Pseudodifferential Operators on Modulation Spaces

Demetrio Labate

Department of Mathematics, Washington University in St.Louis, St.Louis, Missouri 63130-4899. E–mail: dlabate@math.wustl.edu

We establish a connection between certain classes of pseudodifferential operators and Hille–Tamarkin operators. As an application, we find the conditions that guarantee compactness and summability of the eigenvalues of pseudodifferential operators acting on the modulation spaces $M^{p,p}$.

1. Introduction

A pseudodifferential operator can be defined through the Weyl or the Kohn–Nirenberg correspondences by bijectively assigning to any distributional symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$ a linear operator $T_\sigma\colon \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$, so that the properties of the operator are in an appropriate way reflected in the properties of the symbol. Pseudodifferential operators have a wide range of applications in mathematics, physics and engineering.

In this paper, we are interested in pseudodifferential operators whose symbols satisfy certain integrability conditions in the time–frequency plane and are not necessarily smooth. The interest of these classes of operators stems partly from electrical engineering applications, in particular signal processing theory, where operators arising from the Weyl correspondence are used as models for time–frequency or time–varying filters (see, for instance, [6, 13, 16]). In this context, the symbol is the mask of the filter, since it selectively weights the different time–frequency components of the signal.

In order to exactly quantify the time–frequency content of the symbol, we use certain function spaces, called modulation spaces. These spaces were introduced about 1980 by H. Feichtinger by prescribing the decay properties of the Short–Time Fourier Transform (STFT) of a given function or distribution. This class contains a large collection of function spaces, including some classical function spaces such as L^2 or the Sobolev spaces. It was shown in recent papers that modulation spaces provide natural symbol classes for pseudodifferential operators in both the Weyl and the Kohn–Nirenberg correspondences (cf. [11, 14, 18]). Furthermore, some recent results from the literature suggest that the modulation spaces M^p are natural substitutes for the spaces L^p in the study of the action of pseudodifferential operators beyond L^2 : [17] shows that pseudodifferential operators with traditional symbol classes are bounded on the modulation spaces $M_w^{p,q}$, and in [11]) an improvement of the classical Calderòn–Vaillancourt theorem is obtained as a corollary of a result about boundedness of pseudodifferential operators on the modulation spaces $M^{p,p}$.

In our approach, the pseudodifferential operator T_σ is realized as a superposition of elementary rank–one operators through the time–frequency decomposition of the associated symbol σ. Using this time–frequency decomposition of the operator we obtain a fundamental connection between certain classes of pseudodifferential operators and Hille–Tamarkin operators. As an application, we study compactness and spectral properties of pseudodifferential operators acting on the modulation spaces $M^{p,p}$, including $M^{2,2} = L^2$. Using some results from the theory of absolutely summing operators, we obtain conditions that ensure compactness and summability of the eigenvalues of pseudodifferential operators acting on the spaces $M^{p,p}$.

Notation.

Let X, X_0, Y, Y_0 be Banach spaces. $\mathcal{L}(X,Y)$ is the space of all bounded linear operators from X to Y, and $\mathcal{L}(X) = \mathcal{L}(X,X)$. The norm of X is $\|\cdot\|_X$, or simply $\|\cdot\|$ if the context is clear. $B(X)$ is the closed unit ball $\{f \in X : \|f\| \leq 1\}$ in X. The dual space of X is X'. We write $\langle f, g\rangle$ for the action of $g \in X'$ on $f \in X$. T^* is the adjoint operator of T. A linear operator $T : X \to Y$ is *compact* if, given any bounded sequence (x_n) in X, (Tx_n) has a norm convergent subsequence in Y. T is *weakly compact* if, given any bounded sequence (x_n) in X, (Tx_n) has a weakly convergent subsequence in Y. T is *completely continuous* if it maps weakly convergent sequences in X to norm convergent sequences in Y. A linear subspace $\mathcal{A}(X,Y)$ of $\mathcal{L}(X,Y)$ is an *operator ideal* if $UTV \in \mathcal{A}(X,Y)$ whenever $U \in \mathcal{L}(Y,Y_0)$, $T \in \mathcal{A}(X,Y)$, and $V \in \mathcal{L}(X_0,X)$. The classes of compact, weakly compact, and completely continuous operators are operator ideals.

