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The Theory of Wavelets with Composite Dilations

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Summary. A wavelet with composite dilations is a function generating an orthonormal basis or a Parseval frame for $L^2(\mathbb{R}^n)$ under the action of lattice translations and dilations by products of elements drawn from non-commuting sets of matrices A and B. Typically, the members of B are matrices whose eigenvalues have magnitude one, while the members of A are matrices expanding on a proper subspace of $\mathbb{R}^n$. The theory of these systems generalizes the classical theory of wavelets and provides a simple and flexible framework for the construction of orthonormal bases and related systems that exhibit a number of geometric features of great potential in applications. For example, composite wavelets have the ability to produce "long and narrow" window functions, with various orientations, well-suited to applications in image processing.

Dedicated to John J. Benedetto.

11.1 Introduction

We assume familiarity with the basic properties of separable Hilbert spaces. $L^2(\mathbb{R}^n)$, the space of all square integrable functions on $\mathbb{R}^n$, and $L^2(\mathbb{T}^n)$, the space of all square integrable $\mathbb{Z}^n$-periodic functions, are such spaces.

The construction and the study of orthonormal bases and similar collections of functions are of major importance in several areas of mathematics and applications, and have been a very active area of research in the last few decades.

Historically, there are two basic methods for constructing orthonormal bases of $L^2(\mathbb{R}^n)$. The most elementary approach is the following. Let $g = \chi_{[0,1)}$ and

$$\mathcal{G}(g) = \{e^{2\pi ikx}\, g(x-m) : k, m \in \mathbb{Z}\}.$$

It is easy to see that $\mathcal{G}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$. More generally, let T_y, $y \in \mathbb{R}^n$, be the *translation operator*, defined by

$$T_y\, f(x) = f(x-y),$$

and M_ν, $\nu \in \mathbb{R}^n$, be the *modulation operator*, defined by

$$M_\nu\, f(x) = e^{2\pi i\nu x}\, f(x).$$

The *Gabor* or *Weyl–Heisenberg system* generated by $G = \{g_1, \ldots, g_L\} \subset L^2(\mathbb{R}^n)$ is the family of the form

$$\mathcal{G}(G) = \{M_{bm}\, T_k\, g_\ell : k, m \in \mathbb{Z}^n,\ \ell = 1, \ldots, L\}, \tag{11.1}$$

where $b \in GL_n(\mathbb{R})$. Then the basic question is: what are the sets of functions $G \subset L^2(\mathbb{R}^n)$ such that $\mathcal{G}(G)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$? It turns out that one can construct several examples of such sets of functions G. However, a fundamental result in the theory of Gabor systems—the *Balian-Low Theorem*—shows that such functions are not very well behaved. In fact, if $G = \{g\}$, then g cannot have fast decay both in $\mathbb{R}^n$ and in $\widehat{\mathbb{R}}^n$; if $G = \{g_1, \ldots, g_L\}$, then at least one function g_ℓ cannot have fast decay both in $\mathbb{R}^n$ and in $\widehat{\mathbb{R}}^n$ (see [1] for a nice overview of the Balian–Low Theorem, and [20] for the multiwindow case).

As we mentioned before, there is a second approach for constructing orthonormal bases of $L^2(\mathbb{R}^n)$. Let $A = \{a^i : i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$, and let us replace the modulation operator in (11.1) with the *dilation operator* D_a, defined by

$$D_a\, f(x) = |\det a|^{-1/2}\, f(a^{-1}x).$$

By doing this, we obtain the *affine* or *wavelet systems* generated by $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$, which are the systems of the form

$$\mathcal{A}_A(\Psi) = \{D_{a^i}\, T_k\, \psi_\ell : a \in A,\ i \in \mathbb{Z},\ \ell = 1, \ldots, L\}.$$

If $\mathcal{A}_A(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, then Ψ is called a *multiwavelet* or, simply, a *wavelet* if $\Psi = \{\psi\}$. Even though the first example of a wavelet, the Haar wavelet, was discovered in 1909, the theory of wavelets was actually born in the beginning of the 1980s. Again, as in the case of Gabor systems, the basic mathematical question is: what are the wavelets $\Psi \subset L^2(\mathbb{R}^n)$? It turns out that there are "many" such functions and that, in a certain sense, they are "more plentiful" than the functions $G \subset L^2(\mathbb{R}^n)$ generating an orthonormal Gabor system. This fact, together with the ability of such bases to exhibit a number of properties which are useful in applications, in part explains the considerable success of wavelets in mathematics and applications in the last twenty years. We refer to [13] and [5] for more details, in particular regarding the comparison between Gabor and affine systems.

In the last few years, there has been a considerable interest, both in the mathematical and engineering literature, in the study of variants of the affine systems which contain basis elements with many more locations, scales and directions than the "classical" wavelets (see the papers in [19] for examples of such systems). The motivation for this study comes partly from signal processing, where such bases are useful in image compression and feature extraction, and partly from the investigation of certain classes of singular integral operators. The main subject of this paper is the study of a new class of systems, which we call *affine systems with composite dilations* and which have the form

$$\mathcal{A}_{AB}(\Psi) = \{D_a D_b T_k \Psi : \, k \in \mathbb{Z}^n, \, a \in A, \, b \in B\},$$

where A, $B \subset GL_n(\mathbb{R})$. If $\mathcal{A}_{AB}(\Psi)$ is an orthonormal basis, then Ψ will be called a *composite* or *AB-multiwavelet* (as before we use the term wavelet rather than multiwavelet if $\Psi = \{\psi\}$). As we will show, the theory of these systems generalizes the classical theory of wavelets and provides a simple and flexible framework for the construction of orthonormal bases and related systems that exhibit a number of geometric features of great potential in applications. For example, one can construct composite wavelets with good time-frequency decay properties whose elements contain "long and narrow" waveforms with many locations, scales, shapes and directions [8]. These constructions have properties similar to those of the *curvelets* [2] and *contourlets* [6], which have been recently introduced in order to obtain efficient representations of natural images. The theory of affine systems with composite dilations is more general. In fact, the contourlets can be described as a special case of these systems (see [9]). In addition, our approach extends naturally to higher dimensions and allows a multiresolution construction which is well suited to a fast numerical implementation.

It is of interest to point out that there exist affine systems with composite dilations $\mathcal{A}_{AB}(\Psi)$ that are orthonormal bases (as well as Parseval frames) for $L^2(\mathbb{R}^n)$ when the dilation set A is not known to be associated with an affine system $\mathcal{A}_A(\Psi)$ that is an orthonormal basis (or a Parseval frame) for $L^2(\mathbb{R}^n)$. For example, when $A = \{a^i : i \in \mathbb{Z}\}$, where $a = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, with $|\lambda_1| > 1 > |\lambda_2| > 0$, then, by a result in [16, Prop. 2.2], the Calderòn equation $\sum_{\ell=1}^{L} \sum_{i \in \mathbb{Z}} |\hat{\psi}_\ell(\xi a^i)|^2 = 1$ a.e. fails. This equation is one of the "traditional" equalities that characterize wavelets (see [15], [14]), and it is a necessary condition for MSF wavelets to exist (the MSF wavelets are those wavelets ψ such that $\hat{\psi} = \chi_T$, for some measurable set T). The situation for general wavelets is unknown, but it seems unlikely that wavelets could be obtained without satisfying the Calderòn condition. However, we will show in Section 11.2.2 that there are orthonormal bases of the form $\mathcal{A}_{AB}(\Psi)$ for $B = \{b^j : j \in \mathbb{Z}\}$ where $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

11.1.1 Reproducing Function Systems

Before examining in more detail the properties of the affine systems with composite dilations and their variants, let us make a few observations about the general properties of the collections that form an orthonormal basis for a Hilbert space. We have the following simple proposition.

