given. Assume σ, $\tau\in\mathcal{S}(\mathbf{R}^{2n})$. Let w, $w1$ and $w2$ be nonnegative functions satisfying $w(\alpha,\beta)\le w1(\alpha)\,w2(\beta)$. Then:*

$$\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,q}_w} \;\le\; 2^{3n}\,\|\tilde{S}'_\sigma\|_{L^{r',p}_{w1}}\,\|S'_\tau\|_{L^{r,q}_{w2}},$$

where $\tilde{S}'_\sigma(\eta,\alpha)=S'_\sigma(\alpha,\eta)$.

Proof. For simplicity of notation, define: $a(\alpha)=|A(\phi)(\alpha)|$, $s_\eta(\xi)=|S_\Phi\sigma(N(\xi,\eta))|$, $t_\mu(\nu)=|S_\Phi\tau(N(\mu,\nu))|$, $s'_\alpha(\eta)=(s_\eta*a)(\alpha)=S'_\sigma(\alpha,\eta)$, $t'_\beta(\mu)=(t_\mu*a)(\beta)=S'_\tau(\mu,\beta)$. Since ϕ is even, Proposition 2.2(g) implies that $\|a\|_{L^1}=\int_{\mathbf{R}^{2n}}A(\phi)(\alpha)\,d\alpha=\int_{\mathbf{R}^n}\phi(\frac{p}{2})^2\,dp=2^n$. By Proposition 3.1, Hölder's inequality, and Young's inequality, we have:

$$\begin{aligned}|S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta))| &\le 2^{2n}\,\langle s'_\alpha*a,t'_\beta\rangle\\ &\le 2^{2n}\,\|s'_\alpha*a\|_{L^{r'}}\,\|t'_\beta\|_{L^r}\\ &\le 2^{2n}\,\|a\|_{L^1}\,\|s'_\alpha\|_{L^{r'}}\,\|t'_\beta\|_{L^r} \;=\; 2^{3n}\,\|s'_\alpha\|_{L^{r'}}\,\|t'_\beta\|_{L^r}.\end{aligned} \tag{3.7}$$

Consequently,

$$\begin{aligned}\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,q}_w} &= \left(\int_{\mathbf{R}^{2n}}\left(\int_{\mathbf{R}^{2n}}|S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta))|^p\,w(\alpha,\beta)^{sp}\,d\alpha\right)^{q/p}d\beta\right)^{1/q}\\ &\le 2^{3n}\left(\int_{\mathbf{R}^{2n}}\left(\int_{\mathbf{R}^{2n}}\|s'_\alpha\|^p_{L^{r'}}\,\|t'_\beta\|^p_{L^r}\,w(\alpha,\beta)^{sp}d\alpha\right)^{q/p}d\beta\right)^{1/q}\\ &\le 2^{3n}\left(\int_{\mathbf{R}^{2n}}\left(\int_{\mathbf{R}^{2n}}|S'_\sigma(\alpha,\eta)|^{r'}(w1(\alpha))^{r'}\,d\eta\right)^{p/r'}d\alpha\right)^{1/p}\\ &\qquad\times\left(\int_{\mathbf{R}^{2n}}\left(\int_{\mathbf{R}^{2n}}|S'_\tau(\mu,\beta)|^{r}(w2(\beta))^{r}\,d\mu\right)^{q/r}d\beta\right)^{1/q}\\ &= 2^{3n}\,\|\tilde{S}'_\sigma\|_{L^{r',p}_{w1}}\,\|S'_\tau\|_{L^{r,q}_{w2}}.\quad\square\end{aligned}$$

Lemma 3.6. *Let $1 < p' \le r \le q < \infty$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$, be given. Assume σ, $\tau \in \mathcal{S}(\mathbf{R}^{2n})$. Let w, $w1$ and $w2$ be nonnegative functions, with $w(\alpha,\beta) \le w1(\alpha)\, w2(\beta)$. Then:*

$$\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,q}_w} \;\le\; C_w\, \|S_\Phi\sigma\circ N\|_{L^{p,r'}_{w1}}\, \|S_\Phi\tau\circ \tilde{N}\|_{L^{q,r}_{w2}},$$

where $C_w = 2^{3n}\, \|A(\phi)\|_{L^1_{w1}}\, \|A(\phi)\|_{L^1_{w2}}$.