$L^{p,q}(\mathbf{R}^{2n})$ is the mixed–normed space of functions f on $\mathbf{R}^{2n}$ with norm $\|f\|_{L^{p,q}} =$

$(\int_{\mathbf{R}^n}(\int_{\mathbf{R}^n}|f(x,y)|^p\,dx)^{q/p}\,dy)^{1/q}$. If $p=q$, we have the classical space $L^p(\mathbf{R}^{2n})=L^{p,p}(\mathbf{R}^{2n})$. Similarly, $\ell^{p,q}(\mathbf{Z}^{2n})$ is the space of sequences $a=(a_{km})_{k,m\in\mathbf{Z}^n}$ with norm $\|a\|_{\ell^{p,q}}=(\sum_m(\sum_k|a_{km}|^p)^{q/p})^{1/q}$. If $p=q$, we have the classical sequence space $\ell^p(\mathbf{Z}^{2n})=\ell^{p,p}(\mathbf{Z}^{2n})$. $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space of all infinitely differentiable functions on $\mathbf{R}^n$ decaying rapidly at infinity, and $\mathcal{S}'(\mathbf{R}^n)$ is its topological dual, the space of tempered distributions. $H^s(\mathbf{R}^n)$ is the Sobolev space of functions defined by the norm $\|f\|_{H^s}^2=\int_{\mathbf{R}^n}|\hat{f}(\gamma)|^2\,(1+|\gamma|^2)^s\,d\gamma$. The usual dot product of $x, y\in\mathbf{R}^n$ is denoted by juxtaposition, i.e., $xy=x_1y_1+\cdots+x_ny_n$. The symplectic form on $\alpha=(\alpha_1,\alpha_2)$, $\beta=(\beta_1,\beta_2)\in\mathbf{R}^n\times\mathbf{R}^n$ is $[\alpha,\beta]=\alpha_1\beta_2-\alpha_2\beta_1$. The composition of f and g is $(f\circ g)(t)=f(g(t))$. The inner product of $f, g\in L^2(\mathbf{R}^n)$ is $\langle f,g\rangle=\int_{\mathbf{R}^n}f(t)\,\overline{g(t)}\,dt$; the same notation is used for the extension of the inner product to $\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}'(\mathbf{R}^n)$. The Fourier transform is $\mathcal{F}f(\gamma)=\hat{f}(\gamma)=\int_{\mathbf{R}^n}f(t)\,e^{-2\pi i\gamma t}\,dt$; the inverse Fourier transform is $\check{f}(\gamma)=\hat{f}(-\gamma)$. The Fourier transform maps $\mathcal{S}(\mathbf{R}^n)$ onto itself, and extends to $\mathcal{S}'(\mathbf{R}^n)$ by duality.

2. Background: Time–Frequency Analysis

We briefly review the Schrödinger representation of the Heisenberg group as a tool for constructing and analyzing pseudodifferential operators. We adopt most of the notation and conventions of Folland's book [7].

2.1. The Schrödinger representation. The Schrödinger representation of the Heisenberg group $\mathbf{H}^n=\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}$ is the map ρ from $\mathbf{H}^n$ to the group of unitary operators on $L^2(\mathbf{R}^n)$ defined by $\rho(a,b,t)f(x)=e^{2\pi it}\,e^{\pi iab}\,e^{2\pi ibx}\,f(x+a)$. In many considerations the t-variable is unimportant, so for $(a,b)\in\mathbf{R}^{2n}$ we define $\rho(a,b)f(x)=e^{\pi iab}\,e^{2\pi ibx}\,f(x+a)$. We recall the following useful facts.

Proposition 2.1. *Let $f\in L^2(\mathbf{R}^n)$ and let $a, b, a', b'\in\mathbf{R}^n$. Then:*

(a) $\|\rho(a,b)f\|_{L^2}=\|f\|_{L^2}$,

(b) $(\rho(a,b)f)^\wedge=\rho(-b,a)\hat{f}$,

(c) $(\rho(a,b))^{-1}=(\rho(a,b))^*=\rho(-a,-b)$,

(d) $\rho(a,b)\,\rho(a',b')f = e^{\pi i(ab'-a'b)}\rho(a+a',\, b+b')f.$

The *Fourier-Wigner transform* of $f,\, g \in L^2(\mathbf{R}^n)$ is:

$$A(f,g)(a,b) \;=\; \langle \rho(a,b)f, g\rangle \;=\; \int_{\mathbf{R}^n} e^{\pi iab}\, e^{2\pi ibx}\, f(x+a)\,\overline{g(x)}\, dx.$$

If $f = g$, we write $A(f,f) = A(f)$. A slight change in the definition yields the *Short–Time Fourier Transform* (STFT) of a distribution $f \in \mathcal{S}'(\mathbf{R}^n)$ with respect to a window $g \in \mathcal{S}(\mathbf{R}^n)$:

$$S_g f(a,b) \;=\; \int_{\mathbf{R}^n} f(x)\,\overline{g(x-a)}\, e^{-2\pi ixb}\, dx \;=\; e^{-\pi iab}\langle \rho(a,-b)f, g\rangle \;=\; e^{-\pi iab} A(f,g)(a,-b).$$

The *Wigner transform* of $f,\, g \in L^2(\mathbf{R}^n)$ is the Fourier transform of the Fourier–Wigner transform of f and g:

$$W(f,g)(\xi,x) \;=\; A(f,g)^\wedge(\xi,x) \;=\; \int_{\mathbf{R}^n} e^{-2\pi ip\xi}\, f(x+\tfrac{p}{2})\,\overline{g(x-\tfrac{p}{2})}\, dp. \qquad (2.1)$$

If $f = g$, we write $W(f,f) = W(f)$.