Proposition 11.1. *Let $\mathcal{H}$ be a separable Hilbert space, $T : \mathcal{H} \to \mathcal{H}$ be unitary and $\Phi = \{\phi_1, \ldots, \phi_N\}$, $\Psi = \{\psi_1, \ldots, \psi_M\} \subset \mathcal{H}$, where $N, M \in \mathbb{N} \cup \{\infty\}$. Suppose that $\{T^j \phi_k : j \in \mathbb{Z}, 1 \le k \le N\}$ and $\{T^j \psi_i : j \in \mathbb{Z}, 1 \le i \le M\}$ are orthonormal bases for $\mathcal{H}$. Then $N = M$.*

Proof. It follows from the assumptions that, for each $1 \le k \le N$,

$$\|\phi_k\|^2 = \sum_{j \in \mathbb{Z}} \sum_{i=1}^{M} |\langle \phi_k, T^j \psi_i \rangle|^2.$$

Thus,

$$\begin{aligned} N = \sum_{k=1}^{N} \|\phi_k\|^2 &= \sum_{k=1}^{N} \sum_{j \in \mathbb{Z}} \sum_{i=1}^{M} |\langle \phi_k, T^j \psi_i \rangle|^2 \\ &= \sum_{i=1}^{M} \sum_{j \in \mathbb{Z}} \sum_{k=1}^{N} |\langle T^{-j} \phi_k, \psi_i \rangle|^2 \\ &= \sum_{i=1}^{M} \|\psi_i\|^2 = M. \quad \square \end{aligned}$$

Using this proposition, we can now show that *there are no orthonormal bases for $L^2(\mathbb{R}^n)$ generated using only dilates of a finite family of functions.* Indeed, arguing by contradiction, suppose that there are finitely many functions $\{\phi^1, \ldots, \phi^N\} \subset L^2(\mathbb{R}^n)$ such that $\{D_2^j \phi^\ell : j \in \mathbb{Z}, 1 \le \ell \le N\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. On the other hand, it is known that there exist wavelets $\psi \in L^2(\mathbb{R}^n)$ for which $\{D_2^j T_k \psi : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. This is a contradiction since, by Proposition 11.1, in the first case $N < \infty$, while, in the second case $M = \infty$, and, thus, $M \ne N$. The same argument applies to more general dilation matrices $A \in GL_n(\mathbb{R})$ (namely, all those for which wavelets exist). By applying a similar argument using the Gabor systems rather than the wavelets, one shows that *there are no orthonormal bases for $L^2(\mathbb{R}^n)$ generated using only modulations or only translations of a finite family of functions.* These observations, which we deduce from Proposition 11.1, are special cases of deeper and more general results obtained from the study of the *density* of Gabor and affine systems. Indeed, using this more general approach, one obtains similar results holding not only for orthonormal bases but even for frames (see [3], [12]).

In many situations, the notion of orthonormal basis turns out to be too restrictive and one can consider more general collections of functions, called *frames*, that preserve, as we will now show, many of the properties of bases.

A countable family $\{e_j : j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a *frame* if there exist constants $0 < \alpha \leq \beta < \infty$ satisfying

$$\alpha \|v\|^2 \leq \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2 \leq \beta \|v\|^2$$

for all $v \in \mathcal{H}$. The constants α and β are called *lower* and *upper frame bounds*, respectively. If the right-hand side inequality holds but not necessarily the left-hand side, we say that $\{e_j : j \in \mathcal{J}\}$ is a *Bessel system* with constant β. A frame is *tight* if α and β can be chosen so that $\alpha = \beta$, and is a *Parseval frame* if $\alpha = \beta = 1$. Thus, if $\{e_j : j \in \mathcal{J}\}$ is a Parseval frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2$$

for each $v \in \mathcal{H}$. This is equivalent to the reproducing formula

$$v = \sum_{j\in\mathcal{J}} \langle v, e_j\rangle\, e_j \tag{11.2}$$

for all $v \in \mathcal{H}$, where the series in (11.2) converges in the norm of $\mathcal{H}$. Equation (11.2) shows that a Parseval frame provides a basis-like representation. In general, however, the elements of a frame need not be independent and a frame or Parseval frame need not be a basis. The elements of a frame $\{e_j\}_{j\in\mathcal{J}}$ must satisfy $\|e_j\| \leq \sqrt{\beta}$ for all $j \in \mathcal{J}$, as can easily be seen from

$$\|e_j\|^4 = |\langle e_j, e_j\rangle|^2 \leq \sum_{i\in \mathrm{i}} |\langle e_j, e_i\rangle|^2 \leq \beta\, \|e_j\|^2.$$

In particular, if $\{e_j\}_{j\in\mathcal{J}}$ is a Parseval frame, then $\|e_j\| \leq 1$ for all $j \in \mathcal{J}$, and the frame is an orthonormal basis for $\mathcal{H}$ if and only if $\|e_j\| = 1$ for all $j \in \mathcal{J}$. We refer the reader to [7] or [15, Ch. 8] for more details about frames.

11.1.2 Notation

It will be useful to establish the notation and basic definitions that will be used in this paper. We adopt the convention that $x \in \mathbb{R}^n$ is a column vector, i.e., $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and that $\xi \in \widehat{\mathbb{R}}^n$ is a row vector, i.e., $\xi = (\xi_1, \dots, \xi_n)$. Similarly for the integers, $k \in \mathbb{Z}^n$ is the column vector $k = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$, and $\hat{k} \in \widehat{\mathbb{Z}}^n$ is the row vector $\hat{k} = (\hat{k}_1, \dots, \hat{k}_n)$. A vector x multiplying a matrix $a \in GL_n(\mathbb{R})$ on the right is understood to be a column vector, while a vector ξ multiplying a on the left is a row vector. Thus, $ax \in \mathbb{R}^n$ and $\xi a \in \widehat{\mathbb{R}}^n$. The Fourier transform is defined as

$$\hat{f}(\xi) = (\mathcal{F} f)(x) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi x}\, dx,$$

where $\xi \in \widehat{\mathbb{R}}^n$, and the inverse Fourier transform is

$$\check{f}(x) = (\mathcal{F}^{-1} f)(x) = \int_{\widehat{\mathbb{R}}^n} f(\xi)\, e^{2\pi i x \xi}\, d\xi.$$

For any $E \subset \widehat{\mathbb{R}}^n$, we denote by $L^2(E)^\vee$ the space $\{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset E\}$.

11.2 Affine Systems

Let $A \subset GL_n(\mathbb{R})$ be a countable set and $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$. We introduce the following extension of the previously defined notion of affine systems. A collection of the form

$$\mathcal{A}_A(\Psi) = \{D_a\, T_k\, \Psi : \ a \in A,\, k \in \mathbb{Z}^n\}$$

is a (generalized) *affine system*. A special role is played by those functions Ψ for which the system $\mathcal{A}_A(\Psi)$ is an orthonormal basis or, more generally, a Parseval frame for $L^2(\mathbb{R}^n)$. In particular, Ψ is an *orthonormal A-multiwavelet* if the set $\mathcal{A}_A(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ and is a *Parseval frame A-multiwavelet* if $\mathcal{A}_A(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$. The set A is *admissible* if a Parseval frame A-wavelet exists.

It will also be useful to consider A-multiwavelets which are defined on subspaces of $L^2(\mathbb{R}^n)$ of the form $L^2(S)^\vee$, where $S \subset \widehat{\mathbb{R}}^n$ is a measurable set with positive Lebesgue measure. As we will show, they will play a major role in the construction of the composite wavelets that we mentioned in Section 11.1. If $\mathcal{A}_A(\Psi)$ is a Parseval frame (resp., an orthonormal basis) for $L^2(S)^\vee$, then $\Psi \in L^2(\mathbb{R}^n)$ is called a Parseval frame A-multiwavelet (resp., an orthonormal A-multiwavelet) for $L^2(S)^\vee$. If such a multiwavelet exists, the set A is *S-admissible.*

The affine systems $\mathcal{A}_A(\{\psi\})$ where $|\hat{\psi}| = \chi_T$ for some measurable set $T \subseteq \widehat{\mathbb{R}}^n$ are called *minimally supported in frequency (MSF)* systems, and the corresponding function ψ is called an *MSF wavelet* for $L^2(S)^\vee$ if ψ is a Parseval frame A-wavelet for $L^2(S)^\vee$. One can show (see [10], [16]) that ψ is a Parseval frame A-wavelet for $L^2(S)^\vee$ if and only if $\Omega_T = \bigcup_{\hat{k} \in \widehat{\mathbb{Z}}^n} (T + \hat{k})$ is a disjoint union, modulo null sets, and $S = \bigcup_{a \in A} (T\, a^{-1})$ is also a disjoint union, modulo null sets. In this case, we say that the set T is both a $\widehat{\mathbb{Z}}^n$*-tiling set* for Ω_T and a A^{-1}*-tiling set* for S. In particular, $|T| \leq 1$ since T is contained in a $\widehat{\mathbb{Z}}^n$-tiling set for $\widehat{\mathbb{R}}^n$.

