Wavelets Associated with Composite Dilations

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1 Basic definitions

In this paper, we present a description of a collaboration with K. Guo, W. Lim, A. Savov and E. Wilson. We make use of the Fourier transform $f \mapsto \hat{f}$ that, for $f \in L^1(\mathbb{R}^n)$, is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi x}\, dx.$$

As is well-known, this operator has a unique extension to $L^2(\mathbb{R}^n)$ that is a unitary operator. In the following, we refer to the domain of $\hat{f}$ as the "frequency" domain and denote it by $\widehat{\mathbb{R}}^n$. The elements $\xi \in \widehat{\mathbb{R}}^n$ will be denoted by Greek letters and considered to be row vectors $\xi = (\xi_1, \dots, \xi_n)$, while the points $x \in \mathbb{R}^n$ are column vectors, i.e., $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. If $a = \begin{pmatrix} \alpha_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ is a real $n \times n$ matrix, then $\xi a x$ is defined by the usual matrix multiplication. The Fourier transform is a mapping from $L^2(\mathbb{R}^n)$ to $L^2(\widehat{\mathbb{R}}^n)$, while the inverse Fourier transform, that is defined (at least for a dense subset of $L^2(\widehat{\mathbb{R}}^n)$) by

$$\check{f}(x) = \int_{\widehat{\mathbb{R}}^n} f(\xi)\, e^{2\pi i \xi x}\, d\xi,$$

is a mapping from $L^2(\widehat{\mathbb{R}}^n)$ to $L^2(\mathbb{R}^n)$.

The Fourier transform interacts in particularly simple ways with three important operators that are used for the formation of wavelets: the **dilations**, the **translations** and the **modulations**. It will be convenient to define these operators separately when their action is either on $\mathbb{R}^n$ or on $\widehat{\mathbb{R}}^n$.

Definition 1.1. *On $\mathbb{R}^n$ we have:*

(i) the **dilation** *by $a \in GL_n(\mathbb{R})$ is the operator D_a mapping the function $f(x)$, $x \in \mathbb{R}^n$, into the function $(D_a f)(x) = |\det a|^{-1/2} f(a^{-1}x)$;*

(ii) the **translation** *by $k \in \mathbb{R}^n$ is the operator T_k defined by $(T_k f)(x) = f(x-k)$;*

(iii) the **modulation** *by $k \in \widehat{\mathbb{R}}^n$ is the operator M_k defined by $(M_k f)(x) = e^{2\pi i k x} f(x)$.*

On $\widehat{\mathbb{R}}^n$ we have:

(i') For $a \in GL_n(\mathbb{R})$, the **dilation** *$\widehat{D}_a$ is the operator mapping the function $f(\xi)$, $\xi \in \widehat{\mathbb{R}}^n$, into $(\widehat{D}_a\, f)(\xi) = |\det a|^{-1/2}\, f(\xi a^{-1})$;*

(ii') the **translation** *by $k \in \widehat{\mathbb{R}}^n$ is the operator $\widehat{T}_k$ defined by $(\widehat{T}_k f)(\xi) = f(\xi - k)$;*

(iii') the **modulation** *by $k \in \mathbb{R}^n$is the operator $\widehat{M}_k$ defined by $(\widehat{M}_k f)(\xi) = e^{2\pi i \xi k}\, f(\xi)$.*

The reader can easily verify that $(D_a\, f)^\wedge(\xi) = (\widehat{D}_{a^{-1}}\, \hat{f})(\xi)$, $(M_k\, f)^\wedge(\xi) = (\widehat{T}_k\, \hat{f})(\xi)$, and $(T_k\, f)^\wedge(\xi) = (\widehat{M}_{-k}\, \hat{f})(\xi)$.