Proof. We adopt the notation defined in the proof of Lemma 3.5. Since $1 \le \frac{p}{r'} < \infty$, Minkowski's inequality for integrals implies:

$$\begin{aligned}\|\tilde{S}'_\sigma\|_{L^{r',p}_{w1}} &= \left(\int_{\mathbf{R}^{2n}} \left(\int_{\mathbf{R}^{2n}} |S'_\sigma(\alpha,\eta)|^{r'} (w1(\alpha))^{r'}\, d\eta\right)^{p/r'} d\alpha\right)^{1/p} \\ &\le \left(\int_{\mathbf{R}^{2n}} \left(\int_{\mathbf{R}^{2n}} |S'_\sigma(\alpha,\eta)|^{p} (w1(\alpha))^{p}\, d\alpha\right)^{r'/p} d\eta\right)^{1/r'} \;=\; \|S'_\sigma\|_{L^{p,r'}_{w1}}.\end{aligned} \tag{3.8}$$

It follows from Young's inequality that $\|(s_\eta * a)\|_{L^p_{w1}} \le \|a\|_{L^1_{w1}}\, \|s_\eta\|_{L^p_{w1}}$ for any $1 \le p \le \infty$. Consequently:

$$\|S'_\sigma\|_{L^{p,r'}_{w1}} \;\le\; \|A(\phi)\|_{L^1_{w1}}\, \|S_\Phi\sigma\circ N\|_{L^{p,r'}_{w1}}. \tag{3.9}$$

Similarly for the function S'_τ, since $1 \le \frac{q}{r} < \infty$, Minkowski's inequality for integrals implies:

$$\|S'_\tau\|_{L^{r,q}_{w2}} \le \left(\int_{\mathbf{R}^{2n}} \left(\int_{\mathbf{R}^{2n}} |S'_\tau(\mu,\beta)|^{q}\, (w2(\beta))^{q}\, d\beta\right)^{r/q} d\mu\right)^{1/r} \;=\; \|\tilde{S}'_\tau\|_{L^{q,r}_{w2}}, \tag{3.10}$$

where $\tilde{S}'_\tau(\beta,\mu) = S'_\tau(\mu,\beta)$. Using Young's inequality as before, we obtain:

$$\|\tilde{S}'_\tau\|_{L^{q,r}_{w2}} \;\le\; \|A(\phi)\|_{L^1_{w2}}\, \|S_\Phi\tau\circ\tilde{N}\|_{L^{q,r}_{w2}}. \tag{3.11}$$

Finally, the proof follows by substituting (3.8)–(3.11) into Lemma 3.5. $\square$

The following result is similar in nature but is not contained in Lemma 3.6.

Lemma 3.7. *Assume σ, $\tau \in \mathcal{S}(\mathbf{R}^{2n})$. Let w, $w1$ and $w2$ be nonnegative functions, with $w(\alpha,\beta) \le w1(\alpha)\, w2(\beta)$. Then:*

$$\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{1,1}_w} \;\le\; 2^{4n}\, \|S_\Phi\sigma\circ N\|_{L^{1,1}_{w1}}\, \|S_\Phi\tau\circ N\|_{L^{1,1}_{w2}}.$$