In the following, we will examine the admissibility condition for different dilation sets A. In Section 11.2.1, we will recall the situation of the classical wavelets, where $A = \{a^i : i \in \mathbb{Z}\}$, for some $a \in GL_n(\mathbb{R})$. Next, in Section 11.2.2, we will introduce a new family of admissible dilation sets, where the dilations have the form $AB = \{ab : a \in A, b \in B\}$, and $A, B \subset GL_n(\mathbb{R})$.

11.2.1 Admissibility Condition: The Classical Wavelets

We consider first the case where $A = \{a^i : i \in \mathbb{Z}\}$, for some $a \in GL_n(\mathbb{R})$. This is the situation one encounters in the classical wavelet theory.

Recall that a matrix $a \in GL_n(\mathbb{R})$ is *expanding* if each eigenvalue λ of a satisfies $|\lambda| > 1$. Dai, Larson and Speegle [4] have shown that if $a \in GL_n(\mathbb{R})$ is expanding, then the set $A = \{a^i : i \in \mathbb{Z}\}$ is admissible (observe that, in this case, the set A is also a group). In fact, they have shown that, under this assumption on a, one can find a set $T \subseteq [-1/2, 1/2]^n$ which is both a $\widehat{\mathbb{Z}}^n$-tiling set for Ω_T and an A-tiling set for $\widehat{\mathbb{R}}^n$. Thus, $\psi = (\chi_T)^\vee$ is a tiling A-wavelet for $L^2(\mathbb{R}^n)$. In addition, they construct a set T' for which $(\chi_{T'})^\vee$ is an orthonormal A-wavelet. More generally, Wang [17] has shown that if a set $A \subset GL_n(\mathbb{R})$ admits an A^{-1}-tiling set and A contains an expanding matrix m for which $mA \subseteq A$, then A admits an orthonormal A-tiling wavelet.

Until very recently, all affine systems considered in the literature concerned dilation matrices which are expanding. In [14], however, one finds examples of admissible dilation sets $A = \{a^i : i \in \mathbb{Z}\}$, where the matrix $a \in GL_n(\mathbb{R})$ is not expanding. In particular, a theorem in [14] gives a set of equations characterizing Parseval frame A-wavelets, for a large class of matrices a, where a is not necessarily expanding.

Furthermore, observe that "general" dilation sets may fail to be admissible. Consider, for example, the set $A = \{2^j\, 3^i : i, j \in \mathbb{Z}\}$. Then one can show that there are no Parseval frame A-wavelets for this dilation set. In fact, since $\log 2^j 3^i = j \log 3 + i \log 2$ and $\log 3 / \log 2$ is an irrational number, it follows that each nontrivial orbit of A is dense in $\mathbb{R}$, and this implies that there are no Parseval frame A-wavelets (see [9]) for more details). However, the set $\tilde{A} = \left\{ \begin{pmatrix} 2^j & 0 \\ 0 & 3^i \end{pmatrix} : i, j \in \mathbb{Z} \right\}$ is admissible. In fact, it is easy to see that, for any $\psi(x_1, x_2) = \psi_1(x_1)\, \psi_2(x_2)$, where ψ_1 is a (one-dimensional) dyadic wavelet and ψ_2 is a (one-dimensional) triadic wavelet, then ψ is an $\tilde{A}$-wavelet.

11.2.2 Admissibility Condition: The Composite Wavelets

Let $S_0 \subset \widehat{\mathbb{R}}^n$ be a region centered at the origin, and let $B \subset GL_n(\mathbb{R})$ be a set of matrices mapping S_0 into itself. In many situations, as we will show in more detail later, one can find a subregion $U_0 \subset S_0$ which is a B^{-1}-tiling region for S_0. This implies that

$$\{D_b\, T_k\, (\chi_{U_0})^\vee : \, b \in B,\, k \in \mathbb{Z}^n\}$$

is a Parseval frame for $L^2(S_0)^\vee$.

Next, consider a second set of matrices $A = \{a^i \, : \, i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$. If a is expanding in some direction, i.e., *some* of the eigenvalues λ of a satisfy $|\lambda| > 1$, and satisfies some other simple conditions to be specified later, then we can construct a sequence of B-invariant regions $S_i = S_0\, a^i$, $i \in \mathbb{Z}$, with $S_i \subset S_{i+1}$ and $\lim_{i\to\infty} S_i = \widehat{\mathbb{R}}^n$. This enables us to decompose the space $L^2(\mathbb{R}^n)$ into an exhaustive *disjoint* union of closed subspaces:

$$L^2(\mathbb{R}^n) = \bigoplus_{i\in\mathbb{Z}} L^2(S_{i+1} \setminus S_i)^\vee. \tag{11.3}$$

In general, one can find a region $R \subset S_1 \setminus S_0$ which is a B^{-1}-tiling region for $S_1 \setminus S_0$. If S_0 is sufficiently small, this implies that

$$\{D_b\, T_k\, (\chi_R)^\vee : \, b \in B,\, k \in \mathbb{Z}^n\}$$

is a Parseval frame for $L^2(S_1 \setminus S_0)^\vee$, and, as a consequence, the system

$$\{D_a^i\, D_b\, T_k\, (\chi_R)^\vee : \, b \in B,\, i \in \mathbb{Z},\, k \in \mathbb{Z}^n\}$$

is a Parseval frame for $L^2(\mathbb{R}^n)$.

This shows that the set

$$AB = \{a^i\, b : \, i \in \mathbb{Z},\, b \in B\}$$

is admissible, and that ψ is a Parseval frame AB-wavelet whenever ψ is a Parseval frame B-wavelet for $L^2(S_1 \setminus S_0)^\vee$.

In the following subsections, we will present several examples of such dilation sets which are admissible, and construct several examples of AB-wavelets.

Finite Group *B*

The first situation we consider is the case when B is a finite group. Since B is conjugate to a subgroup of the orthogonal group $O_n(\mathbb{R})$ (i.e., given any finite group B, there is a $P \in GL_n(\mathbb{R})$ and a $\tilde{B} \in O_n(\mathbb{R})$ such that $PBP^{-1} = \tilde{B}$), without loss of generality, we may assume that $B \subset O_n(\mathbb{R})$. Let $S_0 \subset \widehat{\mathbb{R}}^n$ be a compact region, starlike with respect to the origin, with the property that B maps S_0 into itself. In many situations, one can find a lattice $L \subset \mathbb{R}^n$ and a region $U_0 \subseteq S_0$ such that U_0 is both a B-tiling region for S_0 and a Λ-packing region for $\widehat{\mathbb{R}}^n$ (i.e., $\sum_{\lambda\in\Lambda} \chi_{U_0}(\xi + \lambda) \le 1$ for a.e. $\xi \in \widehat{\mathbb{R}}^n$), where $\Lambda = \{\lambda \in \widehat{\mathbb{R}}^n : \, \lambda\, l \, \in \mathbb{Z}, \forall\, l \in L\}$ is the lattice dual to L. Then