The "original" wavelets were defined to be functions $\psi \in L^2(\mathbb{R})$ such that the set $\{\psi_{j,k} = D_2^j\, T_k\, \psi \,:\, j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ [6]. The existence of such ψ, led to the study of general **affine systems** associated with a countable collection $C \subset GL_n(\mathbb{R})$. These are systems of functions of the form

$$\mathcal{A}_C(\Psi) = \{D_c\, T_k\, \Psi : \; k \in \mathbb{Z}^n, c \in C\}, \tag{1.1}$$

where $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$. The object of this study is to establish the existence and to construct sets Ψ such that $\mathcal{A}_C(\Psi)$ is an orthonormal basis or, more generally, a Parseval frame for $L^2(\mathbb{R}^n)$. Recall that $\mathcal{A}_C(\Psi)$ is a **Parseval frame** if

$$\|f\|^2 = \sum_{c \in C} \sum_{k \in \mathbb{Z}^n} \sum_{\ell=1}^{L} |\langle f, D_c\, T_k\, \psi^\ell \rangle|^2$$

for all $f \in L^2(\mathbb{R}^n)$, or, equivalently,

$$f = \sum_{c \in C} \sum_{k \in \mathbb{Z}^n} \sum_{\ell=1}^{L} \langle f, D_c\, T_k\, \psi^\ell \rangle\, D_c\, T_k\, \psi^\ell,$$

for all $f \in L^2(\mathbb{R}^n)$, with convergence in $L^2(\mathbb{R}^n)$.

In this work, we shall focus on a particular class of affine systems which we call **affine systems with composite dilations**. These are affine systems for which $C = AB = \{ab : a \in A, b \in B\}$, where $A \subset GL_n(\mathbb{R})$ consists of elements having some "expanding properties", while $B \subset GL_n(\mathbb{R})$ consists of elements having determinant of absolute value one. In order to gain some understanding and familiarity with such systems, let us look at an example.

2 Example

Let $A = \{a^i : i \in \mathbb{Z}\}$, $B = \{b^j : j \in \mathbb{Z}\}$, where $a = \begin{pmatrix} 2 & 0 \\ 0 & \epsilon \end{pmatrix}$, $\epsilon \neq 0$, and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We shall construct an affine system

$$\mathcal{A}_{AB}(\Psi) = \{\psi^\ell_{i,j,k} = D_a^i\, D_b^j\, T_k\, \psi^\ell : i,j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1,2,3\}, \tag{2.2}$$

that is an orthonormal basis for $L^2(\mathbb{R}^2)$. It will be convenient to work in the frequency domain $\widehat{\mathbb{R}}^2$. Let $S_0 = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : 1 \leq \xi_1 < 1\}$. This is the vertical strip of width 2 bounded by the lines ± 1. Then $S_i = S_0\, a^i$, $i \in \mathbb{Z}$, is the vertical strip $\{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : -2^i \leq \xi_1 < 2^i\}$. Clearly, we have that

(i) $S_i \subset S_{i+1}$,

(ii) $\bigcup_{i\in\mathbb{Z}} S_i = \widehat{\mathbb{R}}^2$,

(iii) $\bigcap_{i\in\mathbb{Z}} S_i = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \xi_1 = 0\}$.

For $S \subset \widehat{\mathbb{R}}^2$, we use the notation $L^2(S) = \{f \in L^2(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset S\}$. ¿From the observations that we made about the sets S_i, it follows that

(i) $L^2(S_i) \subset L^2(S_{i+1})$,

(ii) $\overline{\bigcup_{i\in\mathbb{Z}} L^2(S_i)} = L^2(\widehat{\mathbb{R}}^2)$,

(iii) $\bigcap_{i\in\mathbb{Z}} L^2(S_i) = \{0\}$.

Next let $R_0 = S_1 \backslash S_0$, and $R_i = R_0\, a^i$, $i \in \mathbb{Z}$. Then we can write $L^2(\widehat{\mathbb{R}}^2)$ as the orthogonal direct sum $L^2(\widehat{\mathbb{R}}^2) = \bigoplus_{i\in\mathbb{Z}} L^2(R_i)$.

We shall construct an orthonormal basis for $L^2(R_0)$ of the form $\{\widehat{D}_b^j\, \widehat{M}_k\, \hat{\psi}^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, 2, 3\}$, for an appropriate choice of functions $\hat{\psi}^1$, $\hat{\psi}^2$, $\hat{\psi}^3 \in L^2(R_0)$. Once this is done, our mission is easily accomplished. Indeed, since the operators $\widehat{D}_a^i$, $i \in \mathbb{Z}$, are unitary, it follows that

$$\{\widehat{D}_a^i\, \widehat{D}_b^j\, \widehat{M}_k\, \hat{\psi}^\ell : i, j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, 2, 3\}, \tag{2.3}$$

is an orthonormal basis for $L^2(\widehat{\mathbb{R}}^2)$. The desired orthonormal basis for $L^2(\mathbb{R}^2)$ is then, simply, the inverse Fourier transform of the collection (2.3), which has the form (2.2).