The Fourier–Wigner transform and the Wigner transform extend to a map from $\mathcal{S}(\mathbf{R}^n)\times\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}(\mathbf{R}^{2n})$ and from $\mathcal{S}'(\mathbf{R}^n)\times\mathcal{S}'(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^{2n})$. The following properties will be useful (cf. [7, Chap. 1]).

PROPOSITION 2.2. *Let $f,\, g \in L^2(\mathbf{R}^n)$ and let $a,\, b,\, u_1,\, u_2,\, v_1,\, v_2 \in \mathbf{R}^n$. Let $N\colon \mathbf{R}^{2n}\times\mathbf{R}^{2n} \to \mathbf{R}^{4n}$ be the linear transformation*

$$N(u,v) \;=\; N(u_1,u_2,v_1,v_2) \;=\; \big(\tfrac{u_2+v_2}{2}, -\tfrac{u_1+v_1}{2}, u_1-v_1, u_2-v_2\big), \qquad (2.2)$$

where $u = (u_1,u_2)$, $v = (v_1,v_2) \in \mathbf{R}^n\times\mathbf{R}^n$. Then:

(a) $A(f,g), W(f,g) \in L^2(\mathbf{R}^{2n})$, *with* $\|A(f,g)\|_{L^2} = \|W(f,g)\|_{L^2} = \|f\|_{L^2}\,\|g\|_{L^2}$.

(b) $\|A(f,g)\|_{L^\infty}, \|W(f,g)\|_{L^\infty} \le \|f\|_{L^2}\,\|g\|_{L^2}$.

(c) $W(\rho(u)f, \rho(v)g)(a,b) = \rho(N(u,v))\,W(f,g)$.

(d) $S_g f(a,b) \;=\; e^{2\pi iab}\, S_{\hat g}\hat f(b,-a)$.

2.2 Modulation Spaces. The modulation spaces measure the joint time-frequency distribution of $f \in \mathcal{S}'(\mathbf{R}^d)$. For background and detailed information on their properties we refer to [2, 3, 4, 8].

Let w be a submultiplicative positive weight function on $\mathbf{R}^{2n}$, i.e., $1 \leq w(\alpha) < \infty$, and $w(\alpha+\beta) \leq w(\alpha)\, w(\beta)$ for all $\alpha,\ \beta \in \mathbf{R}^{2n}$, and assume that w has at most polynomial growth, i.e., $w(\alpha) \leq C|\alpha|^N$ for some $C, N \geq 0$ and for all $\alpha \in \mathbf{R}^{2n}$. Let $1 \leq p,\ q \leq \infty$. Given a window function $g \in \mathcal{S}(\mathbf{R}^n)$, denote by $M^{p,q}_w(\mathbf{R}^n)$ the space of all distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ for which the norm

$$\|f\|_{M^{p,q}_w(\mathbf{R}^n)} \;=\; \|S_g f\|_{L^{p,q}(\mathbf{R}^{2n})} \;=\; \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} |S_g f(x,y)|^p\, w(x,y)^p\, dx\right)^{q/p} dy\right)^{1/q}$$

is finite, with obvious modifications if p or $q = \infty$. If $w \equiv 1$ then we write $M^{p,q}(\mathbf{R}^n)$. The result presented in this paper will only make use of modulation spaces with $w \equiv 1$, but here we preferred to present the more general definition, to better motivate the importance of modulation spaces in time–frequency analysis.

$M^{p,q}_w(\mathbf{R}^n)$ is a Banach space whose definition is independent of the choice of window g, i.e., different choices of windows g yield equivalent norms. The assumptions on the weight w guarantee that the modulation spaces are defined in the realm of tempered distributions, and that $\mathcal{S}$ is dense in all modulation spaces $M^{p,q}_w$ for all $1 \leq p, q \leq \infty$ [10, Section 11.1]. For $1 \leq p,\ q < \infty$, the dual space of $M^{p,q}(\mathbf{R}^n)$ is $(M^{p,q}(\mathbf{R}^n))' = M^{p',q'}(\mathbf{R}^n)$, where p', q' satisfy $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$. $M^{p,p}(\mathbf{R}^n)$ is invariant under the Fourier transform. Moreover, the modulation spaces are invariant under the metaplectic representation (cf. [4, Theorem 29]). In particular, multiplication by $e^{-\pi i x y}$ leaves the space $M^{p,q}(\mathbf{R}^n)$ invariant for each $1 \leq p, q \leq \infty$, i.e., $\|f\|_{M^{p,q}} = \|e^{-\pi i x y} f\|_{M^{p,q}}$. We will use this property in Sections 3 to transfer results between Weyl and Kohn–Nirenberg correspondences.

Among the modulation spaces the following well-known function spaces occur.

(a) $M^{2,2}(\mathbf{R}^n) = L^2(\mathbf{R}^n)$.

(b) (Weighted L^2-spaces) If $w(x,y) = (1+|x|)^s$, then $M^{2,2}_w(\mathbf{R}^n) \;=\; L^2_w(\mathbf{R}^n)$.