$$\Phi_B = \{D_b\, T_l\, (\chi_{U_0})^\vee : b \in B,\, l \in L\}$$

is a Parseval frame for $L^2(S_0)^\vee$ (this fact is well known and can be found, for example, in [11]). Next suppose that $A = \{a^i : i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$ is expanding, $a\,B\,a^{-1} = B$ and $S_0 \subseteq S_0\,a = S_1$. These assumptions imply that each region $S_i = S_0\,a^i$, $i \in \mathbb{Z}$, is B-invariant and the family of disjoint regions $S_i \subset S_{i+1}$, $i \in \mathbb{Z}$, tile the plane $\widehat{\mathbb{R}}^n$. Thus, one can decompose $L^2(\mathbb{R}^n)$ according to (11.3). As in the general situation described above, when S_0 is sufficiently small, one can find a region $R \subset S_1 \setminus S_0$ which is a B-tiling region for $S_1 \setminus S_0$, so that

$$\Psi_{AB} = \{D_a^i\, D_b\, T_l\, (\chi_R)^\vee : b \in B,\, i \in \mathbb{Z},\, l \in L\}$$

is a Parseval frame for $L^2(\mathbb{R}^n)$. In addition, when U_0 is a Λ-tiling region for $\widehat{\mathbb{R}}^n$ and $|\det a| \in \mathbb{N}$, one can decompose the region R into a disjoint union of regions $R_1, \ldots, R_N$ in such a way that

$$\widetilde{\Psi}_{AB} = \{D_a^i\, D_b\, T_l\, (\chi_{R_\ell})^\vee : i \in \mathbb{Z},\, b \in B,\, l \in L,\, \ell = 1, \ldots, N\}$$

is a not only a Parseval frame but also an orthonormal AB-multiwavelet for $L^2(\mathbb{R}^n)$. Moreover, in this case, the set Φ_B is an orthonormal basis for $L^2(S_0)^\vee$.

We will present two examples to illustrate the general construction that we have outlined above.

Example 11.2. The first example is illustrated in Fig. 11.1. Let B be the 8-element group consisting of the isometries of the square $[-1, 1]^2$. Namely, $B = \{\pm b_0, \pm b_1, \pm b_2, \pm b_3,\}$ where $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Let U be the parallelogram with vertices $(0, 0), (1, 0), (2, 1)$ and $(1, 1)$ and $S_0 = \bigcup_{b\in B} U\,b$ (see the snowflake region in Fig. 11.1). It is easy to verify that S_0 is B-invariant.

Now let $a = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, and $S_i = S_0\,a^i$, $i \in \mathbb{Z}$. Observe that a is expanding, $a\,B\,a^{-1} = B$ and $S_0 \subseteq S_0\,a = S_1$. In particular, the region $S_1 \setminus S_0$ is the disjoint union $\bigcup_{b\in B} R\,b$, where the region R is the parallelogram illustrated in Fig. 11.1. Also observe that R is a fundamental domain. Thus, as in the general situation that we have described before, it turns out that the set

$$\{D_a^i\, D_b\, T_k\, \psi : i \in \mathbb{Z},\, b \in B,\, k \in \mathbb{Z}^2\},$$

where $\hat{\psi} = \chi_R$, is an orthonormal basis for $L^2(\mathbb{R}^2)$.

Example 11.3. Next consider the situation where $a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and B is as in Example 11.2. Let U and S_i, $i \in \mathbb{Z}$, be defined as before. Also in this case, a

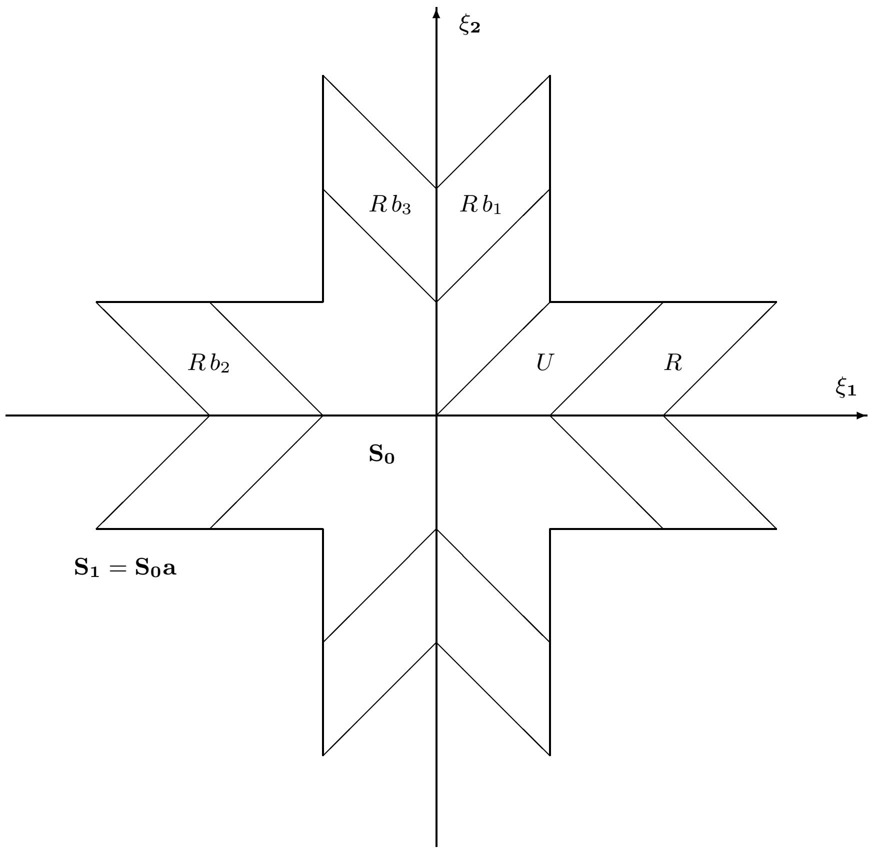

Fig. 11.1. Example 11.2. Admissible AB set; $A = \{a^i : i \in \mathbb{Z}\}$, where a is the quincunx matrix, and B is the group of isometries of the square $[-1, 1]^2$.

is expanding, $a\,B\,a^{-1} = B$ and $S_1 = S_0 a \supset S_0$. A direct computation shows that the region $S_1 \setminus S_0$ is the disjoint union $\bigcup_{b \in B} Rb$, where $R = R_1 \bigcup R_2 \bigcup R_3$ and the regions R_1, R_2, R_3 are illustrated in Fig. 11.2. Observe that each of the regions R_1, R_2, R_3 is a fundamental domain. Thus, the system

$$\{D_a^i\, D_b\, T_k\, \psi^\ell : i \in \mathbb{Z},\, b \in B,\, k \in \mathbb{Z}^2,\, \ell = 1, \ldots, 3\},$$

where $\hat{\psi}^\ell = \chi_{R_\ell}$, $\ell = 1, 2, 3$, is an orthonormal basis for $L^2(\mathbb{R}^2)$.

The examples that we described are two-dimensional. It is clear, however, that not only are there many more examples in dimension $n = 2$, but also that there are similar constructions for $n > 2$. For $n = 2$ or $n = 3$, in particular, every finite group of $O_n(\mathbb{R})$ acts by isometries on a regular polyhedron, and one can apply a construction very similar to the one described in these examples.