Thus, we begin with the construction of the subsystem $\{\widehat{D}_b^j\, \widehat{M}_k\, \hat{\psi}^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, 2, 3\}$. It will be helpful to look at Figure 1. Consider the three sets $I_\ell = I_\ell^+ \cup I_\ell^-$, $\ell = 1, 2, 3$, contained in R_0, where: I_1^+ is the rectangle with vertices $(1, 0)$, $(1, 1/2)$, $(2, 0)$, $(2, 1/2)$; I_2^+ is the rectangle with vertices $(1, 1/2)$, $(1, 1)$, $(2, 1/2)$, $(2, 1)$; I_3^+ is the triangle with vertices $(1, 1)$, $(2, 1)$, $(2, 2)$; and $I_\ell^- = -I_\ell^+$, $\ell = 1, 2, 3$. Each set I_ℓ, $\ell = 1, 2, 3$, is **fundamental**; that is, the collection $\{e^{2\pi i \xi k} \chi_{I_\ell} : k \in \mathbb{Z}^2\}$ is an orthonormal basis for $L^2(I_\ell)$. The sets $I_\ell\, b^j$, for $j \in \mathbb{Z}$, form a disjoint covering of R_0 (they intersect, at most, at the boundaries). Furthermore, each of these sets is also fundamental, since, for any $j \in \mathbb{Z}$, b^j maps $\mathbb{Z}^2$ into itself, and the collection $\{e^{2\pi i \xi b^j k} : k \in \mathbb{Z}^2\}$ is equal to the collection $\{e^{2\pi i \xi k} : k \in \mathbb{Z}^2\}$. Figure 1 illustrates how this covering takes place. The same figure also describes the action of the matrix a on the sets I_ℓ, $\ell = 1, 2, 3$. Thus, the collection $\{I_\ell\, b^j\, a^i : i, j \in \mathbb{Z}, \ell = 1, 2, 3\}$ covers $\widehat{\mathbb{R}}^2$, up to sets of measure zero. If we let ψ^ℓ be defined by $\hat{\psi}^\ell = \chi_{I_\ell}$, $\ell = 1, 2, 3$, we see that the system (2.2) is an orthonormal basis for $L^2(\mathbb{R}^2)$.

Before we introduce other examples of wavelets with composite dilations, we refer the reader to the papers [3, 4, 5]. In these papers, we go more deeply into this subject and discuss some more technical matters. On the other hand, in this presentation we will concentrate on results that do not require much knowledge of the theory of wavelets. The construction we have just described, for example, requires only an elementary knowledge of the Fourier transform. Nevertheless, some of the important features of the affine systems with composite dilations are exhibited by the system (2.2). In particular, we observe that the elements of the dilation set A are matrices expanding or contracting only in one direction, while the elements of B act by volume–preserving maps in a "transverse" direction. It follows that the wavelet functions that we obtain have exactly those geometric properties (e.g., directionality, elongated shapes, scales, oscillations) recently advocated by many authors for multidimensional signal and image processing applications (cf., for example, [2, 1]).

