EDGE DETECTION AND PROCESSING USING SHEARLETS

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ABSTRACT

It is now widely acknowledged that wavelets are not very effective in representing images containing distributed discontinuities such as edges. This paper deals with a new multiscale directional representation called the shearlet transform that has been shown to represent specific classes of images with edges optimally. Techniques based on this transform for edge detection and analysis are presented. Unlike previously developed directional filter based techniques for edge detection, shearlets provide a theoretical basis for characterizing how edges will behave in such representations. Experiments demonstrate that this novel approach is very competitive for the purpose of edge detection and analysis.

***Index Terms*—** Multidimensional digital filters, Image edge analysis, Wavelet transforms

1. INTRODUCTION

Edges are prominent features in images and their detection and analysis is a primary task in a variety of image processing applications. Edges are recognized as those points of an image u where the gradient is noticeably large and can be identified as

$$\Gamma_u(p) = \{x \in \Omega \subset \mathbb{R}^2 : |\nabla u(x)| \geq p\},$$

where p is some suitable threshold.

The main difficulty in using such a naive characterization of edges directly to design an effective edge detection scheme is its high sensitivity to noise. As a consequence, in the most successful edge detector schemes, to watch out for the interference of noise, the image is first mollified. For example, the classic Canny edge detection algorithm [1] consists of the following steps. First, the image is smoothed by convolving with a scalable Gaussian filter:

$$u_a = u * G_a, \tag{1}$$

where $G_a(x) = a^{-1}\, G(a^{-1}x)$, for $a > 0$, and $G(x) = C\, e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}^2$ is the 2-dimensional Gaussian function. This step is followed by computing the gradient of u_a, and then finding the its local maxima. Notice that as the scaling parameter a decreases, the detection of the edge location becomes more accurate. However, as a decreases, also the detector's sensitivity to noise increases. As a result, the performance of the algorithm depends heavily on the scaling factor a.

It was observed in [2, 3] that identifying the edges of an image u using Canny's algorithm is equivalent to detecting the local maxima of the wavelet transform of u, for some particular choices of the analyzing wavelet. In fact, let

$$\psi = \nabla G.$$

It turns out that, up to an appropriate normalization, the function ψ is a *wavelet* and, thus, for each image $u \in L^2(\mathbb{R}^2)$, we have the reproducing formula

$$u(x) = \int W_\psi u(a, y)\, \psi_a(x - y)\, dy,$$

where $\psi_a(y) = a^{-1}\, \psi(a^{-1}x)$, and $W_\psi u(a, y)$ is the *wavelet transform* of u, defined by

$$W_\psi u(a, y) = \int u(y)\, \psi_a(x - y)\, dy = u * \psi_a(x). \tag{2}$$

The significance of this representation is that the wavelet transform provides a space-scale decomposition of the image u, where each $u \in L^2(\mathbb{R}^2)$ is mapped into the elements $W_\psi u(a, y)$ which depend on the location $y \in \mathbb{R}^2$ and the scaling factor $a > 0$.

The following simple observation shows that the wavelet transform $W_\psi u(a, x)$ is indeed proportional to the gradient of the image u smoothed at the scale a:

$$\nabla u_a(x) = u * \nabla G_a(x) = u * \psi_a(x) = W_u(a, x).$$

Thus, finding the maxima of the gradient of the smoothed image u_a is equivalent to finding the maxima of the wavelet transform $W_u(a, x)$. The advantages of the wavelet point of view is that it allows one to take advantage of several computationally efficient algorithms available for the implementation of the wavelet transform. Furthermore, it provides a proper mathematical framework for the multiscale analysis of edges [2, 3].