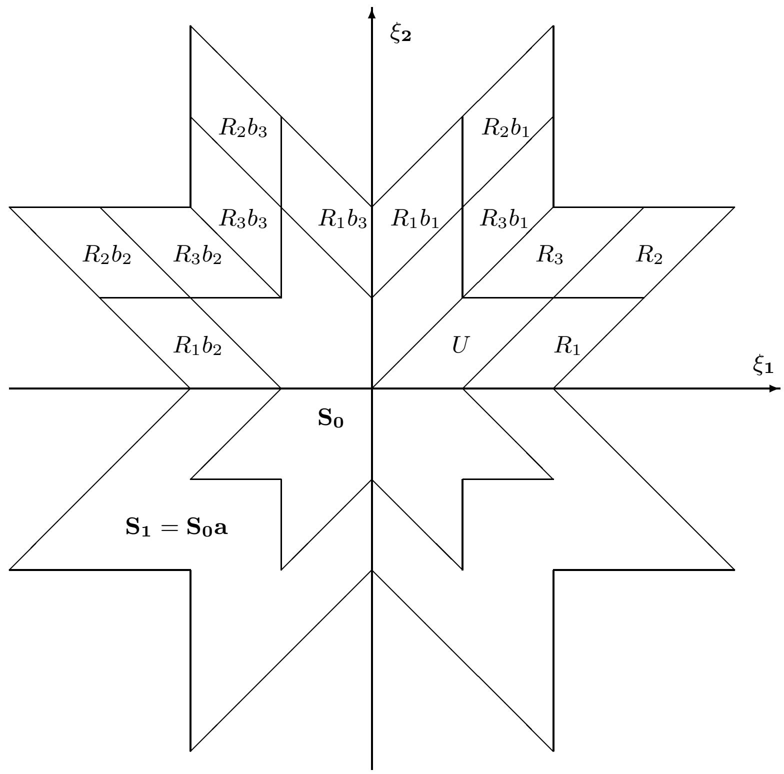

Fig. 11.2. Example 11.3. Admissible AB set; $A = \{a^i : i \in \mathbb{Z}\}$, where $a = 2I$, and B is the group of isometries of the square $[-1, 1]^2$.

Shear Matrices B

We consider now the case of an infinite group. For simplicity, let us restrict ourselves, for the moment, to the two-dimensional situation and consider the group

$$B = \{b^j : j \in \mathbb{Z}\}, \quad \text{where } b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Observe that $b^j = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$. Since $(\xi_1, \xi_2)\, b^j = (\xi_1, \xi_2 + j\xi_1)$, the right action of the matrix $b^j \in B$ maps the line through the origin of slope m to the line through the origin of slope $m + j$. Observe that the matrices B are not expanding (all eigenvalues have magnitude one): they are called *shear matrices*. For $0 \le \alpha < \beta$, let $S(\alpha, \beta) = \{\xi = (\xi_1, \xi_2) \in \widehat{\mathbb{R}}^2 : \alpha \le |\xi_1| < \beta\}$ and let $T(\alpha, \beta) = T^+(\alpha, \beta) \bigcup T^-(\alpha, \beta)$, where $T^+(\alpha, \beta)$ is the trapezoid with vertices $(\alpha, 0)$, (α, α), $(\beta, 0)$ and (β, β), and $T^-(\alpha, \beta) = \{\xi \in \widehat{\mathbb{R}}^2 : -\xi \in T^+(\alpha, \beta)\}$. A simple computation shows that $T(\alpha, \beta)$ is a B-tiling region for $S(\alpha, \beta)$, and, thus, for β sufficiently small, the function $\phi = (\chi_{T(\alpha,\beta)})^\vee$ is a Parseval frame B-wavelet for $L^2(S(\alpha,\beta))^\vee$. This shows that the set B is $S(\alpha,\beta)$-admissible.

Now fix $0 < c < 1$ and let $A = \{a^i : i \in \mathbb{Z}\}$, where $a = \begin{pmatrix} c^{-1} & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}$ and $a_{1,2}, a_{2,2} \in \mathbb{R}$, with $a_{2,2} \neq 0$. Let ψ be defined by $\hat{\psi} = \chi_{T(c,1)}$. Since $T(c,1)$ is a B-tiling region for $L^2(S(c,1))$ and a $\widehat{\mathbb{Z}}^n$-packing region for $\widehat{\mathbb{R}}^n$, it follows that ψ is a Parseval frame B-wavelet for $L^2(S(c,1))^\vee$. A direct computation shows that the sets $S(c,1)\,a^{-i}$, $i \in \mathbb{Z}$, tile $\widehat{\mathbb{R}}^2$. Hence

$$\Psi_{AB} = \{D_a^i\, D_b\, T_k\, \psi : i \in \mathbb{Z},\, b \in B,\, k \in \mathbb{Z}^2\}$$

is a Parseval frame for $L^2(\mathbb{R}^2)$. This shows that the set $AB = \{a^i\, b : i \in \mathbb{Z},\, b \in B\}$ is admissible and ψ is a Parseval frame AB-wavelet for $L^2(\mathbb{R}^2)$. Observe that the Parseval frame AB-wavelet ψ depends only on $a_{11} = c^{-1}$ (on the other hand, the elements of the system Ψ_{AB}, when $i \neq 0$, do depend also on the other entries of the matrix a). Also observe that, if $|a_{22}| < 1$, no member of AB is an expanding matrix.

The following example shows how to obtain an orthonormal basis rather than a Parseval frame, by using a slightly different construction.

Example 11.4. Observe that, when $\alpha = 0$ and $\beta \in \mathbb{N}$, then $T(0,\beta)$ is the union of two triangles, which satisfies

$$\big(T^-(0,\beta) + (\beta, b)\big) \bigcup T^+(0,\beta) = [0,\beta)^2.$$

Let $\alpha = 0$ and $\beta = 1$. Then $U_0 = T(0,1)$ is a fundamental domain of $\mathbb{Z}^2$ and $\phi = (\chi_{U_0})^\vee$ is an orthonormal B-wavelet for $L^2(S(0,1))^\vee$ (see Fig. 11.3). Next let $A = \{a^i : i \in \mathbb{Z}\}$, where $a = \begin{pmatrix} 2 & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix}$ and $a_{1,2}, a_{2,2} \in \mathbb{R}$, with $a_{2,2} \neq 0$. Similarly to the situation we described above, we have that the strip domains $S(1,2)\,a^{-i}$, $i \in \mathbb{Z}$, tile $\widehat{\mathbb{R}}^2$ and the set $T(1,2)$ is a B-tiling region for $L^2(S(1,2))$.

However, unlike the situation we described before, $T(1,2)$ is not a $\widehat{\mathbb{Z}}^n$-packing region for $\widehat{\mathbb{R}}^n$; in fact, its area is 3. On the other hand, we can split $T(1,2)$ into three regions of area one, which are fundamental domains of $\widehat{\mathbb{Z}}^2$. This can be done in several ways, for example, by introducing the trapezoids R_1, R_2, R_3 of Fig. 11.3, where $T(1,2) = R_1 \cup R_2 \cup R_3$. It then follows that the system

$$\Psi_{AB} = \{D_a^i\, D_b\, T_k\, \psi^\ell : i \in \mathbb{Z},\, b \in B,\, k \in \mathbb{Z}^2,\, \ell = 1, \ldots, 3\},$$

where $\hat{\psi}^\ell = \chi_{R_\ell}$, $\ell = 1,2,3$, is an orthonormal basis for $L^2(\mathbb{R}^2)$.

For $n > 2$, there are several generalizations of the constructions described in this section. For example, let $n = k + l$ and $B = \left\{ b_j = \begin{pmatrix} 1 & j \\ 0 & I_\ell \end{pmatrix} : j \in \mathbb{Z}^\ell \right\}$.