(c) (Sobolev spaces) If $w(x,y) = (1+|y|)^s$, then $M^{2,2}_w(\mathbf{R}^n) \;=\; H^s(\mathbf{R}^n)$.

(d) If $w(x,y) = (1+|x|+|y|)^s$, then $M^{2,2}_w(\mathbf{R}^n) \;=\; L^2_s(\mathbf{R}^n) \cap H^s(\mathbf{R}^n)$.

(e) (Feichtinger's algebra) $M^{1,1}(\mathbf{R}^n) = S_0(\mathbf{R}^n)$.

We recall that the space S_0 is contained in L^2, and that it is an algebra under both convolution and pointwise multiplication. The space S_0 plays an important role in abstract harmonic analysis (cf. [5]).

2.3 Time–Frequency Expansion of Pseudodifferential Operators.

The *Weyl correspondence* is the 1-1 correspondence between a distributional symbol $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$ and the pseudodifferential operator $L_\sigma = \sigma(D,X)\colon \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ defined implicitly by:

$$\langle L_\sigma f, g\rangle \;=\; \langle \hat{\sigma}, A(g,f)\rangle \;=\; \langle \sigma, W(g,f)\rangle,$$

where $f, g \in \mathcal{S}(\mathbf{R}^n)$. L_σ is the *Weyl transform* of σ. The *Kohn–Nirenberg correspondence* assigns to a symbol τ the operator $K_\tau = \sigma(D,X)_{KN}$ defined implicitly by:

$$\langle K_\tau f, g\rangle \;=\; \langle \hat{\tau}, e^{\pi i \xi x} A(g,f)\rangle. \tag{2.3}$$

K_τ is the *Kohn–Nirenberg transform* of τ. Equation (2.3) shows that the operators L_σ in the Weyl correspondence and K_τ in the Kohn–Nirenberg are equal if and only if their symbols are related by $\hat{\sigma}(\xi,x) = \hat{\tau}(\xi,x)e^{-\pi i\xi x}$. Therefore, statements invariant under multiplication by $e^{-\pi i \xi x}$ will be valid for one correspondence if and only if they are valid for the other.

We expand the Weyl operator L_σ by realizing the symbol σ of the operator as a superposition of time–frequency shifts. The fundamental result needed for the time–frequency expansion is the following inversion formula (cf. [2]).

THEOREM 2.3. *If $\Phi \in \mathcal{S}(\mathbf{R}^{2n})$ with $\|\Phi\|_{L^2} = 1$, and $\sigma \in M^{p,q}(\mathbf{R}^{2n})$ with $1 \le p, q < \infty$, then:*

$$\sigma \;=\; \iint_{\mathbf{R}^{4n}} \langle \sigma, \rho(\alpha,\beta)\Phi\rangle\, \rho(\alpha,\beta)\Phi \, d\alpha\, d\beta, \tag{2.4}$$

where the integral converges in the norm of $M^{p,q}(\mathbf{R}^{2n})$. If $p = \infty$ or $q = \infty$ or if $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$, then (2.4) holds with weak convergence of the integral.

The following consequence of Theorem 2.3 is proved in [11, Lemma 3.2].

THEOREM 2.4. *Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ with $\|\phi\|_{L^2} = 1$, and let $\Phi = W(\phi)$. Let $\sigma \in M^{p,q}(\mathbf{R}^{2n})$, with $1 \le p, q \le \infty$. Let N be the linear transformation defined in Proposition 2.2, and let $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \mathbf{R}^n \times \mathbf{R}^n$. Then, for $f \in \mathcal{S}(\mathbf{R}^n)$ we have:*

$$L_\sigma f \;=\; \iint_{\mathbf{R}^{4n}} S_\Phi \sigma(N(\xi,\eta))\, e^{\pi i [\xi,\eta]}\, \langle f, \rho(\eta)\phi\rangle\, \rho(\xi)\phi \, d\xi\, d\eta. \tag{2.5}$$

This integral converges as in Theorem 2.3.

From now on we will let ϕ denote an arbitrary but fixed function in $\mathcal{S}(\mathbf{R}^n)$ such that $\phi(t) = \phi(-t)$ and $\|\phi\|_{L^2} = 1$. For example, we could take $\phi(x) = 2^{n/4}\, e^{-\pi x^2}$. We set $\Phi = W(\phi)$. We will let N denote the linear transformation defined in Proposition 2.2, and $\tilde{N}$ the linear transformation $\tilde{N}(\xi,\eta) = N(\eta,\xi)$.

3. Pseudodifferential Operators on Modulation Spaces

We will prove the following result in this section.

THEOREM 3.1. *Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

(a) *If $\sigma \in M^1$, then L_σ is a weakly compact and completely continuous operator mapping $M^1(\mathbf{R}^n)$ into itself and the eigenvalues of L_σ are 2-summable with*

$$\Big(\sum |\lambda_n|^2\Big)^{1/2} \;\le\; C\, \|\sigma\|_{M^1},$$

where $C > 0$ is independent of σ.

