Wavelets with composite dilations and their MRA properties

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Abstract

Affine systems are reproducing systems of the form

\[ \mathcal{A}_C = \{ D_c T_k \psi^\ell : 1 \leq \ell \leq L, k \in \mathbb{Z}^n, c \in C \}, \]

which arise by applying lattice translation operators \( T_k \) to one or more generators \( \psi^\ell \) in \( L^2(\mathbb{R}^n) \), followed by the application of dilation operators \( D_c \), associated with a countable set \( C \) of invertible matrices. In the wavelet literature, \( C \) is usually taken to be the group consisting of all integer powers of a fixed expanding matrix. In this paper, we develop the properties of much more general systems, for which \( C = \{ c = a b : a \in A, b \in B \} \) where \( A \) and \( B \) are not necessarily commuting matrix sets. \( C \) need not contain a single expanding matrix. Nonetheless, for many choices of \( A \) and \( B \), there are wavelet systems with multiresolution properties very similar.

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to those of classical dyadic wavelets. Typically, $A$ expands or contracts only in certain directions, while $B$ acts by volume-preserving maps in transverse directions. Then the resulting wavelets exhibit the geometric properties, e.g., directionality, elongated shapes, scales, oscillations, recently advocated by many authors for multidimensional signal and image processing applications. Our method is a systematic approach to the theory of affine-like systems yielding these and more general features.

**Key words:** Affine systems, frames, multiwavelets, wavelets

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### 1 Introduction

There is considerable interest, both in mathematics and its applications, in the study of efficient representations of multidimensional functions. The motivation comes partly from signal processing, where such representations are useful in image compression and feature extraction, and from the investigation of certain classes of singular integral operators. For example, it was pointed out in several recent research papers that oriented oscillatory waveforms play a fundamental role in the construction of representations for multidimensional functions and signals (cf. [3], [5], [8], and articles in [20]). In particular, it was shown that, in order to be optimally sparse in a certain sense, such representations must contain basis elements with many more locations, scales, shapes and directions than the “classical” wavelets (cf. [4]).

In this paper, we introduce a new class of representation systems which have exactly the features we have described, as well as several other properties which are closely analogous to the properties of systems constructed in [7], and, therefore, for the reasons explained there as well as in [4,5,8], have great potential in applications. We call these systems **affine systems with composite dilations**, and they have the form

$$\mathcal{A}_{AB}(\Psi) = \{ D_a D_b T_k \Psi : k \in \mathbb{Z}^n, b \in B, a \in A \},$$

where $\Psi = (\psi^1, \ldots, \psi^L) \subset L^2(\mathbb{R}^n)$, $T_k$ are the translations, defined by

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$T_k f(x) = f(x-k)$, $D_a$ are the dilations, defined by $D_a f(x) = |\det a|^{-1/2} f(a^{-1} x)$, and $A, B$ are countable subsets of $GL_n(\mathbb{R})$. By choosing $\Psi, A$, and $B$ appropriately, we can make $A_{AB}(\Psi)$ an orthonormal (ON) basis or, more generally, a Parseval frame (PF) for $L^2(\mathbb{R}^n)$. In this case, we call $\Psi$ an ON $AB$-multiwavelet or a PF $AB$-multiwavelet, respectively. If the system has only one generator, that is, $\Psi = \{\psi\}$, then we use the expression wavelet rather than multiwavelet in this definition.

As we will show, the mathematical theory of these systems provides a simple and flexible framework for the construction of several classes of orthonormal bases and Parseval frames. For example, in Section 5, we construct $AB$ PF wavelets with good time-frequency decay properties, whose elements contain “long and narrow” waveforms with many locations, scales, shapes and directions. These examples have similarities to the curvelets [4] and contourlets [7], which have been recently introduced in order to obtain efficient representations of natural images. Our approach is more general and presents a simple method for obtaining several such orthonormal bases and Parseval frames that exhibit these and other geometric features. In particular, our approach extends naturally to higher dimensions and allows a multiresolution construction which is well suited to a fast numerical implementation.

The paper will be organized as follows. In Section 2 we introduce the study of $AB$ multiwavelets by constructing some examples of such systems in $L^2(\mathbb{R}^2)$. In Section 3 we examine the conditions on $A, B \in GL_n(\mathbb{R})$ that ensure the existence of $AB$ multiwavelets and present several classes of these systems for $L^2(\mathbb{R}^n)$. In Sections 4 and 5, we describe the $AB$ multiwavelets generated using a generalization of the classical MRA. Finally, in Section 6, we describe an example of a singly generated orthonormal $AB$ wavelet.