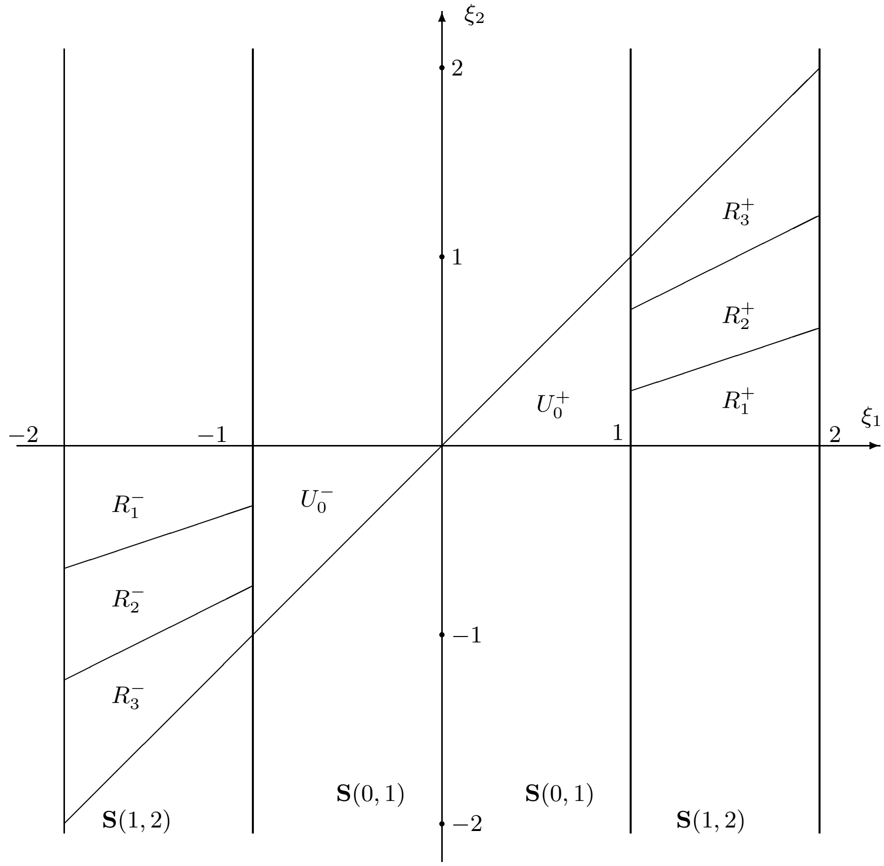

Fig. 11.3. Example 11.4. Admissible AB set; $A = \{a^i : i \in \mathbb{Z}\}$, where $a_{11} = 2$ and $a_{2,2} \neq 0$, and B is the group of shear matrices. In this figure, $U_0 = U_0^- \bigcup U_0^+$, $R_\ell = R_\ell^- \cup R_\ell^+$, $\ell = 1, 2, 3$, and $T(1, 2) = R_1 \cup R_2 \cup R_3$.

Let $A = \{a^i : i \in \mathbb{Z}\}$, where $a = \begin{pmatrix} a_0 & a_1 \\ 0 & a_2 \end{pmatrix} \in GL_n(\mathbb{R})$, and $a_0 \in GL_k(\mathbb{R})$ is an expanding matrix. We then construct a 'trapezoidal' region T in $\widehat{\mathbb{R}}^n = \widehat{\mathbb{R}}^k \times \widehat{\mathbb{R}}^\ell$, which has for base the set $T_0 \subset \widehat{\mathbb{R}}^k$ and for which T is a B-tiling region for the strip $T_0 \times \widehat{\mathbb{R}}^\ell$. If T_0 is contained in sufficiently small neighborhood of the origin, we can ensure that the sets Sa^i, $i \in \mathbb{Z}$, tile $\widehat{\mathbb{R}}^n$ and thereby guarantee that $\psi = \chi_T^\vee$ is a Parseval frame AB-wavelet for $L^2(\mathbb{R}^n)$, where $AB = \{a^i b : i \in \mathbb{Z}, b \in B\}$ (thus, AB is admissible). In particular, the function $\psi = (\chi_R)^\vee$, where $R = \{(\xi_0, \dots, \xi_\ell) \in \mathbb{R}^n : \frac{1}{2|a_0|} \leq |\xi_0| < \frac{1}{2}, 0 \leq \xi_i/\xi_0 < 1,$ for $1 \leq i \leq \ell\}$, is such a Parseval frame MRA AB wavelet. Observe that, as in the construction for $n = 2$, the set T and, thus, the wavelet ψ depends only on a_0 and B.

11.2.3 Further Remarks

Observe that, while all the examples of composite wavelets described in this section are of MSF type, that is, the magnitude of their Fourier transform is the characteristic function of a set, there are examples of composite wavelets that are not of this type. In [8], there are examples of composite wavelets $\psi \in L^2(\mathbb{R}^2)$ where $\hat{\psi}$ is in $C^\infty(\mathbb{R}^2)$. This implies that there is a $K_N > 0$ such that $|\psi(x)| \leq K_N\,(1+|x|)^{-N}$, for any $N \in \mathbb{N}$ and, thus, ψ is well localized both in $\mathbb{R}^2$ and $\widehat{\mathbb{R}}^2$.

11.3 *AB*-Multiresolution Analysis

As in the classical theory of wavelets, it turns out that one can introduce a general framework based on a multiresolution analysis for constructing *AB*-wavelets. Let B be a countable subset of $\widetilde{SL}_n(\mathbb{Z}) = \{b \in GL_n(\mathbb{R}) : |\det b| = 1\}$ and $A = \{a^i : i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{Z})$ (notice that a is an *integral* matrix). Also assume that *a normalizes* B, that is, $a\,b\,a^{-1} \in B$ for every $b \in B$, and that the quotient group $B/(aBa^{-1})$ has finite order. Then the sequence $\{V_i\}_{i\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is an *AB-multiresolution analysis (AB-MRA)* if the following holds:

(i) $D_b\,T_k\,V_0 = V_0$, *for any* $b \in B$, $k \in \mathbb{Z}^n$,
(ii) *for each* $i \in \mathbb{Z}$, $V_i \subset V_{i+1}$, *where* $V_i = D_a^{-i}\,V_0$,
(iii) $\bigcap V_i = \{0\}$ *and* $\overline{\bigcup V_i} = L^2(\mathbb{R}^n)$,
(iv) *there exists* $\phi \in L^2(\mathbb{R}^n)$ *such that* $\Phi_B = \{D_b\,T_k\,\phi : b \in B,\, k \in \mathbb{Z}^n\}$ *is a semi-orthogonal Parseval frame for* V_0*, that is,* Φ_B *is a Parseval frame for* V_0 *and, in addition,* $D_b\,T_k\,\phi \perp D_{b'}\,T_{k'}\,\phi$ *for any* $b \neq b'$, $b, b' \in B$, $k, k' \in \mathbb{Z}^n$.

The space V_0 is called an *AB scaling space* and the function ϕ is an *AB scaling function* for V_0. In addition, if Φ_B is an orthonormal basis for V_0, then ϕ is an *orthonormal AB scaling function*.

Observe that the main difference in the definition of *AB*-MRA with respect to the classical MRA is that, in the *AB*-MRA, the space V_0 is invariant with respect to the integer translations *and* with respect to the B-dilations. On the other hand, in the classical MRA, the space V_0 is only invariant with respect to the integer translation, and, as a consequence, the set of generators of V_0 is simply of the form $\Phi = \{T_k\,\phi : k \in \mathbb{Z}^n\}$.

As in the classical MRA, let W_0 be the orthogonal complement of V_0 in V_1, that is, $W_0 = V_1 \bigcap (V_o)^\perp$. Then, $V_1 = V_0 \bigoplus W_0$ and we have the following elementary result.

Theorem 11.5. (i) *Let* $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ *be such that* $\{D_b\,T_k\,\psi^\ell : b \in B,\, \ell = 1, \ldots, L,\, k \in \mathbb{Z}^n\}$ *is a Parseval frame for* W_0*. Then* Ψ *is a Parseval frame AB-multiwavelet.*

(ii) *Let* $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$ *be such that* $\{D_b T_k \psi^\ell : b \in B, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}$ *is an orthonormal basis for* W_0. *Then* Ψ *is an orthonormal AB-multiwavelet.*

Proof. As in the classical MRA case, we define the spaces W_j as $W_j = V_{j+1} \bigcap (V_j)^\perp$, $j \in \mathbb{Z}$. It follows from the definition of AB-MRA that $L^2(\mathbb{R}^n) = \bigoplus_{j\in\mathbb{Z}} W_j$. Since $\{D_b T_k \psi^\ell : b \in B, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\}$ is a Parseval frame for W_0, then $\{D_a^i D_b T_k \psi^\ell : b \in B, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\}$ is a Parseval frame for W_i. Thus $\{D_a^i D_b T_k \psi^\ell : b \in B, i \in \mathbb{Z}, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\}$ is a Parseval frame for $L^2(\mathbb{R}^n)$.