(b) *If $1 < p \le 2$ and $\sigma \in M^p$, then L_σ is a compact operator mapping $M^p(\mathbf{R}^n)$ into itself and the eigenvalues of L_σ are 2-summable, with*

$$\Big(\sum |\lambda_n|^2\Big)^{1/2} \;\le\; C\, \|\sigma\|_{M^p},$$

where $C > 0$ is independent of σ.

(c) *If $2 \le p < \infty$ and $\sigma \in M^{p'}$, then L_σ is a compact operator mapping $M^p(\mathbf{R}^n)$ into itself and the eigenvalues of L_σ are r-summable, where $r = \max\{2, p\}$, with*

$$\Big(\sum |\lambda_n|^r\Big)^{1/r} \;\le\; C\, \|\sigma\|_{M^{p'}},$$

where $C > 0$ is independent of σ.

Remark 3.2. (i) Since $\|\sigma\|_{M^2} = \|\sigma\|_{L^2}$, in the special case $p=2$, Theorem 3.1 reduces to the statement that L_σ is a Hilbert–Schmidt operator if $\sigma \in L^2$. This recovers a classical result of Pool [15].

(ii) Recall that, by [9], if $\sigma \in M^1$, then L_σ is a trace–class operator (hence, it is compact on L^2 with summable singular values).

(iii) The results of Theorem 3.1 can be transferred to the Kohn–Nirenberg correspondence without any changes. In fact, by equation (2.3) (and the comment thereafter), $K_\omega = L_{T\omega}$ where $(T\omega)^\wedge = e^{\pi i x\xi}\hat{\omega}$, and by the observation in Section 2.2, $\|(T\omega)^\wedge\|_{M^{p,q}} = \|\hat{\omega}\|_{M^{p,q}}$.

Before proving Theorem 3.1, we recall some background on p-summing operators.

3.1 Absolutely Summing Operators.

Let X, Y be Banach spaces, $T \in \mathcal{L}(X,Y)$, and $1 \le p < \infty$. Then T is *absolutely p-summing* or simply *p-summing* if there is a constant $c \ge 0$ such that for all sequences $(f_i)_{i=1}^m$ in X we have:

$$\Big(\sum_{i=1}^m \|Tf_i\|_Y^p\Big)^{1/p} \le c \sup_{g\in B(X')} \Big(\sum_{i=1}^m |\langle g, f_i\rangle|^p\Big)^{1/p}.$$

The least c for which the inequality holds is denoted by $\pi_p(T)$. The set of p-summing operators from X to Y is denoted by $\Pi_p(X,Y)$. The collection $\Pi_p(X,Y)$ is a Banach space with norm $\pi_p(T)$, it is an operator ideal, and it coincides with the class of Hilbert–Schmidt operators if X and Y are Hilbert spaces (cf. [1, Ch. 2]). The following standard result will be useful (cf. [12, Section 2.b]).

Proposition 3.3. *Let X, Y be Banach spaces. If $T \in \Pi_p(X,Y)$, then T is weakly compact and completely continuous and the eigenvalues of T are r-summable, where $r = \max\{2,p\}$, with*

$$\Big(\sum |\lambda_n|^r\Big)^{1/r} \le \pi_p(T).$$

Furthermore, if X is reflexive, then T is compact.

It follows from Proposition 3.3 that the composition of a p-summing with a q-summing operator is compact, no matter how we choose $1 \le p, q < \infty$.

The following example of p-summing operators will be useful. Let (Ω, μ) be a measure space. Assume $k : \Omega \times \Omega \to \mathbf{R}$ is $\mu \times \mu$ measurable and that

$$\|k\|_{p,q} = \left(\int_\Omega \left(\int_\Omega |k(\omega, \omega')|^q \, d\mu(\omega') \right)^{p/q} d\mu(\omega) \right)^{1/p} < \infty.$$

The operators of the form $T_k f(\omega) = \int_\Omega k(\omega, \omega') \, f(\omega') \, d\mu(\omega')$ are called *Hille-Tamarkin operators*. The following is a classical result. We include a proof as certain steps in it will be useful later.

PROPOSITION 3.4. *Let $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If $\|k\|_{p,p'} < \infty$, then the Hille-Tamarkin operator T_k maps $L^p(\mu)$ into itself and is a p–summing operator with $\pi_p(T_k) \le \|k\|_{p,p'}$.*

Proof. Let $(f_i)_{i=1}^m$ be a sequence in $L^p(\mu)$. By direct calculation we have:

$$\begin{aligned}
\sum_{i=1}^n \|T_k f_i\|_{L^p}^p &= \sum_{i=1}^n \int_\Omega \left| \int_\Omega k(\omega, \omega') \, f_i(\omega') \, d\mu(\omega') \right|^p d\mu(\omega) \\
&= \int_\Omega \left(\int_\Omega |k(\omega, \nu)|^{p'} d\mu(\nu) \right)^{p/p'} \sum_{i=1}^n \left| \int_\Omega f_i(\omega') \frac{k(\omega, \omega')}{(\int_\Omega |k(\omega, \nu)|^{p'} d\mu(\nu))^{1/p'}} d\mu(\omega') \right|^p d\mu(\omega) \\
&\le \|k\|_{p,p'}^p \sup_\omega \sum_{i=1}^n \left| \int_\Omega f_i(\omega') \, \frac{k(\omega, \omega')}{(\int_\Omega |k(\omega, \nu)|^{p'} \, d\mu(\nu))^{1/p'}} \, d\mu(\omega') \right|^p . \qquad (3.1)
\end{aligned}$$