Proof. We adopt the notation defined in the proof of Lemma 3.5. From (3.7), observing that $\|a\|_{L^\infty} \le 1$ (Proposition 2.2(b)), the Young's inequality implies that:

$$\begin{aligned}|S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta))| \;&\le\; 2^{2n}\, \|s'_\alpha * a\|_{L^\infty}\, \|t'_\beta\|_{L^1} \;\le\; 2^{2n}\, \|a\|_{L^\infty}\, \|s'_\alpha\|_{L^1}\, \|t'_\beta\|_{L^1} \\ &\le\; 2^{2n}\, \|s'_\alpha\|_{L^1}\, \|t'_\beta\|_{L^1}.\end{aligned} \tag{3.12}$$

By Proposition 2.2 we have $\int_{\mathbf{R}^{2n}} a(\alpha)\,d\alpha = 2^n$, and so:

$$\int \|s'_\alpha\|_{L^1}\,d\alpha = \iiint s_\eta(\xi)\,a(\xi-\alpha)\,d\xi\,d\eta\,d\alpha = 2^n \iint s_\eta(\xi)\,d\xi\,d\eta = 2^n\,\|S_\Phi\sigma\circ N\|_{L^{1,1}}.$$

Similarly, we can show that $\int_{\mathbf{R}^{2n}} \|t'_\beta\|_{L^1}\,d\beta = 2^n\,\|S_\Phi\tau\circ N\|_{L^{1,1}}$. Consequently, from equation (3.12) we obtain:

$$\begin{aligned}
\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{1,1}_w(\mathbf{R}^{4n})} &= \iint_{\mathbf{R}^{4n}} |S_\Phi(\sigma\sharp\tau)(N(\alpha,\beta))|\,w(\alpha,\beta)\,d\alpha\,d\beta\\
&\le 2^{2n}\int_{\mathbf{R}^{2n}} \|s'_\alpha\|_{L^1}\,w1(\alpha)\,d\alpha \int_{\mathbf{R}^{2n}} \|t'_\beta\|_{L^1}\,w2(\beta)\,d\beta\\
&= 2^{4n}\,\|S_\Phi\sigma\circ N\|_{L^{1,1}_{w1}(\mathbf{R}^{4n})}\|S_\Phi\tau\circ N\|_{L^{1,1}_{w2}(\mathbf{R}^{4n})}. \qquad \square
\end{aligned}$$

Now, we examine the effect of the linear transformations N and $\tilde{N}$ on the $L^{p,q}_w$-norm of a function of the form $F\circ N$ or $F\circ\tilde{N}$. Indeed, in general, $\|f\|_{M^w_{p,q}} = \|S_\Phi f\|_{L^{p,q}_w} \neq \|S_\Phi f\circ N\|_{L^{p,q}_w}$ (and similarly when N is replaced by $\tilde{N}$). The following lemma shows that in some special cases the $L^{p,q}_w$-norm of $S_\Phi f\circ N$ is controlled by some modulation space norm of f. Parts (c) and (d) of the following lemma are not needed in the proof of Theorem 3.2, but are included for completeness.

Lemma 3.8. *Let $1\le p\le\infty$ and $s\ge 0$ be given. Let $w(\alpha,\beta) = (1+|\alpha|^2+|\beta|^2)^{s/2}$. Then:*

(a) $2^{-s/2}\,\|S_\Phi\sigma\circ N\|_{L^{p,p}_w} \le \|\sigma\|_{M^{p,p}_w} \le 2^{s/2}\,\|S_\Phi\sigma\circ N\|_{L^{p,p}_w}$,

(b) $\|S_\Phi\sigma\circ N\|_{L^{p,p}} = \|\sigma\|_{M^{p,p}}$,

(c) $\|S_\Phi\sigma\circ N\|_{L^{1,\infty}} \le 2^{2n}\,\|\hat{\sigma}\|_{M^{\infty,1}}$,

(d) $\|S_\Phi\sigma\circ N\|_{L^{1,\infty}} \le \|\sigma\|_{M^{\infty,1}}$.

