Math 4377/6308 Advanced Linear Algebra
1.2 Vector Spaces
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Many concepts concerning vectors in \( \mathbb{R}^n \) can be extended to other mathematical systems.

- Parallelogram law for vector addition.
- Reading: §1.1.
We can think of a vector space in general, as a collection of objects that behave as vectors do in $\mathbb{R}^n$. The objects of such a set are called vectors.

Field

Let $F$ be a field, whose elements are referred to as scalars.

- $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (rational numbers), etc.
- Reading: Appendix C.
A vector space over $F$ is a nonempty set $V$, whose elements are referred to as vectors, together with two operations.

- The first operation, called addition and denoted by $+$, assigns to each pair $(u, v)$ of vectors in $V$ a vector $u + v$ in $V$ (Axiom 1).
- The second operation, called scalar multiplication and denoted by juxtaposition, assigns to each pair $(a, u) \in F \times V$ a vector $au$ in $V$ (Axiom 6).

Furthermore, the following properties must be satisfied:

(VS 1) (Commutativity of addition) (Axiom 2) For all vectors $u, v \in V$,

$$u + v = v + u.$$
(VS 2) \textbf{(Associativity of addition)} (Axiom 3) For all vectors \( u, v, w \in V \),
\[
(u + v) + w = u + (v + w)
\]

(VS 3) \textbf{(Existence of a zero)} (Axiom 4) There is a vector (called the zero vector) \( 0 \) in \( V \) such that
\[
u + 0 = u.
\]
for all vectors \( u \in V \).

(VS 4) \textbf{(Existence of additive inverses)} (Axiom 5) For each vector \( u \) in \( V \), there is a vector in \( V \) (called the additive inverse of \( u \)), denoted by \(-u\), satisfying
\[
u + (-u) = 0.
\]
Vector Space (cont.)

(VS 5-8) **(Properties of scalar multiplication)** (Axioms 7-10) For all scalars $a, b \in F$ and for all vectors $u, v \in V$,

1. $1u = u$.
2. $(ab)u = a(bu)$.
3. $a(u + v) = au + av$.
4. $(a + b)u = au + bu$.

A vector space over a field $F$ is sometimes called an $F$-space. A vector space over the real field is called a **real vector space** and a vector space over the complex field is called a **complex vector space**.
Example

The set $F^n$ of all ordered $n$-tuples whose components lie in a field $F$, is a vector space over $F$, with addition and scalar multiplication defined componentwise:

$$(a_1, \cdots, a_n) + (b_1, \cdots, b_n) = (a_1 + b_1, \cdots, a_n + b_n)$$

and

$$c(a_1, \cdots, a_n) = (ca_1, \cdots, ca_n)$$

When convenient, we will also write the elements of $F^n$ in column form

$$
\begin{pmatrix}
a_1 \\
\vdots \\
a_n
\end{pmatrix}
$$
Example

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are real} \right\}$

In this context, note that the 0 vector is $\begin{bmatrix} \end{bmatrix}$. 
Vector Spaces: \( m \times n \) Matrices

Example

The set \( \mathcal{M}_{m,n}(F) \) of all \( m \times n \) matrices with entries in a field \( F \) of the form:

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]

with \( a_{ij} \in F \) for \( 1 \leq i \leq m, \ 1 \leq j \leq n \), is a vector space over \( F \), under the operations of matrix addition and scalar multiplication:

\[
(A + B)_{ij} = A_{ij} + B_{ij},
\]

\[
(cA)_{ij} = cA_{ij},
\]

for \( 1 \leq i \leq m, \ 1 \leq j \leq n \).
Vector Spaces: Sequences

Example

Many sequence spaces are vector spaces. The set \( \text{Seq}(F) \) of all infinite sequences with members from a field \( F \) is a vector space under the componentwise operations

\[
\{s_n\} + \{t_n\} = \{s_n + t_n\}
\]

and

\[
a\{s_n\} = \{as_n\}
\]

Example (\( c_0 \))

In a similar way, the set \( c_0 \) of all sequences of complex numbers that converge to 0 is a vector space.