Since $g_\omega(\omega') = \frac{k(\omega, \omega')}{(\int_\Omega |k(\omega, \nu)|^{p'} d\mu(\nu))^{1/p'}} \in L^{p'}(\mu)$, and since $\|g_\omega\|_{L^{p'}} = 1$ for each ω, from (3.1) we obtain:

$$\begin{aligned}
\sum_{i=1}^n \|T_k f_i\|_{L^p}^p &\le \|k\|_{p,p'}^p \sup_\omega \sum_{i=1}^n \left| \int_\Omega f_i(\omega') \, g_\omega(\omega') \, d\mu(\omega') \right|^p \qquad (3.2) \\
&\le \|k\|_{p,p'}^p \sup_{g \in B(L^{p'})} \sum_{i=1}^n \left| \int_\Omega f_i(\omega') \, g(\omega') \, d\mu(\omega') \right|^p . \quad \square
\end{aligned}$$

3.2 Proof of Theorem 3.1.

First, we briefly sketch the idea of the proof. Using Theorem 2.4, we show that the Weyl transform L_σ acting on $M^p(\mathbf{R}^n)$ induces an integral kernel operator T_k acting on a closed subspace of $L^p(\mathbf{R}^{2n})$. Under the assumptions of Theorem 3.1, T_k will be a p-summing Hille–Tamarkin operator. Using the properties of modulation spaces, we then show that L_σ is also a p-summing operator.

We start with the following technical lemma.

LEMMA 3.5. *Let $\sigma \in M^{p,q}(\mathbf{R}^{2n})$, with $1 \le p,q \le \infty$. For $\xi = (\xi_1, \xi_2) \in \mathbf{R}^{2n}$, define $\sigma_\xi = \rho(\frac{\xi_2}{2}, -\frac{\xi_1}{2}, -\xi_1, -\xi_2)\sigma$. Then the integral $s_\xi = \iint_{\mathbf{R}^{2n}} \hat{\sigma}_\xi(x,y)\, \overline{\rho(-x,-y)\phi(t)}\, dx\, dy$ converges weakly, and*

$$S_\Phi\sigma(N(\xi,\eta)) \;=\; e^{-\pi i[\xi,\eta]}\, e^{\pi i \eta_1\eta_2}\, S_\phi s_\xi(\eta_1, -\eta_2).$$

Proof. Using Proposition 2.1, Parseval's identity, and Proposition 2.2, we obtain:

$$\begin{aligned} S_\Phi\sigma(N(\xi,\eta)) \;&=\; e^{-\pi i(\frac{\xi_2+\eta_2}{2}, -\frac{\xi_1+\eta_1}{2})(\xi_1-\eta_1, \xi_2-\eta_2)} \langle \sigma, \rho(-\tfrac{\xi_2+\eta_2}{2}, \tfrac{\xi_1+\eta_1}{2}, \xi_1-\eta_1, \xi_2-\eta_2)\Phi\rangle \\ &=\; e^{-\pi i[\xi,\eta]}\, \langle \sigma_\xi, \rho(-\tfrac{\eta_2}{2}, \tfrac{\eta_1}{2}, -\eta_1, -\eta_2)\Phi\rangle \\ &=\; e^{-\pi i[\xi,\eta]}\, \langle \hat{\sigma}_\xi, \rho(-\eta_1, -\eta_2, \tfrac{\eta_2}{2}, -\tfrac{\eta_1}{2}) A(\phi)\rangle \\ &=\; e^{-\pi i[\xi,\eta]} \int_{\mathbf{R}^n} \left(\iint_{\mathbf{R}^{2n}} \hat{\sigma}_\xi(x,y) \overline{\rho(-x,-y)\phi(t)}\, dx\, dy \right) \rho(-\eta_1,-\eta_2)\phi(t)\, dt. \end{aligned} \tag{3.3}$$

Since $\phi \in \mathcal{S}(\mathbf{R}^n)$ and $\sigma \in \mathcal{S}'(\mathbf{R}^{2n})$, then $s_\xi = \int_{\mathbf{R}^{2n}} \hat{\sigma}_\xi(x,y)\, \overline{\rho(-x,-y)\phi(t)}\, dx\, dy$ is in $\mathcal{S}'(\mathbf{R}^{2n})$. Hence, from equation (3.3) we obtain:

$$S_\Phi\sigma(N(\xi,\eta)) \;=\; e^{-\pi i[\xi,\eta]}\, \langle s_\xi, \rho(-\eta_1,-\eta_2)\phi\rangle \;=\; e^{-\pi i[\xi,\eta]}\, e^{\pi i\eta_1\eta_2}\, S_\phi s_\xi(\eta_1,-\eta_2). \quad \square$$

Now, we can prove the following result.