2 Example

In this paper, we shall present a variety of affine systems with composite dilations. Perhaps, the most efficient way of entering into the study of these systems is to examine in some detail a particular example of such a system.

Throughout this paper, we shall consider the points $x \in \mathbb{R}^n$ to be column vectors, i.e., $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and the points $\xi \in \hat{\mathbb{R}}^n$ (the frequency domain) to be row vectors, i.e., $\xi = (\xi_1, \ldots, \xi_n)$. A vector $x$ multiplying a matrix $a \in GL_n(\mathbb{R})$ on the right, is understood to be a column vector, while a vector $\xi$ multiplying $a$
on the left is a row vector. Thus, \( ax \in \mathbb{R}^n \) and \( \xi a \in \mathbb{R}^n \). The Fourier transform is defined as
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} \, dx,
\]
where \( \xi \in \mathbb{R}^n \), and the inverse Fourier transform is
\[
\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \xi x} \, d\xi.
\]
Let \( a = \begin{pmatrix} 2 & 0 \\ 0 & \epsilon \end{pmatrix} \), where \( \epsilon \neq 0 \), \( b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( G = \{ (b^j, k) : j \in \mathbb{Z}, k \in \mathbb{Z}^2 \} \).

Then \( G \) is a group with group multiplication:
\[
(b^j, m) (b^j, k) = (b^{j+j}, k + b^{-j} m).
\]
In particular, we have \( (b^j, k)^{-1} = (b^{-j}, -b^j k) \). The multiplication (1) is consistent with the operation that maps \( x \in \mathbb{R}^2 \) into \( b^j (x + k) \in \mathbb{R}^2 \). This is clarified by introducing the unitary representation \( \pi \) of \( G \), acting on \( L^2(\mathbb{R}^2) \), defined by
\[
(\pi(b^j, k)f)(x) = f(b^{-j}x - k) = \left( D_{b^j}^j T_k f \right)(x),
\]
for \( f \in L^2(\mathbb{R}^2) \). The observation that
\[
(D_{b^j}^j T_m) (D_{b^j}^j T_k) = (D_{b^j}^{j+j} T_{k+b^{-j}m}),
\]
where \( \ell, j \in \mathbb{Z}, k, m \in \mathbb{Z}^2 \), shows how the group operation (1) is associated with the unitary representation (2).

Let \( S_0 = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi| \leq 1 \} \) and define
\[
V_0 = L^2(S_0)^\vee = \{ f \in L^2(\mathbb{R}^n) : \text{supp} \hat{f} \subset S_0 \}.
\]
Since, for all \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^2 \), we have
\[
(\pi(b^j, k)f)^{\wedge}(\xi) = \left( D_{b^j}^j T_k f \right)^{\wedge}(\xi) = e^{-2\pi i \xi b^j k} \hat{f}(\xi b^j),
\]
and \( \xi b^j = (\xi_1, \xi_2)b^j = (\xi_1, \xi_2 + j \xi_1) \), then the action of \( b^j \) maps the vertical strip domain \( S_0 \) into itself and, thus, the space \( V_0 \) is invariant under the action of \( \pi(b^j, k) \). The same invariance property holds similarly for the vertical strips
\[
S_i = S_0 a^i = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi| \leq 2^i \},
\]
i \( \in \mathbb{Z} \), and, as a consequence, the spaces \( V_i = L^2(S_i)^\vee \) are also invariant under the action of the operators \( \pi(b^j, k) \). The spaces \( \{ V_i \}_{i \in \mathbb{Z}} \) also satisfy the basic MRA properties: (1) \( V_i \subset V_{i+1}, i \in \mathbb{Z} \); (2) \( D_{a^{-1}} V_0 = V_i \); (3) \( \bigcap V_i = \{0\} \); and (4)
\[ \bigcup V_i = L^2(\mathbb{R}^n) . \]

The complete definition of an MRA includes the assumption that \( V_0 \) is generated by the integer translates of a \( \phi \in V_0 \), called the scaling function, and that these translates \( \{ T_k \phi : k \in \mathbb{Z} \} \) are an orthonormal basis of \( V_0 \). In our situation, as we will discuss later on, there is an analogous property that will replace the ‘scaling’ property.