Proof. (a) The change of variables $(\alpha,\beta) = N(\xi,\eta)$ yields:

$$\begin{aligned}
\|S_\Phi\sigma\circ N\|_{L^{p,p}_w} &= \left(\iint_{\mathbf{R}^{4n}} |S_\Phi\sigma(N(\xi,\eta))|^p\,(1+|\xi|^2+|\eta|^2)^{sp/2}\,d\xi\,d\eta\right)^{1/p}\\
&= \left(\iint_{\mathbf{R}^{4n}} |S_\Phi\sigma(\alpha,\beta))|^p\,(1+2|\alpha|^2+\tfrac{|\beta|^2}{2})^{sp/2}\,d\alpha\,d\beta\right)^{1/p}\\
&\le 2^{s/2}\left(\iint_{\mathbf{R}^{4n}} |S_\Phi\sigma(\alpha,\beta))|^p\,(1+|\alpha|^2+|\beta|^2)^{sp/2}\,d\alpha\,d\beta\right)^{1/p}\\
&= 2^{s/2}\,\|\sigma\|_{M^{p,p}_w}.
\end{aligned}$$

The other inequality is obtained in a similar way.

(b) This case is part (a) when $s = 0$.

(c) Denote $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \mathbf{R}^n \times \mathbf{R}^n$. By direct calculation:

$$\begin{aligned}
\|S_\Phi\sigma \circ N\|_{L^{1,\infty}} &= \sup_\eta \int_{\mathbf{R}^{2n}} |S_\Phi\sigma(N(\xi,\eta))|\, d\xi \\
&\le \sup_{\eta_1,\eta_2} \iint_{\mathbf{R}^{2n}} \sup_\theta |S_\Phi\sigma((\tfrac{\xi_2+\eta_2}{2}, -\tfrac{\xi_1+\eta_1}{2}), \theta)|\, d\xi_1\, d\xi_2 \\
&= \int_{\mathbf{R}^{2n}} \sup_\theta |S_\Phi\sigma(\tfrac{\xi}{2}, \theta)|\, d\xi \\
&= 2^{2n} \int_{\mathbf{R}^{2n}} \sup_\theta |S_\Phi\sigma(\xi,\theta)|\, d\xi \\
&= 2^{2n} \int_{\mathbf{R}^{2n}} \sup_\theta |S_{\hat\Phi}\hat\sigma(\theta,\xi)|\, d\xi \;=\; 2^{2n}\, \|\hat\sigma\|_{M^{\infty,1}}.
\end{aligned}$$

(d) Similar to part (c). $\square$

Remark 3.9. Lemma 3.8 holds when N is replaced by $\tilde N$ without any changes. The proof is exactly the same.

Now we are ready to prove Theorem 3.2 and Corollary 3.3.

Proof of Theorem 3.2. We begin by assuming σ, $\tau \in \mathcal{S}(\mathbf{R}^{2n})$. Since the space $\mathcal{S}(\mathbf{R}^{2n})$ is dense in $M_w^{p,q}(\mathbf{R}^{2n})$, for any $1 \le p$, $q < \infty$ (as discussed in Section 2.2), the extension to the case σ, $\tau \in M_w^{p,p}$ follows by a standard continuity argument.

(a) It is sufficient to prove the cases $p = 1$ and $p = 2$. The theorem then follows by interpolation. In fact, by [1], [2]:

$$(M_w^{1,1}(\mathbf{R}^{2n}), M_w^{2,2}(\mathbf{R}^{2n}))_\theta \;=\; M_w^{p,p}(\mathbf{R}^{2n})$$

with $\frac{1}{p} = (1-\theta) + \frac{\theta}{2}$, $\theta \in [0,1]$.