THEOREM 3.6. *Let $1 \le p < \infty$, and assume $\mathcal{S}_\Phi\sigma \circ \tilde{N} \in L^{p',p}(\mathbf{R}^{4n})$. Then L_σ is a p–summing operator from $M^p(\mathbf{R}^n)$ into itself.*

Proof. By Theorem 2.4, the Weyl operator L_σ can be expressed as:

$$L_\sigma f \;=\; \iint_{\mathbf{R}^{4n}} S_\Phi\sigma(N(\xi,\eta))\, e^{\pi i[\xi,\eta]}\, \langle f, \rho(\eta)\phi)\rangle\, \rho(\xi)\phi\, d\xi\, d\eta.$$

Let $f \in M^p(\mathbf{R}^n)$ and introduce the following definitions:

$$\begin{aligned} F(\eta) \;&=\; \langle f, \rho(\eta)\phi)\rangle \;=\; e^{-\pi i\eta_1\eta_2}\, S_\phi f(-\eta_1, \eta_2), \\ k_\sigma(\xi,\eta) \;&=\; e^{\pi i[\xi,\eta]}\, S_\Phi\sigma(N(\xi,\eta)), \\ T_\sigma F(\xi) \;&=\; \int_{\mathbf{R}^{2n}} k_\sigma(\xi,\eta)\, F(\eta)\, d\eta. \end{aligned}$$

In this notation we have:

$$L_\sigma f \;=\; \int_{\mathbf{R}^{2n}} T_\sigma F(\xi)\,\rho(\xi)\phi\,d\xi. \tag{3.4}$$

Since $F \in L^p(\mathbf{R}^{2n})$ and since

$$\|k_\sigma\|_{p,p'} = \left(\int_{\mathbf{R}^{2n}} \left(\int_{\mathbf{R}^{2n}} |S_\Phi\sigma(N(\xi,\eta))|^{p'}\,d\eta\right)^{p/p'} d\xi\right)^{1/p} = \|S_\Phi\sigma \circ \tilde{N}\|_{L^{p',p}} < \infty,$$

Proposition 3.3 implies that T_σ is a Hille–Tamarkin operator on $L^p(\mathbf{R}^{2n})$, and that it is p-summing with $\pi_p(T_\sigma) = \|k_\sigma\|_{p,p'} = \|S_\Phi\sigma \circ \tilde{N}\|_{L^{p',p}}$. More precisely, the inequality (3.2) from the proof of Proposition 3.4 implies that

$$\sum_{i=1}^{n} \|T_\sigma F_i\|_{L^p}^p \;\le\; \|k_\sigma\|_{p,p'}^p \,\sup_\xi \sum_{i=1}^{n} \left|\int_{\mathbf{R}^{2n}} F_i(\eta) G_\xi(\eta)\,d\eta\right|^p, \tag{3.5}$$

where $F_i(\eta) = \langle f_i, \rho(\eta)\phi\rangle$ and $G_\xi(\eta) = \frac{k_\sigma(\xi,\eta)}{(\int |k_\sigma(\xi,\nu)|^{p'}\,d\nu)^{1/p'}}$. Note that $G_\xi(\eta) \in B(L^{p'})$ for each ξ.

Now, fix $\xi \in \mathbf{R}^{2n}$ and define $c_\xi = (\int_{\mathbf{R}^{2n}} |k_\sigma(\xi,\eta)|^{p'}\,d\eta)^{-1/p'}$, so:

$$G_\xi(\eta) \;=\; c_\xi\, e^{\pi i[\xi,\eta]}\, S_\Phi\sigma(N(\xi,\eta)).$$

By Lemma 3.5, if we set $g_\xi = \iint_{\mathbf{R}^{2n}} \hat{\sigma}_\xi(x,y)\,\rho(-x,-y)\phi(t)\,dx\,dy$, then

$$G_\xi(\eta) \;=\; c_\xi\, e^{\pi i \eta_1\eta_2}\, S_\phi g_\xi(\eta_1,-\eta_2) \;=\; c_\xi\,\langle g_\xi, \rho(\eta)\phi\rangle. \tag{3.6}$$

In particular,

$$\|c_\xi\, g_\xi\|_{M^{p',p'}} \;=\; \|c_\xi\, S_\phi g_\xi\|_{L^{p',p'}} \;=\; \|G_\xi\|_{L^{p'}} = 1. \tag{3.7}$$