Example (\( l^\infty \))

The set \( l^\infty \) of all bounded complex sequences is a vector space.
Example ($l^p$)

If $1 \leq p < \infty$, then the set $l^p$ of all complex sequences \( \{s_n\} \) for which

\[
\sum_{n=1}^{\infty} |s_n|^p < \infty
\]

is a vector space under componentwise operations. To see that addition is a binary operation on $l^p$, one verifies Minkowski’s inequality

\[
\left( \sum_{n=1}^{\infty} |s_n + t_n|^p \right)^{1/p} \leq \left( \sum_{n=1}^{\infty} |s_n|^p \right)^{1/p} + \left( \sum_{n=1}^{\infty} |t_n|^p \right)^{1/p}
\]

which we will not do here.
Example

Let $\mathcal{F}(S, F)$ denote the set of all functions from a nonempty set $S$ to a field $F$. This is a vector space over $F$, under the operations of ordinary addition and scalar multiplication of functions:

$$(f + g)(s) = f(s) + g(s),$$

and

$$(af)(s) = a[f(s)],$$

for each $s \in S$. 
Vector Spaces: Polynomials

Example

Let \( n \geq 0 \) be an integer and let \( P_n = \) the set of all polynomials of degree at most \( n \geq 0 \).

Members of \( P_n \) have the form

\[
p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n
\]

where \( a_0, a_1, \ldots, a_n \) are real numbers and \( t \) is a real variable. The set \( P_n \) is a vector space.

We will just verify 3 out of the 10 axioms here.

Let \( p(t) = a_0 + a_1 t + \cdots + a_n t^n \) and \( q(t) = b_0 + b_1 t + \cdots + b_n t^n \) (set higher coefficients to zero if different degrees). Let \( c \) be a scalar.
Axiom 1:

The polynomial $p + q$ is defined as follows:

$$(p + q)(t) = p(t) + q(t).$$

Therefore,

$$p + q$$

is also a polynomial of degree at most $n$. So

$p + q$ is in $P_n$. 
Vector Spaces: Polynomials (cont.)

**Axiom 4:**

\[
0 = 0 + 0t + \cdots + 0t^n
\]

(zero vector in \( P_n \))

\[
(p + 0)(t) = p(t) + 0 = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n
\]

\[
= a_0 + a_1 t + \cdots + a_n t^n = p(t)
\]

and so \( p + 0 = p \)
Axiom 6:

\[(cp)(t) = cp(t) = (\underline{\phantom{0}}) + (\underline{\phantom{0}})t + \cdots + (\underline{\phantom{0}})t^n\]

which is in \(P_n\).

The other 7 axioms also hold, so \(P_n\) is a vector space.
Vector Spaces: True or False

1. Every vector space contains a zero vector.
2. A vector space may have more than one zero vector.
3. In any vector space, $ax = bx$ implies that $a = b$.
4. In any vector space, $ax = ay$ implies that $x = y$.
5. A vector in $F^n$ may be regarded as a matrix in $M_{n\times 1}(F)$.
6. An $m \times n$ matrix has $m$ columns and $n$ rows.
7. In $P(F)$, only polynomials of the same degree may be added.
8. In $f$ and $g$ are polynomials of degree $n$, then $f + g$ is a polynomial of degree $n$.
9. If $f$ is a polynomial of degree $n$ and $c$ is nonzero scalar, then $cf$ is a polynomial of degree $n$.
10. A nonzero scalar of $F$ may be considered to be a polynomial in $P(F)$ having degree zero.
11. Two functions in $F(S, F)$ are equal if and only if they have the same value at each element of $S$. 
Theorem (1.1 Cancellation Law for Vector Addition)

If \( x, y, z \) are vectors in a vector space \( V \) such that \( x + z = y + z \), then \( x = y \).
Corollary 1 (Uniqueness of the Zero Vector)

The vector $0$ described in (VS 3) is unique (the zero vector).
Corollary 2 (Uniqueness of the Additive Inverse)

The vector \(-u\) described in (VS 4) is unique (the additive inverse).
Theorem (1.2)

In any vector space \( V \), the following statements are true:

(a) \( 0x = 0 \) for each \( x \in V \).

(b) \( (-a)x = -(ax) = a(-x) \) for each \( a \in F \) and \( x \in V \)

(c) \( a0 = 0 \) for each \( a \in F \)