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The difficulty of edge detection is emphasized when several edges are close together or cross each other as, for example, in the case of 2-dimensional projections of 3-dimensional objects. In these conditions, various limitations of the Canny algorithm or the equivalent wavelet approach become evident. For example, the isotropic Gaussian filtering causes edges running close together to be blurred into a single curve and so it becomes difficult to separate close edges. Another major problem is that the isotropic Gaussian filtering is not very accurate in detecting edge orientations. To address these difficulties one needs to better take advantage of the anisotropic nature of the edge lines and curves. In several studies, this task is attempted by replacing the scalable collection of isotropic Gaussian filters $G_a(x_1, x_2)$, $a > 0$ in (1) with a family of steerable and scalable anisotropic Gaussian filters

$$G_{a_1,a_2,\theta}(x_1, x_2) = a_1^{-1/2}\, a_2^{-1/2} R_\theta\, G(a_1^{-1}x_1, a_2^{-1}x_2),$$

where $a_1, a_2 > 0$ and R_θ is the rotation matrix by θ [4, 5, 6]. Unfortunately, it is not obvious how to design such systems in a computationally efficient way.

The approach described in this paper is based on a new multiscale transform called the *shearlet trasform*, which takes advantage of some recent developments in the theory multidimensional wavelets. Indeed, it widely acknowledged that traditional wavelets are not very effective in dealing with distributed discontinuities such as edges. This is due to the fact that traditional multidimensional wavelets are obtained as tensor products of one-dimensional ones and, as a result, they have limited directional sensitivity. While these limitations have been known for a while, only recently a proper mathematical framework has been introduced for constructing truly multidimensional multiscale systems having the ability to deal with multidimensional signal efficiently. The *curvelets* for example, introduced by Donoho and Candès [7], form a collection of analyzing waveforms defined not only as various scales and locations, like wavelets, but also at various orientations with highly anisotropic shapes and are provably optimal in approximating images with smooth edges. The *shearlets*, introduced more recently by the authors and their collaborators [8, 9], share the same optimality properties and enjoy similar geometrical properties. In addition, shearlets provide a simplified mathematical structure and an added flexibility which is particularly useful in the applications described in this paper.

In this paper, we will use the *shearlet transform* to provide an accurate and computationally efficient tool for the analysis and detection of edges. This approach can be viewed as an extension and refinement of the wavelet-based approach in [2, 3], where the isotropic wavelet transform $W_u(a, x)$ is replaced by a truly multidimensional multiscale and directional transform. As we will show, the shearlet approach exhibits a number of most useful features, including the ability to exactly identify the location and orientation of edges. In addition, the transform is based on a rigorous mathematical framework, and has efficient numerical implementations [10].

2. SHEARLET TRANSFORM

Let G be a subgroup of the group of 2×2 invertible matrices. The affine systems generated by $\psi \in L^2(\mathbb{R}^2)$ are the collections of functions of the form

$$\big\{\psi_{M,t}(x) = |\det M|^{-\frac{1}{2}}\,\psi(M^{-1}(x-t)) : t \in \mathbb{R}^2, M \in G\big\}.$$

If, for all $f \in L^2(\mathbb{R}^n)$, we have the reproducing formula:

$$f = \int_{\mathbb{R}^n}\int_G \langle f, \psi_{M,t}\rangle\, \psi_{M,t}\, d\lambda(M)\, dt,$$

where λ is a measure on G, then ψ is a *continuous wavelet*, and the *continuous wavelet transform* is the mapping

$$f \to \mathcal{W}_\psi f(M, t) = \langle f, \psi_{M,t}\rangle, \quad (M, t) \in G \times \mathbb{R}^2.$$

There are a variety of examples of wavelet transforms. The simplest case is when the matrices M are *isotropic*, that is, they have the form $a\,I$, where $a > 0$. In this situation, the continuous wavelet transform of f is simply

$$\mathcal{W}_\psi f(a, t) = a^{-1}\int_{\mathbb{R}} f(x)\, a^{-1}\psi(a^{-1}(x-t))\, dx.$$

Notice that the expression (2) is of this form.