We are now ready to relate the properties of T_σ to the properties of L_σ. By [2, Corollary 4.5], applied to (3.4), there is a constant C independent of f such that:

$$\|L_\sigma f\|_{M^{p,p}}^p \;\le\; C\,\|T_\sigma F\|_{L^p}^p. \tag{3.8}$$

Therefore, using equations (3.5)–(3.8):

$$\begin{aligned}
\sum_{i=1}^{n} \|L_\sigma f_i\|_{M^{p,p}}^p &\le C \sum_{i=1}^{n} \|T_\sigma F_i\|_{L^p}^p \\
&\le C\,\|k_\sigma\|_{p,p'}^p \sup_{\xi} \sum_{i=1}^{n} \left| \int_{\mathbf{R}^{2n}} F_i(\eta)\, G_\xi(\eta)\, d\eta \right|^p \\
&= C\,\|k_\sigma\|_{p,p'}^p \sup_{\xi} \sum_{i=1}^{n} \left| \int_{\mathbf{R}^{2n}} \langle f_i, \rho(\eta)\phi\rangle \, \langle c_\xi g_\xi, \rho(\eta)\phi\rangle \, d\eta \right|^p \\
&\le C\,\|k_\sigma\|_{p,p'}^p \sup_{g \in B(M^{p',p'})} \sum_{i=1}^{n} \left| \int_{\mathbf{R}^{2n}} \langle f_i, \rho(\eta)\phi\rangle \, \langle g, \rho(\eta)\phi\rangle \, d\eta \right|^p \\
&= C\,\|k_\sigma\|_{p,p'}^p \sup_{g \in B(M^{p',p'})} \sum_{i=1}^{n} \left| \int_{\mathbf{R}^{2n}} f_i(t)\, g(t)\, dt \right|^p \qquad \text{by (2.3).} \quad \square
\end{aligned}$$

Using the properties of p–summing operators, from Theorem 3.6 we obtain the following result.

PROPOSITION 3.7. *Let* $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, *and assume* $S_\Phi \sigma \circ \tilde{N} \in L^{p',p}(\mathbf{R}^{4n})$. *Then:*

(a) *If* $p = 1$, *then* L_σ *is a weakly compact and completely continuous operator mapping* $M^{1,1}(\mathbf{R}^n)$ *into itself, and the eigenvalues of* L_σ *are 2-summable with*

$$\left(\sum |\lambda_n|^2\right)^{1/2} \le C\,\|S_\Phi \sigma \circ \tilde{N}\|_{L^{\infty,1}},$$

where $C > 0$ *is independent of* σ.

(b) *If* $1 < p < \infty$, *then* L_σ *is a compact operator mapping* $M^{p,p}(\mathbf{R}^n)$ *into itself, and the eigenvalues of* L_σ *are* r*-summable, where* $r = \max\{2, p\}$, *with*

$$\left(\sum |\lambda_n|^r\right)^{1/r} \le C\,\|S_\Phi \sigma \circ \tilde{N}\|_{L^{p',p}},$$

where $C > 0$ *is independent of* σ.

Proof. By Theorem 3.6, L_σ is a p–summing operator mapping $M^p(\mathbf{R}^n)$ into itself. Therefore, by Proposition 3.3, L_σ is weakly compact, completely continuous, and the eigenvalues are r-summable, with $r = \max\{2, p\}$. Furthermore, if $1 < p < \infty$, then $M^p(\mathbf{R}^n)$ is reflexive (cf. Section 2.2), and so, by Proposition 3.3, L_σ is also compact. $\square$

Finally, the proof of Theorem 3.1 follows from Proposition 3.7.

Proof of Theorem 3.1. By [8], $\|S_\Phi\sigma\circ\tilde N\|_{L^{p,q}} \approx \|\big((S_\Phi\sigma\circ\tilde N)(k,m)\big)_{k,m}\|_{\ell^{p,q}}$ and, therefore, since $\ell^{p1,q1}\subset\ell^{p2,q2}$ whenever $p_1\le p_2$, $q_1\le q_2$ (cf. [10, Sec. 12.2]), this implies that $\|S_\Phi\sigma\circ\tilde N\|_{L^{p_2,q_2}}\le\|S_\Phi\sigma\circ\tilde N\|_{L^{p_1,q_1}}$ whenever $p_1\le p_2$, $q_1\le q_2$,. In particular, if $1\le p\le 2$, since $\tilde N$ is a linear transformation, we obtain:

$$\|S_\Phi\sigma\circ\tilde N\|_{L^{p',p}} \;\le\; \|S_\Phi\sigma\circ\tilde N\|_{L^{p,p}} \;\le\; c\,\|\sigma\|_{M^p}, \tag{3.9}$$

and, if $2\le p<\infty$, then we have:

$$\|S_\Phi\sigma\circ\tilde N\|_{L^{p',p}} \;\le\; \|S_\Phi\sigma\circ\tilde N\|_{L^{p',p'}} \;\le\; c'\,\|\sigma\|_{M^{p'}}, \tag{3.10}$$

where c, c' are constants independent of σ.

Part (a) now follows directly from equation (3.9) with $p=1$ and Proposition 3.7(a).

Part (b) follows from equation (3.9) with $1<p\le 2$ and Proposition 3.7(b).

Finally, part (c) follows from equation (3.10) and Proposition 3.7(b). □

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