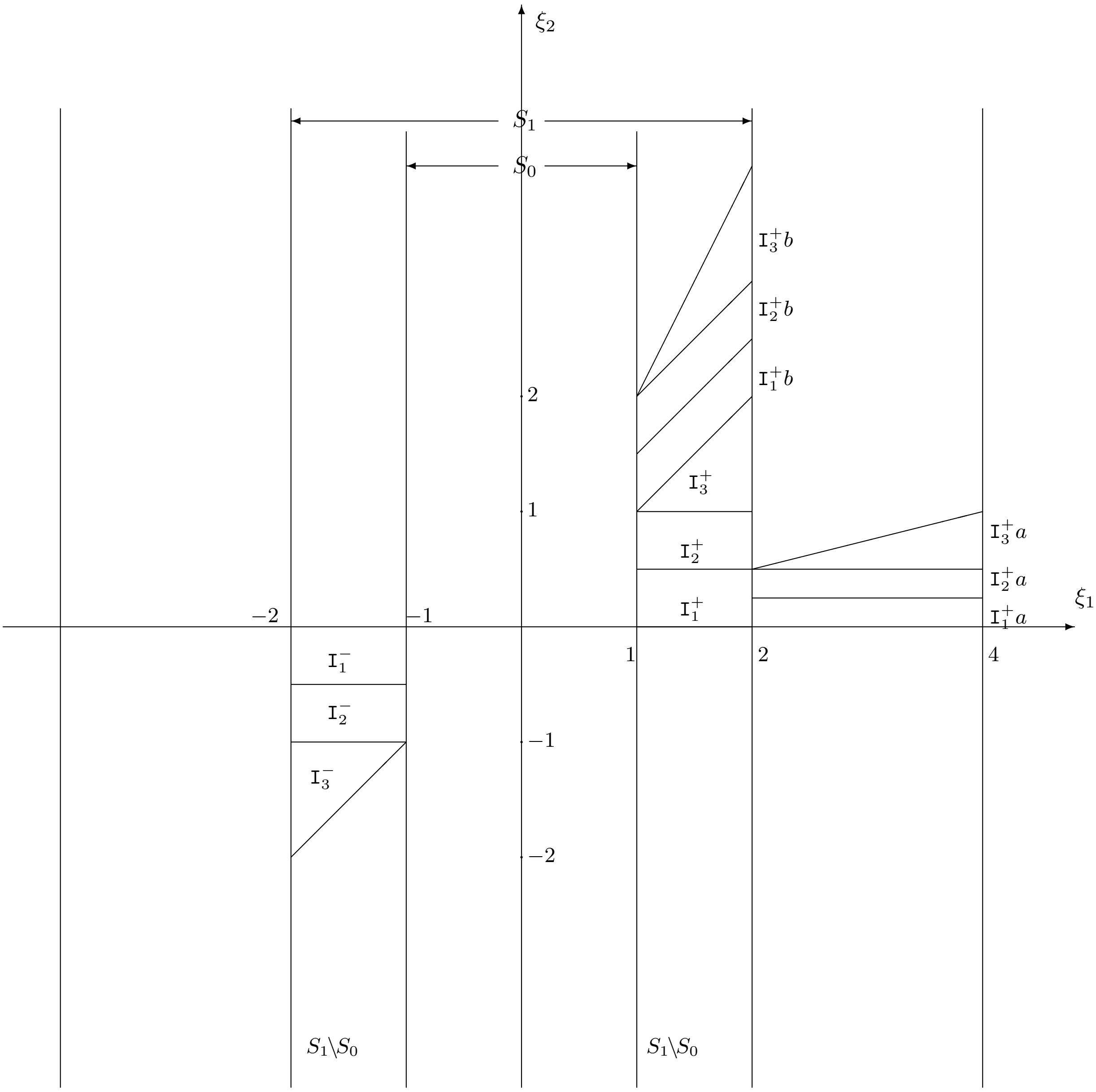

Figure 1: Example of orthonormal AB wavelet.

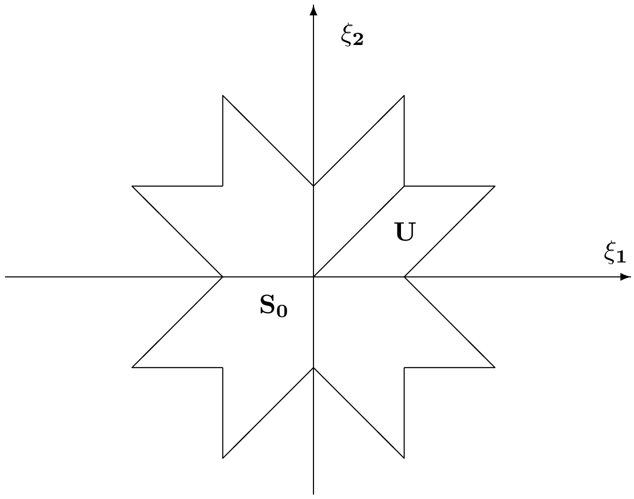

Figure 2: The set S_0.

3 Other examples

In higher dimensions there are natural extensions of the systems we have constructed in the previous section. Observe that the matrix b we introduced before formula (2.2) satisfies $(b - I_2)^2 = 0$, where I_2 is the 2×2 identity matrix. We say that a matrix $b \in \mathbb{R}^{n \times n}$ is a **shear matrix** if

$$(b - I_n)^2 = 0.$$

In [5], we study these matrices in detail and, together with appropriate "expanding" collections of matrices A, we obtain a large class of orthonormal bases, and, more generally, Parseval frames for $L^2(\mathbb{R}^n)$ that have the form (2.2).