The proof for the orthonormal case is exactly the same. □

In the situation described by the hypotheses of Theorem 11.5 (where Ψ is not only a Parseval frame for $L^2(\mathbb{R}^n)$, but is also derived from an AB-MRA), we say that Ψ is a *Parseval frame MRA AB-wavelet* or an *orthonormal MRA AB-wavelet*, respectively.

It is easy to see that all the examples of wavelets with composite dilations described in Section 11.2 are indeed examples of MRA AB-multiwavelets. In Examples 11.2 and 11.3 from Section 11.2.2, the spaces $\{V_i = L^2(S_i)^\vee : i \in \mathbb{Z}\}$ form an AB-MRA, with orthonormal AB scaling function $\phi = (\chi_U)^\vee$, where U is a parallelogram (see Fig. 11.1 and Fig. 11.2). Observe that the AB scaling function ϕ as well as the AB scaling space $V_0 = L^2(S_0)^\vee$ are the same in both examples (the other spaces V_i, $i \neq 0$, are different because $S_i = S_0 a^i$, and a is different in the two examples). Similarly, in Example 11.4 from Section 11.2.2, the spaces $\{V_i = L^2(S_i)^\vee : i \in \mathbb{Z}\}$ form an AB-MRA, with orthonormal AB scaling function $\phi = (\chi_U)^\vee$, where U is the union of two triangles (see Fig. 11.3).

It turns out that, while it is possible to construct a Parseval frame AB-wavelet using a single generator $\Psi = \{\psi\}$ (examples of such singly generated AB-wavelets can be found in [8]), in the case of *orthonormal* MRA AB-multiwavelets, multiple generators are needed in general, that is, $\Psi = \{\psi^1, \ldots, \psi^L\}$, where $L > 1$. The following result establishes the number of generators needed to obtain an orthonormal MRA AB-wavelet.

Theorem 11.6. *Let* $\Psi = \{\psi^1, \ldots, \psi^L\}$ *be an orthonormal MRA AB-multiwavelet for* $L^2(\mathbb{R}^n)$, *and let* $N = |B/aBa^{-1}|$ (= *the order of the quotient group* B/aBa^{-1}). *Assume that* $|\det a| \in \mathbb{N}$. *Then* $L = N\,|\det a| - 1$.

Observe that this result generalizes the corresponding result for the classical MRA, where the number of generators needed to obtain an orthonormal MRA A-wavelet is given by $L = |\det a| - 1$ (cf., for example, [18]). As a simple application of Theorem 11.6, let us re-examine the examples described in Section 11.2. Also observe that, if B is a finite group, then $N = |B/aBa^{-1}| = 1$, and so, in this situation, $L = |\det a| - 1$. Applying Theorem 11.6 to Example 11.2 in Section 11.2.2, we obtain that the number of generators is $L = 1$ since a is the quincunx matrix and $\det a = 2$. In the case of Example 11.3, the number of generators is $L = 3$ since $a = 2I$ and $\det = 4$. In the case of Example 11.4, where B is the two-dimensional group of shear matrices and $A = \{a^i : i \in \mathbb{Z}\}$, with $a = \begin{pmatrix} 2 & a_{1,2} \\ 0 & a_{2,2} \end{pmatrix} \in GL_2(\mathbb{Z})$, a calculation shows that $|B/aBa^{-1}| = 2|a_{2,2}|^{-1}$ and, thus, the number of generators is $L = 2|a_{2,2}|^{-1}\, 2|a_{2,2}| - 1 = 3$.

In order to prove Theorem 11.6, we need some additional notation and construction.

We recall that a $\mathbb{Z}^n$-*invariant space* (or a *shift-invariant space*) of $L^2(\mathbb{R}^n)$ is a closed subspace $V \subset L^2(\mathbb{R}^n)$ for which $T_k V = V$ for each $k \in \mathbb{Z}^n$. For $\phi \in L^2(\mathbb{R}^n) \setminus \{0\}$, we denote by $\langle \phi \rangle$ the shift-invariant space generated by ϕ, that is,

$$\langle \phi \rangle = \overline{\text{span}}\{T_k\, \phi : \, k \in \mathbb{Z}^n\}.$$

Given $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$, their *bracket product* is defined as

$$[\phi_1, \phi_2](x) \,=\, \sum_{k \in \mathbb{Z}^n} \phi_1(x-k)\, \overline{\phi_2(x-k)}. \tag{11.4}$$

The bracket product plays a basic role in the study of shift-invariant spaces. The following properties are easy to verify, and they can be found, for example, in [18, Sec. 3].

Proposition 11.7. *Let ϕ, ϕ_1, $\phi_2 \in L^2(\mathbb{R}^n)$.*

(i) *The series* (11.4) *converges absolutely a.e. to a function in $L^1(\mathbb{T}^n)$.*
(ii) *The spaces $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ are orthogonal if and only if $[\hat{\phi}_1, \hat{\phi}_2](\xi) = 0$ a.e.*
(iii) *Let $V(\phi) = \{T_k\, \phi : k \in \mathbb{Z}^n\}$. Then $V(\phi)$ is an orthonormal basis for $\langle \phi \rangle$ if and only if $[\hat{\phi}, \hat{\phi}](\xi) = 1$ a.e.*

The theory of shift-invariant spaces plays a basic role in the study of the MRA precisely because the scaling space V_0, as well as all other scaling spaces $V_i = D_a^{-i}\, V_0$, are shift-invariant spaces. In the AB-MRA, the AB-scaling space satisfies a different invariance property, and this motivates the following definition. If B is a subgroup of $\widetilde{SL}_n(\mathbb{Z})$, a $B \ltimes \mathbb{Z}^n$-*invariant space* of $L^2(\mathbb{R}^n)$ is a closed subspace $V \subset L^2(\mathbb{R}^n)$ for which $D_b\, T_k\, V = V$ for each $(b,k) \in B \ltimes \mathbb{Z}^n$, where $B \ltimes \mathbb{Z}^n$ is the semi-direct product of B and $\mathbb{Z}^n$ (that is, it is the group obtained from the set $B \times \mathbb{Z}^n$, with natural group action on $b \times \{0\}$ and $\{I_n\} \times \mathbb{Z}^n$, and with $(b,0)(I_n,k)(b,0)^{-1} = (I_n, bk)$).

For $b \in \widetilde{SL}_n(\mathbb{Z})$, we have

$$\{D_b\, T_k : \, k \in \mathbb{Z}^n\} = \{T_{k'}\, D_b : \, k' \in \mathbb{Z}^n\},$$

and, as a consequence, $D_b\, \langle \phi \rangle = \langle D_b\, \phi \rangle$ for each $\phi \in L^2(\mathbb{R}^n)$. We also have that $\widehat{\mathbb{Z}}^n b = \widehat{\mathbb{Z}}^n$ and, thus,

$$[\hat{D}_b\, \hat{\phi}_1, \hat{\phi}_2](\xi) = [\hat{\phi}_1, \hat{D}_{b^{-1}}\, \hat{\phi}_2](\xi b), \tag{11.5}$$

for each ϕ_1, $\phi_2 \in L^2(\mathbb{R}^n)$ and $\xi \in \widehat{\mathbb{R}}^n$.