Let \( A = \{ a^i : i \in \mathbb{Z} \} \) and \( B = \{ b^j : j \in \mathbb{Z} \} \), and \( W_0 \) be the orthogonal complement of \( V_0 \) in \( V_1 \), that is, \( V_1 = V_0 \oplus W_0 \). We shall now show how to construct an ON \( AB \)-multiwavelet generated by three mutually orthogonal functions \( \psi_1, \psi_2, \psi_3 \in W_0 \) of norm 1. It will be convenient to work in the Fourier domain. Thus, \( \hat{V}_1 = \hat{V}_0 \oplus \hat{W}_0 \) and, consequently, \( \hat{W}_0 = L^2(R_0) \), where \( R_0 = S_1 \setminus S_0 = \{ \xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 < |\xi_1| \leq 2 \} \). We begin by constructing a particular orthonormal basis of \( W_0 \) that it is mapped into itself by the representation \( \pi \). To do this, define the following subsets of \( R_0 = S_1 \setminus S_0 \):

\[ I_1 = I_1^+ \cup I_1^-, \ I_2 = I_2^+ \cup I_2^-, \ I_3 = I_3^+ \cup I_3^- , \]

where

\[ I_1^+ = \{ \xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 < \xi_1 \leq 2, 0 \leq \xi_2 < 1/2 \} , \]

\[ I_2^+ = \{ \xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 < \xi_1 \leq 2, 1/2 \leq \xi_2 < 1 \} , \]

\[ I_3^+ = \{ \xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 < \xi_1 \leq 2, 1 \leq \xi_2 < \xi_1 \} , \]

and \( I_\ell^- = \{ \xi \in \hat{\mathbb{R}}^2 : -\xi \in I_\ell^+ \} , \ \ell = 1, 2, 3 \). These sets are shown in Figure 1. We then define \( \psi_\ell , \ \ell = 1, 2, 3 \) by setting \( \hat{\psi_\ell} = \chi_{I_\ell} \), \( \ell = 1, 2, 3 \). Observe that each set \( I_\ell \) is a fundamental domain of \( \mathbb{Z}^2 \), that is, the functions \( \{ e^{2\pi i k} : k \in \mathbb{Z} \} \)
restricted to $I_\ell$, form an orthonormal basis of $L^2(I_\ell)$. It follows that the collection
\[ \{ e^{2\pi i \xi k} \hat{\psi}^\ell(\xi) : k \in \mathbb{Z}^2 \} \]
is an orthonormal basis of $L^2(I_\ell)$, $\ell = 1, 2, 3$. A simple direct calculation shows that the sets $\{ I_\ell b^j : j \in \mathbb{Z}, \ell = 1, 2, 3 \}$ are a partition of $R_0$, that is,
\[ \bigcup_{\ell=1}^3 \bigcup_{j \in \mathbb{Z}} I_\ell b^j = R_0, \]
where the union is disjoint. It follows that the collection
\[ \{ e^{2\pi i \xi k} \hat{\psi}^\ell(\xi b^j) : k \in \mathbb{Z}^2, j \in \mathbb{Z}, \ell = 1, 2, 3 \} \quad (4) \]
is an orthonormal basis of $L^2(R_0)$ and, thus, by taking the inverse Fourier transform of (4), we have that
\[ \{ \pi(b^j, k) \psi^\ell : k \in \mathbb{Z}^2, j \in \mathbb{Z}, \ell = 1, 2, 3 \} \quad (5) \]
is an orthonormal basis of $W_0 = L^2(R_0)$. Notice that, since, for each $j \in \mathbb{Z}$ fixed, $b^j$ maps $\mathbb{Z}^2$ into itself, the collection $\{ e^{2\pi i \xi k} : k \in \mathbb{Z}^2 \}$ is equal to the collection $\{ e^{2\pi i \xi k} : k \in \mathbb{Z}^2 \}$.