Let us now consider a different type of affine systems with composite dilations for which the set of dilations B is a finite group. Again we construct such system in dimension 2. Let B be the 8-element group consisting of the isometries of the square $[-1,1]^2$. Namely, $B = \{\pm b_0, \pm b_1, \pm b_2, \pm b_3, \}$ where $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Let U be the parallelogram with vertices $(0,0), (1,0), (2,1)$ and $(1,1)$ and

$$S_0 = \bigcup_{b \in B} U\, b.$$

This region in shown Figure 2. It is easy to verify that S_0 is B–invariant.

Now let $a = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ be the **quincunx** matrix and $S_i = S_0\, a^i$, $i \in \mathbb{Z}$. Observe that a is expanding, and $S_0 \subseteq S_0\, a = S_1$. Let R be the parallelogram with vertices $(1,0)$, $(2,0)$, $(3,1)$, $(2,1)$. Then the region $S_1 \setminus S_0$ is the disjoint union $\bigcup_{b \in B} R\, b$ (see Figure 3). Also observe that R is a fundamental domain. It then follows, by the same reasoning we gave in Section 2, that the set

$$\{D_a^i\, D_b\, T_k\, \psi : \ i \in \mathbb{Z},\ b \in B,\ k \in \mathbb{Z}^2\}, \tag{3.4}$$

where $\hat{\psi} = \chi_R$, is an orthonormal basis for $L^2(\mathbb{R}^2)$.

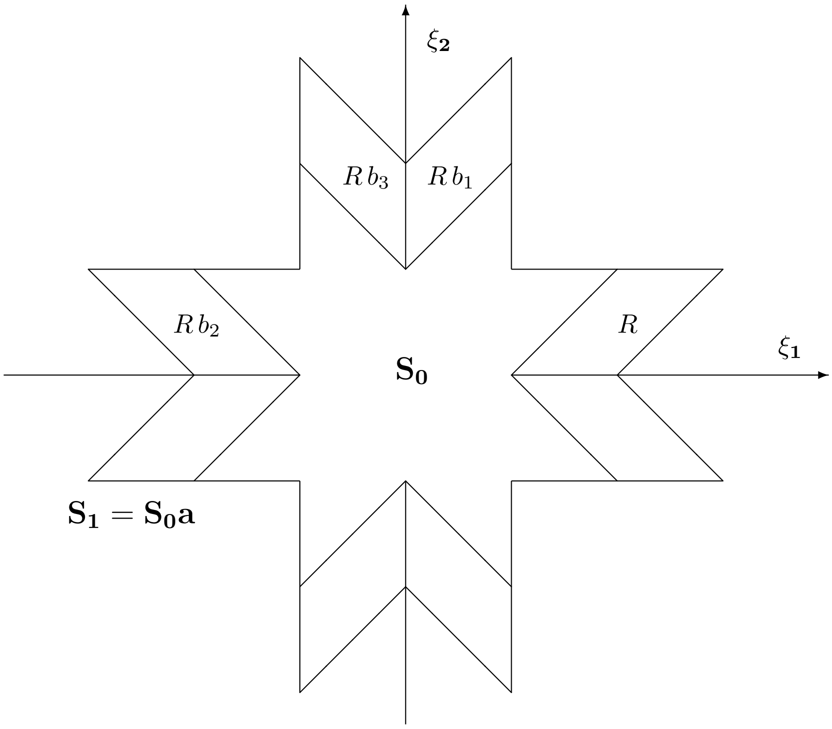

Figure 3: The regions S_0 and $S_1 = S_0\, a$, where a is the quincunx matrix.

One difference between the system from Section 2 and this system is that the first is generated by three functions ψ^1, ψ^2, ψ^3, while the second is generated by a single function.