The isotropic wavelet transform has the ability to identify the singularities of a signal. In fact, suppose that f is smooth apart from a discontinuity at a point x_0. Then, provided ψ is a “nice” continuous wavelet, the wavelet transform $\mathcal{W}_\psi f(a, t)$ decays rapidly as $a \to 0$, unless t is near x_0 [11]. Thus, $\mathcal{W}_\psi f(a, t)$ is able to resolve the *singular support* of a distribution f, that, to identify the set of points where f is not regular. This fact provides the theoretical justification for the ability of the wavelet transform to detect edges, as described in Section 1.

However, the isotropic wavelet transform is unable to provide additional information about the *geometry* of the set of singularities of f. In many situations, it is useful to not only identify the location of singularities, but also its geometrical properties, such as, for example, the orientation of a discontinuity curve. To do that, one needs to consider wavelet transforms associated to more general groups G. In particular, we will employ the *continuous shearlet transform*, defined as the mapping

$$\mathcal{SH}_\psi f(a, s, t) = \langle f, \psi_{ast}\rangle,\ a > 0, s \in \mathbb{R}, t \in \mathbb{R}^2,$$

where $\psi_{ast}(x) = |\det M_{as}|^{-\frac{1}{2}}\,\psi(M_{as}^{-1}(x-t)$, and $M_{as} = \begin{pmatrix} a & s \\ 0 & \sqrt{a}\end{pmatrix}$. Observe each matrix M_{as} can be factorized as $B_s\, A_a$, where $B_s = \begin{pmatrix} 1 & -s \\ 0 & 1\end{pmatrix}$ is a *shear matrix* and $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a}\end{pmatrix}$ is an anisotropic dilation matrix. The analyzing

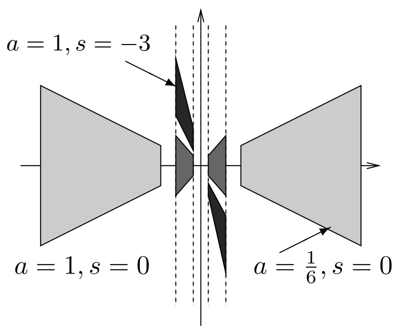


Fig. 1. Frequency support of the shearlets for different values of a and s.

function ψ has to be chosen appropriately. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_2 \neq 0$, let ψ be given by

$$\hat{\psi}(\xi) = \hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(\tfrac{\xi_2}{\xi_1}),$$

where $\psi_1 \in L^2(\mathbb{R})$ satisfies the Calderòn condition

$$\int_0^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a} = 1 \quad \text{for a.e. } \xi \in \mathbb{R},$$

and $\|\psi_2\|_{L^2} = 1$. Then, for each $f \in L^2(\mathbb{R}^2)$, we have [12]:

$$f = \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \int_0^\infty \langle f, \psi_{ast} \rangle\, \psi_{ast} \frac{da}{a^3}\, ds\, dt.$$

Notice that there are a many possible functions ψ generating such systems. In the following, we will assume that ψ is a well localized function. In particular, we assume $\hat{\psi}_1 \in C_0^\infty(\mathbb{R})$ with supp$\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and $\hat{\psi}_2 \in C_0^\infty(\mathbb{R})$ with supp$\hat{\psi}_2 \subset [-1, 1]$ and $\hat{\psi}_2 > 0$ on $(-1, 1)$. Unlike the isotropic wavelet transform $\mathcal{W}_\psi f(a, t)$ which depends only on scale and translation, the shearlet transform is a function of three variables: the *scale* a, the *shear* s and the *translation* t. Hence, the shearlets ψ_{ast} form a collection of well-localized waveforms at various scales, orientations and locations. The frequency support of the shearlets is illustrated in Figure 1.

It turns out that the Continuous Shearlet Transform can be used to very precisely describe the geometry of the singularities of a 2-dimensional function f. In fact, the decay rate of $\mathcal{SH}_\psi f(a, s, t)$ describes not only the location, but also the orientation of the singularities of f. Consider, for example, $B(x_1, x_2) = \chi_D(x_1, x_2)$, where $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. We have the following:

Theorem. *If* $t_1^2 + t_2^2 = 1$ *and* $s = \frac{t_2}{t_1}$, $t_1 \neq 0$, *then*

$$\mathcal{SH}_\psi B(a, s, t) \sim a^{\frac{3}{4}} \quad \text{as } a \to 0.$$

In all other cases, $\mathcal{SH}_\psi B(a, s, t)$ *decays rapidly as* $a \to 0$.