Observe that the number of generators, three, of the orthonormal basis (5) of $W_0$ is independent of the choice of the functions $\psi^\ell$. That is, if
\[ \{ \pi(b^j, k) \phi^\ell : k \in \mathbb{Z}^2, j \in \mathbb{Z}, \ell = 1, \ldots, L \}, \]
for some functions $\phi^\ell \in L^2(\mathbb{R}^2)$, is an orthonormal basis of $L^2(R_0)$, then $\ell$ must range through the set $\{1, 2, 3\}$. This is a consequence of the following general result:

**Proposition 1** Let $G$ be a countable set and, for each $u \in G$, let $T_u$ be a unitary operator acting on a Hilbert space $\mathcal{H}$. Assume that, for each $T_u$, there is a unique $u^* \in G$ such that $T_{u^*} = T_u^*$. Suppose $\Phi = \{ \phi^1, \ldots, \phi^N \}$, $\Psi = \{ \psi^1, \ldots, \psi^M \} \subset \mathcal{H}$, where $N, M \in \mathbb{N} \cup \{\infty\}$. If $\{ T_u \phi^k : u \in G, 1 \leq k \leq N \}$ and $\{ T_u \psi^i : u \in G, 1 \leq i \leq M \}$ are each orthonormal bases for $\mathcal{H}$, then $N = M$.

**Proof.** It follows from the assumptions that, for each $1 \leq k \leq N$:
\[ \|\phi^k\|^2 = \sum_{u \in G} \sum_{i=1}^M |\langle \phi^k, T_u \psi^i \rangle|^2. \]

Thus, using the unitary property of $T_u$, we have:
\[ N = \sum_{k=1}^N \|\phi^k\|^2 = \sum_{k=1}^N \sum_{u \in G} \sum_{i=1}^M |\langle \phi^k, T_u \psi^i \rangle|^2 \]
\[
\sum_{i=1}^{M} \sum_{u^* \in G} \sum_{k=1}^{N} |(T_{u^*} \phi^k, \psi^i)|^2 = \sum_{i=1}^{M} \|\psi^i\|^2 = M. \quad \square
\]

In order to obtain the desired ON \(AB\)–affine system for \(L^2(\mathbb{R}^2)\), we apply the dilations \(D_{a^i}\), \(i \in \mathbb{Z}\) to the orthonormal system (5). This is easily seen in the Fourier domain, since the action of these dilations on the region \(R_0\) generates the sets

\[R_i = R_0 a^i = \{ \xi = (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 2^i < |\xi_1| \leq 2^{i+1} \},\]

and we have that \(\bigcup_{i \in \mathbb{Z}} R_i = \mathbb{R}^2\), where the union is disjoint. Since the dilations \(D_{a^i}\) are unitary operators, they map an orthonormal basis into an orthonormal basis and, thus, for each \(i \in \mathbb{Z}\), the set \(\{D_{a^i} \pi(b^j, k) \psi^\ell : k \in \mathbb{Z}^2, j, \ell \in \mathbb{Z}, \ell = 1, 2, 3\}\) is an orthonormal basis of \(L^2(R_i)\). Since the spaces \(L^2(R_i)\) (and thus the spaces \(W_i\)) are mutually orthogonal, it follows that the system

\[
\{D_{a^i} \pi(b^j, k) \psi^\ell : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, \ell = 1, 2, 3\} = \{D_{a^i} D_{a^j} T_k \psi^\ell : k \in \mathbb{Z}^2, i, j \in \mathbb{Z}, \ell = 1, 2, 3\}
\]

is an orthonormal basis of \(L^2(\mathbb{R}^2) = \bigoplus_{i \in \mathbb{Z}} W_i\), that is, \(\Psi = \{\psi^1, \psi^2, \psi^3\}\) is an ON \(AB\)–multiwavelet.

The number of generators of this ON \(AB\)–multiwavelet is fixed. Indeed, by Proposition 1, if we could replace \(\Psi\) in (5) by a \(\Phi = \{\phi^1, \ldots, \phi^L\}\), then \(L = 3\), and this applies to (6) as well. As we will show later on, the Fourier transform of the multiwavelets \(\hat{\phi}^\ell\) need not be characteristic functions.