On the other hand, if we use the dyadic matrix $a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ rather than the quincunx matrix, we obtain an affine system with composite dilations that is an orthonormal basis for $L^2(\mathbb{R}^2)$ and is generated by *three* functions. This is easily understood if we examine Figure 4. Also in this case, a is expanding and $S_1 = S_0 a \supset S_0$. Let R_1, R_2, R_3 be the parallelograms illustrated in Figure 4 (for example, R_1 is the parallelogram with vertices $(1,0)$, $(2,0)$, $(3,1)$, $(2,1)$). A direct computation shows that the region $S_1 \setminus S_0$ is the disjoint union $\bigcup_{b \in B} R\, b$, where $R = R_1 \bigcup R_2 \bigcup R_3$. Observe that each of the regions R_1, R_2, R_3 is a fundamental domain. Again, we can apply the reasoning we used in the other two constructions to show that the system

$$\{D_a^i\, D_b\, T_k\, \psi^\ell \,:\, i \in \mathbb{Z},\, b \in B,\, k \in \mathbb{Z}^2,\, \ell = 1, \ldots, 3\}, \tag{3.5}$$

where $\hat{\psi}^\ell = \chi_{R_\ell}$, $\ell = 1, 2, 3$, is an orthonormal basis for $L^2(\mathbb{R}^2)$.

It is natural to investigate further the question of how many generators $\psi^1, \psi^2, \ldots, \psi^L$ are needed in order to obtain such orthonormal bases. The following simply proved result provides a powerful tool to answer this question.

Theorem 3.1. *Let $\mathcal{H}$ be a separable Hilbert space, G be a countable set and, for each $u \in G$, let $T_u : \mathcal{H} \to \mathcal{H}$ be unitary. Moreover, assume that for each T_u there exists a unique $u^* \in G$ such that $T_{u^*} = T_u^*$ (= the adjoint of T_u). Suppose that $\Phi = \{\phi_1, \phi_2, \ldots\}$ and $\Psi = \{\psi_1, \psi_2, \ldots\}$ are two subsets of $\mathcal{H}$ such that $\{T_u\, \phi_k \,:\, u \in G, \phi_k \in \Phi\}$ and $\{T_u\, \psi_i \,:\, u \in G, \psi \in \Psi\}$ are orthonormal bases for $\mathcal{H}$. Then card Φ = card Ψ.*

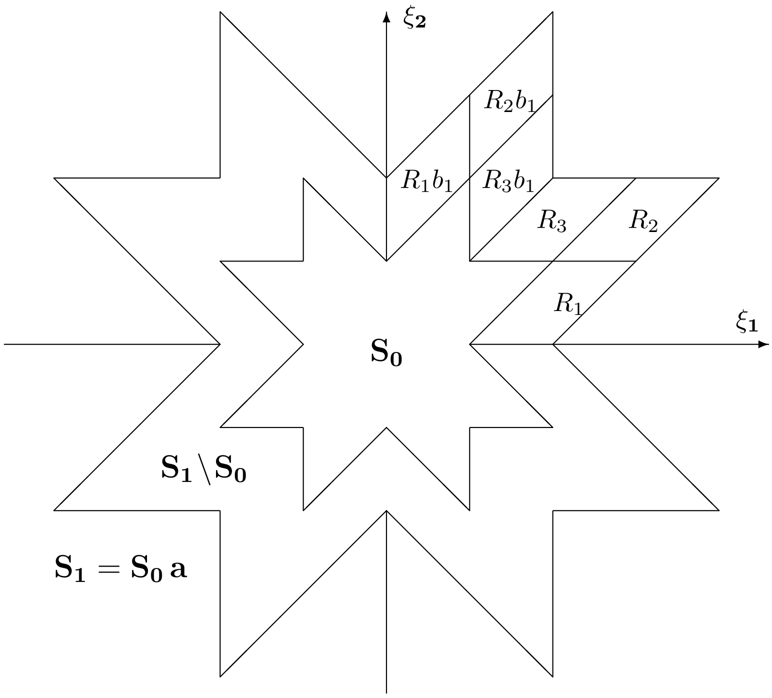

Figure 4: The regions S_0 and $S_1 = S_0\, a$, where $a = 2\, I_2$.