Similar results hold for more general domains D. These results sets the foundation for the application of the shearlet transform to the analysis of edges.

3. DETECTION OF EDGE ORIENTATION

Let u be an image containing an edge. As mentioned in Section 1, the information about the edge points of u can be analyzed from the properties of the corresponding wavelet tranform $W_\psi u(a, t)$. In particular, for a given scale a, the edge orientation can be estimated by

$$\angle(u * \nabla G_a(\tau)) = \text{atan}\left(\frac{u * \psi_a^y(\tau)}{u * \psi_a^x(\tau)}\right) \qquad (3)$$

where τ is an edge point, $\psi_a^x = \frac{\partial G_a}{\partial x}$,$\psi_a^y = \frac{\partial G_a}{\partial y}$ and G_a is a dilated Gaussian function.

In the practical numerical computations, the operator $\frac{\partial}{\partial x}$ is approximated by a finite difference. This is a significant source of inaccuracies in the estimate of the edge orientation, especially at fine scales (a small). Other methods estimate the edge orientation from the estimated edge points, after the edge points have been fitted to a curve. The main problem here is that, since the orientation detection is based on the edge points, the error in segmentation will seriously affect the orientation estimation.

The advantage of the shearlet transform is that, by decomposing an image as function of scale, location and orientation, it allows one to extract directly the information about the orientation of the edges. In the following ,we will run several numerical tests to demonstrate the superior performance of the shearlet transform in detecting edge orientations.

As a first experiment, we will measure the directional sensitivity of the shearlet transform applied to a collection of test images u_θ, representing half-planes at various orientations:

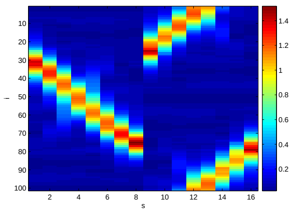
$$u_\theta(x, y) = \chi_D(x, y), \quad D = \{(x, y) : \tfrac{y}{x} \leq \tan(\theta)\}.$$

Let E be the set of the edge points of u_θ and $|E|$ be the number of elements in the set E. Then we define the directional response of shearlet transform at the edge points as:

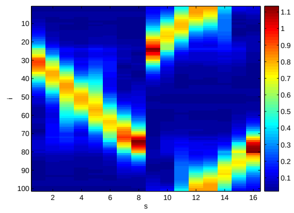
$$DR(\theta, s, a) = \frac{1}{|E|} \cdot \sum_{t \in E} |\mathcal{SH}_\psi u_\theta(a, s, t)|. \qquad (4)$$

Thanks to its directional sensitivity, for each fixed scale a, the shearlet transform will have a significant magnitude only for orientations s in a very small interval. This is illustrated in Figure 2, for two different values of the scale a.

As a second test we will to compare the accuracy in detecting the orientation of edges using the shearlet transform versus the wavelet method. As mentioned above, in the wavelet approach, the edge orientation is estimated using (3). Recall that this is equivalent to the estimation obtained from Canny's algorithm. Using the shearlet transform $\mathcal{SH}_\psi u(a, s, t)$, we will estimate the edge orientations by identifying the values of the orientation parameter s which maximize the magnitude of the transform. Results are illustrated in Figure 3 using as test image the characteristic



(a) scale $a = 1$



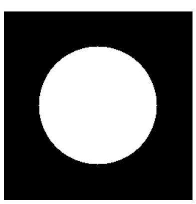
(b) scale $a = 16$

Fig. 2. Directional response of the shearlet transform $DR(\theta, s, a)$ at scales $a = 1$ and $a = 16$. The orientations of the half-planes range over $\theta = \frac{i}{100}\pi$, $i = 1, \ldots, 100$.