It is clear that $w(\alpha,\beta) \le w1(\alpha)\, w2(\beta)$ and that

$$w1(\alpha), w2(\beta) \;\le\; w(\alpha,\beta) \tag{3.13}$$

for all α, $\beta \in \mathbf{R}^{2n}$. The case $p = 1$ then follows from the following estimates:

$$\begin{aligned}
\|S_\Phi(\sigma\sharp\tau)\|_{L^1_w} &\le 2^{\frac{s}{2}}\, \|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^1_w} && \text{by Lemma 3.8} \\
&\le 2^{\frac{s}{2}+4n}\, \|S_\Phi\sigma\circ N\|_{L^1_{w1}}\, \|S_\Phi\tau\circ N\|_{L^1_{w2}} && \text{by Lemma 3.7} \\
&\le 2^{\frac{s}{2}+4n}\, \|S_\Phi\sigma\circ N\|_{L^1_w}\, \|S_\Phi\tau\circ N\|_{L^1_w} && \text{by equation (3.13)} \\
&\le 2^{\frac{3}{2}s+4n}\, \|S_\Phi\sigma\|_{L^1_w}\, \|S_\Phi\tau\|_{L^1_w} && \text{by Lemma 3.8} \\
&= 2^{\frac{3}{2}s+4n}\, \|\sigma\|_{M^1_w}\, \|\tau\|_{M^1_w}.
\end{aligned}$$

For the case $p = 2$ we have:

$$\begin{aligned}
\|S_\Phi(\sigma\sharp\tau)\|_{L^2_w} &\le 2^{s/2}\,\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{2,2}_w} && \text{by Lemma 3.8} \\
&\le 2^{s/2}\,C_w\,\|S_\Phi\sigma\circ N\|_{L^{2,2}_{w1}}\,\|S_\Phi\tau\circ\tilde{N}\|_{L^{2,2}_{w2}} && \text{by Lemma 3.6} \\
&\le 2^{s/2}\,C_w\,\|S_\Phi\sigma\circ N\|_{L^{2,2}_{w}}\,\|S_\Phi\tau\circ\tilde{N}\|_{L^{2,2}_{w}} && \text{by equation (3.13)} \\
&\le 2^{\frac{3}{2}s}\,C_w\,\|\sigma\|_{M^{2,2}_w}\,\|\tau\|_{M^{2,2}_w}, && \text{by Lemma 3.8}
\end{aligned}$$

where $C_w = 2^{3n}\,\|A(\phi)\|_{L^1_{w1}}\,\|A(\phi)\|_{L^1_{w2}}$. Recall that $\|A(\phi)\|_{L^1} = 2^n$ (by Proposition 2.2(b)). Furthermore, we have that $\|A(\phi)\|_{L^1_{w1}}, \|A(\phi)\|_{L^1_{w2}} \ge \|A(\phi)\|_{L^1} = 2^n$. Therefore $C_w \ge 2^{5n}$ and $C_w = 2^{5n}$ if $s = 0$.

(b) From Lemma 3.6 and equation (3.13) we have:

$$\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,p'}_w} \le C_w\,\|S_\Phi\sigma\circ N\|_{L^{p,p}_w}\,\|S_\Phi\tau\circ\tilde{N}\|_{L^{p',p'}_w}. \tag{3.14}$$

By [6], $\|S_\Phi\sigma\circ N\|_{L^{p,p1}_w} \approx \|\big((S_\Phi\sigma\circ N)(k,m)\big)_{k,m}\|_{\ell^{p,p1}_w}$ and, therefore, since $\ell^{p,p1}_w \subset \ell^{p,p2}_w$ if $p1 \le p2$ (cf. [8, Chapter 12]), this implies that:

$$\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,p}_w} \le \|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,p'}_w}. \tag{3.15}$$

Using Lemma 3.8 and equations (3.13) and (3.15) into equation (3.14) we obtain:

$$\begin{aligned}
\|\sigma\sharp\tau\|_{M^{p,p}_w} &\le 2^{s/2}\,\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,p}_w} \\
&\le 2^{s/2}\|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{p,p'}_w} \\
&\le 2^{\frac{3}{2}s}\,C_w\,\|\sigma\|_{M^{p,p}_w}\,\|\tau\|_{M^{p',p'}_w}. \quad \square
\end{aligned}$$

The proof of Corollary 3.3 follows easily.