Let V be a $B \ltimes \mathbb{Z}^n$-invariant space of $L^2(\mathbb{R}^n)$. The set $\Phi = \{\phi^1, \ldots, \phi^N\}$, with $N \in \mathbb{N} \cup \{\infty\}$, is a *$B \ltimes \mathbb{Z}^n$-orthonormal set of generators* for V if the set $\{D_b T_k \phi^i : (b,k) \in B \ltimes \mathbb{Z}^n, 1 \le i \le N\}$ is an orthonormal basis for V. Equivalently, we have that $[\widehat{D}_b \hat{\phi}^i, \hat{\phi}^j] = \delta_{i,j}\, \delta_{b,I_n}$ a.e. In addition, if this is the case, we have that

$$V = \bigoplus_{j=1}^{N} \overline{\mathrm{span}}\{D_b T_k \phi^j : b \in B, k \in \mathbb{Z}^n\}.$$

It follows from this observation and from the properties of shift-invariant spaces, that, if $f \in V$ and Φ is a $B \ltimes \mathbb{Z}^n$-orthonormal set of generators for V, then

$$\hat{f} = \sum_{j=1}^{N} \sum_{b \in B} [\hat{f}, \widehat{D}_b\, \hat{\phi}^j]\, \widehat{D}_b\, \hat{\phi}^j, \tag{11.6}$$

with convergence in $L^2(\mathbb{R}^n)$.

We can now show the following simple result, whose proof is similar to that of Proposition 11.1.

Proposition 11.8. *Let $\Phi = \{\phi^1, \ldots, \phi^N\}$ and $\Psi = \{\psi^1, \ldots, \psi^M\}$ be two $B \ltimes \mathbb{Z}^n$-orthonormal sets of generators for the same $B \ltimes \mathbb{Z}^n$-invariant spaces V. Then $M = N$.*

Proof. By (11.6), we have that

$$\hat{\psi}^i = \sum_{j=1}^{N} \sum_{b \in B} [\hat{\psi}^i, \widehat{D}_b\, \hat{\phi}^j]\, \widehat{D}_b\, \hat{\phi}^j,$$

with convergence in $L^2(\mathbb{R}^n)$, for each $i = 1, \ldots, M$. Thus,

$$1 = \|\psi^i\|^2 = \sum_{j=1}^{N} \|[\hat{\psi}^i, \hat{\phi}^j]\|^2_{L^2(\mathbb{T}^n)}$$

and, as a consequence,

$$M = \sum_{i=1}^{M} \|\psi^i\|^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \|[\hat{\psi}^i, \widehat{D}_b\, \hat{\phi}^j]\|^2_{L^2(\mathbb{T}^n)}. \tag{11.7}$$

On the other hand, using the same argument on ϕ^j, $1 \le j \le N$, we obtain

$$N = \sum_{j=1}^{N} \|\phi^j\|^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \|[\hat{\phi}^j, \widehat{D}_b\, \hat{\psi}^i]\|^2_{L^2(\mathbb{T}^n)}. \tag{11.8}$$

By (11.5), using the fact that B is a group, it follows that

$$\sum_{b \in B} \|[\hat{\phi}^j, \widehat{D}_b\, \hat{\psi}^i]\|^2_{L^2(\mathbb{T}^n)} = \sum_{b \in B} \|[\hat{\phi}^j, \widehat{D}_{b^{-1}}\, \hat{\psi}^i]\|^2_{L^2(\mathbb{T}^n)} = \sum_{b \in B} \|[\widehat{D}_b\, \hat{\phi}^j, \hat{\psi}^i]\|^2_{L^2(\mathbb{T}^n)},$$

and, thus, by comparing (11.7) and (11.8), it follows that $M = N$. $\square$

We are now ready to prove Theorem 11.6.

Proof of Theorem 11.6. Let $\{V_i\}_{i\in\mathbb{Z}}$, where $V_i = D_a^{-i}V_0$, be an AB-MRA, and, for each $i \in \mathbb{Z}$, let $W_i = V_{i+1} \bigcap (V_i)^{\perp}$. It follows that $L^2(\mathbb{R}^n) = \bigoplus_{i\in\mathbb{Z}} W_i$. By the definition of AB-MRA, the space V_0 is $B \ltimes \mathbb{Z}^n$-invariant and there is an AB-scaling function ϕ which is an orthonormal $B \ltimes \mathbb{Z}^n$ generator of V_0. Since a normalizes B and a is an integral matrix, it follows that

$$\begin{aligned}\{D_b\, T_k\, D_a^{-1} : k \in \mathbb{Z}^n,\, b \in B\} &= \{D_a^{-1}\, D_{aba^{-1}}\, T_{ak} \,:\, k \in \mathbb{Z}^n,\, b \in B\} \\ &\subseteq \{D_a^{-1}\, D_b\, T_k \,:\, k \in \mathbb{Z}^n,\, b \in B\}.\end{aligned}$$

Thus, the spaces V_1 and W_0 are also $B \ltimes \mathbb{Z}^n$-invariant.

The functions $\psi^1, \ldots, \psi^L$ are $B \ltimes \mathbb{Z}^n$ orthonormal generators for W_0 and so $\{\phi, \psi^1, \ldots, \psi^L\}$ are $B \ltimes \mathbb{Z}^n$ orthonormal generators for $V_1 = V_0 \bigoplus W_0$.

Next, take a complete collection of distinct representatives $\beta_0, \ldots, \beta_{N-1}$ for $B/(a\, B\, a^{-1})$, where $N = |B/aBa^{-1}|$. Thus, each $b \in B$ uniquely determines $b' \in B$ and $j \in \{0, \ldots, N-1\}$ for which $b = (a\, b'\, a^{-1})\, \beta_j$. Then

$$D_a^{-1} D_b \,\langle\phi\rangle = D_{a^{-1}b}\,\langle\phi\rangle = D_{b'}\, D_{a^{-1}} D_{\beta_j}\,\langle\phi\rangle = D_{b'}\, D_{a^{-1}}\,\langle D_{\beta_j}\phi\rangle. \tag{11.9}$$

Also, take a complete collection of distinct representatives $\alpha_0, \ldots, \alpha_{M-1}$ for $\mathbb{Z}^n/(a\,\mathbb{Z}^n)$, where $M = |\det a|$. Each $k \in \mathbb{Z}^n$ uniquely determines $k' \in \mathbb{Z}^n$ and $i \in \{0, \ldots, M-1\}$, for which $k = ak' + \alpha_i$. For any $\phi \in L^2(\mathbb{R}^n) \setminus \{0\}$, the space $D_a^{-1}\,\langle\phi\rangle$ is then the shift-invariant space generated by $\Phi = \{\phi^i = D_a^{-1}\, T_{\alpha_i}\,\phi : 0 \le i \le M-1\}$. Since D_a^{-1} is unitary, then Φ is a $\mathbb{Z}^n$-orthonormal generating set for $D_a^{-1}\,\langle\phi\rangle$ if and only if ϕ is a $\mathbb{Z}^n$-orthonormal generating set for $\langle\phi\rangle$, and this holds if and only if $[\hat{\phi}, \hat{\phi}] = 1$ a.e. Thus, if Φ is a $\mathbb{Z}^n$-orthonormal generating set for $D_a^{-1}\,\langle\phi\rangle$, we have

$$D_a^{-1}\,\langle\phi\rangle = \bigoplus_{i=0}^{M-1} \langle\phi^i\rangle. \tag{11.10}$$

By equation (11.10), we have

$$D_{a^{-1}}\,\langle D_{\beta_j}\phi\rangle = \bigoplus_{i=0}^{M-1} \langle\phi_{i,j}\rangle,$$

where $\phi_{i,j} = D_a^{-1} D_{\alpha_i} D_{\beta_j}\,\phi$. Using the last equality and (11.9), we have

$$\begin{aligned}&D_{a^{-1}}\,\overline{\text{span}}\{D_b\, T_k\,\phi : b \in B, k \in \mathbb{Z}^n\} \\ &= \bigoplus_{i=0}^{M-1}\bigoplus_{j=0}^{N-1} \overline{\text{span}}\{D_{b'}\, T_{k'}\,\phi_{i,j} : b' \in B,\, k' \in \mathbb{Z}^n,\, 0 \le i \le M-1,\, 0 \le j \le N-1\}.\end{aligned}$$

This shows that $\{\phi_{i,j} \,:\, 0 \le i \le M-1,\, 0 \le j \le N-1\}$ is a $B \ltimes \mathbb{Z}^n$ set of orthonormal generators for V_1, with NM elements. By Proposition 11.8, $NM = L+1$ and this completes the proof. □

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