Proof. Let $N = \text{card}\ \Phi$ and $M = \text{card}\ \Psi$ (M and N can be either positive integers or ∞). Then, using the fact that $T_{u^*} = T_u^*$ and to any $u \in G$ we associate a unique $u^* \in G$, we have that

$$\begin{aligned} N = \sum_{k=1}^{N} \|\phi_k\|^2 &= \sum_{k=1}^{N} \sum_{u \in G} \sum_{i=1}^{M} |\langle \phi_k, T_u\, \psi_i \rangle|^2 \\ &= \sum_{i=1}^{M} \sum_{u \in G} \sum_{k=1}^{N} |\langle T_{u^*}\, \phi_k, \psi_i \rangle|^2 \\ &= \sum_{i=1}^{M} \sum_{u \in G} \sum_{k=1}^{N} |\langle T_u\, \phi_k, \psi_i \rangle|^2 \\ &= \sum_{i=1}^{M} \|\psi_i\|^2 = M. \qquad \square \end{aligned}$$

An important special case of this theorem occurs when G is a countable group and the mapping $u \mapsto \pi_u$ is a unitary representation of G acting on $\mathcal{H}$. In the examples of composite dilations wavelets we have presented, $G = \{(b,k) : b \in B, k \in \mathbb{Z}^2\}$ is, indeed, a countable group, where the group operation is $(b,k) \cdot (c,m) = (bc, m + c^{-1}k)$, and the mapping $(b,k) \mapsto \pi_{(b,k)} = D_b\, T_k$ is a unitary representation of G acting on $\mathcal{H} = L^2(\mathbb{R}^2)$.

Let us see what Theorem 3.1 tells us about the number of generators L that we used in each of the affine systems of composite dilations (2.2), (3.4) and (3.5) (in which cases, we had $L = 3$, $L = 1$ and $L = 3$, respectively). In all of these cases the spaces $V_i = L^2(S_i)$, where $S_i = S_0\, a^i$, $i \in \mathbb{Z}$, are closed subspaces of $L^2(\mathbb{R}^2)$ having the following properties:

(i) $V_i \subset V_{i+1}$, $i \in \mathbb{Z}$;

(ii) $f \in V_i$ if and only if $f(a\cdot) \in V_{i+i}$;

(iii) $\overline{\bigcup V_i} = L^2(\mathbb{R}^n)$;

(iv) $\bigcap V_i = \{0\}$.

These are four of the basic properties of what is known as a **multiresolution analysis** (MRA) in $L^2(\mathbb{R}^n)$. There is a fifth important property that asserts that there exists a function $\phi \in V_0$, called a **scaling function**, such that $\{T_k \phi : k \in \mathbb{Z}^n\}$ is an orthonormal basis for V_0. These properties give rise to a very elegant method for constructing an affine system $\{D_{2^i I} T_k \psi : i \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ that is an orthonormal basis for $L^2(\mathbb{R}^n)$. The crucial step of the MRA method is to examine the subspace $W_0 \subset V_1$ that is the orthogonal complement of V_0 (that is, $V_1 = W_0 \bigoplus V_0$), and then construct functions $\psi^1, \ldots, \psi^\ell \in W_0$, called **wavelets**, such that $\{T_k \psi^\ell : k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ is an orthonormal basis for W_0. Since $L^2(\mathbb{R}^n)$ is the orthogonal direct sum $\bigoplus_{i\in\mathbb{Z}} W_i$, where $W_i = D_{2I}^i W_0$ (this follows from the properties of the MRA), it follows that $\{D_{2I}^i T_k \psi^\ell : i \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

In the three examples we have developed, we have proceeded in a very similar way to the MRA method. Given $V_0 = L^2(S_0)$ and $V_1 = L^2(S_0\, a)$, the orthogonal complement of V_0 in V_1 is the space $W_0 = L^2(S_1 \setminus S_0)$. Then we constructed an orthonormal basis for $L^2(W_0)$ of the form $\{D_b T_k \psi^\ell : b \in B, k \in \mathbb{Z}^2, \ell = 1, \ldots, L\}$ (observe that, unlike the "classical" MRA method, we need both the integer translations and the dilations by $b \in B$ to obtain this basis). Recall that in the example from Section 2, we used $\hat{\psi}^\ell = \chi_{I_\ell}$, $\ell = 1, 2, 3$; in the other two examples in Section 3, we used $\hat{\psi} = \chi_R$ $(L = 1)$, and $\hat{\psi}^\ell = \chi_{R_\ell}$, $\ell = 1, 2, 3$, respectively. Since $\{(b, k) : b \in B, k \in \mathbb{Z}^2\}$ is a countable group, and $\pi_{(b,k)} = D_b T_k$ is unitary, an application of Theorem 3.1 tells us that $L = 3$, $L = 1$, and $L = 3$, respectively, are *always* the number of generators for each affine system $\{D_b T_k \psi^\ell : b \in B, k \in \mathbb{Z}^2, \ell = 1, \ldots, L\}$ that forms an orthonormal basis for $L^2(W_0)$. Observe that, in general, the functions $\hat{\psi}^\ell$ need not be characteristic functions of subsets of $\widehat{\mathbb{R}}^2$.