Proof of Corollary 3.2.

(a) Recall that $S_0 = M^{1,1}$. Since $\langle\cdot,\rho(\nu)\phi\rangle\rho(\xi)\phi$ is a rank–one operator, it follows that $\|\langle\cdot,\rho(\nu)\phi\rangle\rho(\xi)\phi\|_{\mathcal{I}_1} \le 1$. Applying this observation to equation (2.5), we obtain:

$$\begin{aligned}
\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_1} &\le \|\langle\cdot,\rho(\nu)\phi\rangle\rho(\xi)\phi\|_{\mathcal{I}_1} \iint_{\mathbf{R}^{4n}} |S_\Phi(\sigma\sharp\tau)(N(\xi,\nu))|\,d\nu\,d\xi \\
&\le \|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{1,1}}.
\end{aligned} \tag{3.16}$$

Using equation (3.16) and Lemmas 3.7 and 3.8(b), we obtain:

$$\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_1} \le \|S_\Phi(\sigma\sharp\tau)\circ N\|_{L^{1,1}} \le 2^{4n}\,\|S_\Phi\sigma\circ N\|_{L^{1,1}}\,\|S_\Phi\tau\circ N\|_{L^{1,1}} = 2^{4n}\,\|\sigma\|_{S_0}\,\|\tau\|_{S_0}.$$

(b) Recall that $M^{2,2}_w = L^2_s \cap H^s$, $M^{2,2}_{w1} = L^2_s$, and $M^{2,2}_{w2} = H^s$, with $w1(\alpha,\beta) = (1+|\alpha|^2)^{s/2}$, $w2(\alpha,\beta) = (1+|\beta|^2)^{s/2}$. By [10, Proposition 5.4], if $s > n$, then $\sigma\sharp\tau \in L^2_s \cap H^s$ implies $L_{\sigma\sharp\tau} \in \mathcal{I}_1$. Consequently, from Theorem 3.2 we have:

$$\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_1} \leq c\,\|S_\Phi(\sigma\sharp\tau)\|_{L^{2,2}_w} \leq c\,2^{\frac{3}{2}s}\,C_w\,\|\sigma\|_{M^{2,2}_s}\,\|\tau\|_{M^{2,2}_s},$$

where c is a constant which does not depend on σ or τ, and $C_w = 2^{3n}\,\|A(\phi)\|_{L^1_{w1}}\,\|A(\phi)\|_{L^1_{w2}}$.

(c) By a classical result of Pool ([13]), $\|L_\sigma\|_{\mathcal{I}_2} = \|\sigma\|_{L^2}$. Therefore, by Theorem 3.2 with $s = 0$, we have:

$$\|L_{\sigma\sharp\tau}\|_{\mathcal{I}_2} \;=\; \|\sigma\sharp\tau\|_{L^2} \;=\; \|\sigma\sharp\tau\|_{M^{2,2}} \;=\; \|S_\Phi(\sigma\sharp\tau)\|_{L^{2,2}} \;\leq\; 2^{5n}\,\|\sigma\|_{M^{2,2}}\,\|\tau\|_{M^{2,2}}. \quad \square$$

A closer look at the proof of Theorem 3.2 shows that it is possible to generalize the choice of the weight function w. In fact, if w is any positive subadditive function with at most polynomial growth and $w1$, $w2$ are positive functions satisfying $w(\alpha,\beta) \leq w1(\alpha)w2(\beta)$, $w1(\alpha)$, $w2(\beta) \leq w(\alpha,\beta)$ and $w(N^{-1}(\alpha,\beta)) \leq C\,w(\alpha,\beta)$ for some $C > 0$ and for all $\alpha,\beta \in \mathbf{R}^{2n}$, then the conclusions of Theorem 3.2 hold and the proof is essentially the same.

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