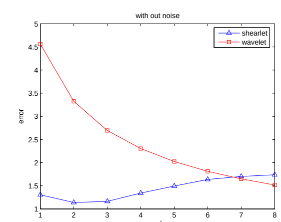
function of a disc, with and without noise. The figure shows, as a function of the scale, the average angle error defined as

$$\frac{1}{|E|} \cdot \sum_{t \in E} |\hat{\theta}(t) - \theta(t)|$$

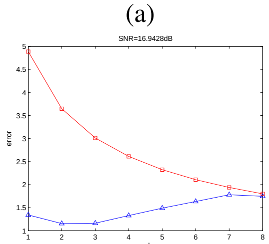
where E is the set of edge points, θ is the exact angle and $\hat{\theta}$ the estimated angle. As the figure shows, the shearlet approach significantly outperforms the wavelet method, and is extremely robust to noise.



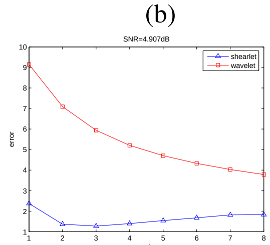
(a)



(b)



(c)



(d)

Fig. 3. *(a) Test image. (b-d) Comparison of the average error in angle estimation using the wavelet method versus the shearlet method, as a function of the scale, for various SNR.*

4. CONCLUSION

This study shows that the shearlet transform provides a multiscale directional framework which is very competitive for the purpose of edge detection and analysis. This approach is based on a simple and rigorous mathematical theory which provides a theoretical basis for characterizing the geometrical properties of edges. In particular, it provides an accurate method for extracting the information about edge orientation, even in presence of noise. This opens the door to a number of further applications in image analysis. For example, the angle function along the contour of an image is widely used as a representation of shapes. Early tests (not shown here for brevity) show the potential of the shearlet approach for this investigation.

5. REFERENCES

[1] F. J. Canny, “A computational approach to edge detection,” *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 8, no. 6, pp.679-698, 1986.

[2] S. Mallat and W. L. Hwang, “Singularity detection and processing with wavelets,” *IEEE Trans. Inf. Theory*, vol. 38, no. 2, pp.617643, 1992.

[3] S. Mallat and S. Zhong, “Characterization of signals from multiscale edges,” *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 14, no. 7, pp.710-732, 1992.

[4] P. Perona, “Steerable-scalable kernels for edge detection and junction analysis,” *Image Vis. Comput.* , vol. 10, pp. 663-672, 1992.

[5] J. Weickert, “Foundations and applications of nonlinear anisotropic diffusion filtering,” *Zeitschr. Angewandte Math. Mechan.*, vol. 76, pp.283-286, 1996.

[6] J. Geusebroek, A. W. M. Smeulders, and J. van de Weijer,“Fast Anisotropic Gauss Filtering,”, *IEEE Trans. on Imag. Processing*, vol. 8, pp.938-943, 2003.

[7] E. J. Candès and D. L. Donoho, “New tight frames of curvelets and optimal representations of objects with C^2 singularities,” *Comm. Pure Appl. Math.* vol. 56, pp.219-266, 2004.

[8] K. Guo and D. Labate, “Optimally sparse multidimensional representation using shearlets,” *SIAM J. Math. Anal.*, vol. 39, pp.298-318, 2007.

[9] D. Labate, W. Lim, G. Kutyniok, and G. Weiss, “Sparse multidimensional representation using shearlets,” *Wavelets XI*, SPIE Proc. vol. 5914, pp. 254-262, SPIE, Bellingham, WA, 2005.

[10] G. Easley, D. Labate, and W-Q. Lim “Sparse Directional Image Representations using the Discrete Shearlet Transform”, to appear in *Appl. Computat. Harmon. Anal.*, 2007.

[11] Y. Meyer, *Wavelets and Operators*, Cambridge Stud. Adv. Math. vol. 37, Cambridge Univ. Press, Cambridge, UK, 1992.

[12] G. Kutyniok and D. Labate, “Resolution of the Wavefront Set using Continuous Shearlets”, preprint 2006.