These observations extend a well-known fact about dyadic affine MRA wavelet systems on $L^2(\mathbb{R}^n)$: they are generated by $2^n - 1$ functions. More generally, for wavelet systems involving dilations that are integer powers of an expanding matrix $a \in GL_n(\mathbb{Z})$, the number of generators is $L = |\det a| - 1$ (notice that 2^n is the determinant of the matrix $2I_n$). All this is discussed in greater detail in [5].

4 More constructions

Many more constructions of affine systems with composite dilations can be found in [3, 4, 5, 8]. For example, in [3, 5] there are examples of singly generated wavelets with composite dilations: these are non–MRA constructions. In [5] there are extensions of the shear matrices B used in the construction of Section 2 to higher dimensions, while in [8] is considered the case of finite groups B in higher dimensions. In [5] it is shown how the "traditional" MRA constructions, involving low–pass and high–pass filters, can be adapted to the setting of composite wavelets. In the same paper, there are examples of well–localized wavelets with composite dilations, that is, systems that have fast decay both in $\mathbb{R}^n$ and in $\widehat{\mathbb{R}}^n$. It is clear that the Fourier transform of the generators of these systems are not characteristic functions.

Beside the groups B we have considered in this paper, there are many other classes of matrices B that provide geometric properties different from the ones we have described. For example, the integral Heisenberg group

$$B = \left\{ b_{(i,j,k)} = \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} : \, i,j,k \in \mathbb{Z} \right\}$$

provides interesting classes of wavelets with composite dilations. A simple example of yet another class of sets B is provided by the powers of a matrix of the form $\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, $\mu \neq 0$. These extend to higher dimensions and are discussed in [5], where they are referred to as *hyperbolic* matrices. Also, let us mention that we need not restrict ourselves to orthonormal wavelets. Several Parseval frame wavelets can be constructed by the methods that we have described.

Finally, one can consider a continuous version of the composite wavelets, that is, systems of the form

$$\{\psi_{abt} = D_{A_a} \, D_{B_b} \, T_t \, \psi : \, t \in \mathbb{R}^2, a, b \in \mathbb{R}\},$$

where the matrices A_a, B_b depend continuously on $a, b \in \mathbb{R}$. For example, one can choose

$$A = \begin{pmatrix} a & 0 \\ & \\ 0 & \sqrt{a} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b \\ & \\ 0 & 1 \end{pmatrix}.$$

Systems of this form are examined in [7].

References

[1] E.J. Candès and D.L. Donoho, New tight frames of curvelets and optimal representations of objects with C^2 singularities, *Comm. Pure Appl. Math.* **56** (2004), 219–266.

[2] M.N. Do and M. Vetterli, The contourlet transform: an efficient directional multiresolution image representation, to appear in: *IEEE Transactions Image on Processing*, 2005.

[3] K. Guo, W-Q. Lim, D. Labate, G. Weiss and E. Wilson, Wavelets with composite dilations, *Electr. Res. Announc. of AMS* **10** (2004), 78–87.

[4] K. Guo, W-Q. Lim, D. Labate, G. Weiss and E. Wilson, The theory of wavelets with composite dilations, to appear in *Harmonic Analysis and Applications*, C. Heil (ed.), Birkäuser, Boston (2005).

[5] K. Guo, W-Q. Lim, D. Labate, G. Weiss and E. Wilson, Wavelets with composite dilations and their MRA properties, to appear in *Appl. Comput. Harmon. Anal.* (2005).

[6] E. Hernández, G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton FL, 1996.

[7] G. Kutiniok and D. Labate, Resolution of the wavefront set using shearlets, preprint, 2005

[8] A. Savov, Finite groups associated with composite dilation wavelets, Senior Honor Thesis, Washington Univesity in St.Louis, 2005

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