Recall that a countable family \(\{e_j : j \in J\}\) of elements in a separable Hilbert space \(\mathcal{H}\) is a \textbf{frame} if there exist constants \(0 < A \leq B < \infty\) satisfying

\[A \|v\|^2 \leq \sum_{j \in J} |\langle v, e_j \rangle|^2 \leq B \|v\|^2\]

for all \(v \in \mathcal{H}\). A frame is \textbf{tight} if \(A\) and \(B\) can be chosen so that \(A = B\), and is a \textbf{Parseval frame} (PF) (also called \textbf{normalized tight frame}) if \(A = B = 1\). Thus, if \(\{e_j : j \in J\}\) is a Parseval frame in \(\mathcal{H}\), then

\[\|v\|^2 = \sum_{j \in J} |\langle v, e_j \rangle|^2\]

for each \(v \in \mathcal{H}\). This is equivalent to the reproducing formula

\[v = \sum_{j \in J} \langle v, e_j \rangle e_j\]

(7)
for all \( v \in \mathcal{H} \), where the series in (7) converges in the norm of \( \mathcal{H} \). Equations (7) shows that a Parseval frame provides a basis-like representation. In general, however, a PF need not be a basis. We refer the reader to [9,14] for more details about frames.

We will now show how to construct a PF of \( L^2(\mathbb{R}^2) \) by modifying the construction that we described above. Let \( T = T^+ \cup T^- \), where \( T^+ \) is the trapezoid with vertices \((1/2,0)\), \((1/2,1/2)\), \((1,0)\) and \((1,1)\), and \( T^- = \{ \xi \in \mathbb{R}^2 : -\xi \in T^+ \} \); and let \( R = S_0 \setminus S_- = \{ \xi = (\xi_1,\xi_2) \in \mathbb{R}^2 : 1/2 < |\xi_1| \leq 1 \} \). This is illustrated in Figure 2. A direct computation shows that \( \bigcup_{j \in \mathbb{Z}} T b^j = R \), where the union is disjoint. It follows from the Plancherel theorem (using the fact that \( T \) is contained inside a fundamental domain) that the function \( \chi_T(\xi) \) satisfies \( \sum_{k \in \mathbb{Z}^2} |\langle \hat{f}, e^{2\pi i \langle \cdot, k \rangle} \chi_T \rangle|^2 = \| \hat{f} \|^2 \) for all \( \hat{f} \in L^2(R) \), and thus the collection

\[
\{ D^j_k e^{2\pi i \xi k} \chi_T(\xi) : k \in \mathbb{Z}^2, j \in \mathbb{Z} \}
\]

is a Parseval frame of \( L^2(T) \). Similarly to the construction above, we have that \( \bigcup_{i \in \mathbb{Z}} R a^i = \mathbb{R}^2 \), where the union is disjoint, and so it follows that the set

\[
\{ D^j_k \pi(b^j, k) \psi : k \in \mathbb{Z}^2, i, j \in \mathbb{Z} \} = \{ D^j_k D^i_k T \psi : k \in \mathbb{Z}^2, i, j \in \mathbb{Z} \}
\]

where \( \psi = (\chi_T)^\vee \) is a PF of \( L^2(\mathbb{R}^2) = \bigoplus_{i \in \mathbb{Z}} L^2(R a^i)^\vee \), that is, \( \psi \) is a Parseval frame \( AB \)-wavelet.

Observe that, unlike the case of ON \( AB \) multiwavelets that we constructed above, this Parseval frame \( AB \) wavelet is singly generated \( (\psi = (\chi_T)^\vee) \). It is not hard to see, however, that, by modifying the function \( \psi \), one can obtain singly generated ON \( AB \) wavelets (cf. [11]). It is important to point out that, as we will discuss in Sections 5 and 6, those singly generated ON \( AB \) wavelets
are not of MRA type. These remarks make clear that the construction of AB Parseval frames is simpler than the corresponding construction of ON AB multiwavelets. Because of this fact, and because Parseval frames are as effective as ON bases in many applications, in the following we will concentrate mostly on the construction of Parseval frames AB wavelets, that are not necessarily orthonormal bases.

We end this section by stating some basic properties of the translation and dilation operators, that will be used throughout the paper.

**Proposition 2** Let

\[ G = \{ U = D_a T_y : (a, y) \in GL_n(\mathbb{R}) \times \mathbb{R}^n \}. \]

\( G \) is a subgroup of the group of unitary operators on \( L^2(\mathbb{R}^n) \) which is preserved by the action of the operator \( U \mapsto \hat{U} \), where \( \hat{U} \hat{f} = (Uf)^\wedge \). In particular, we have:

(i) \( D_a T_y = T_y D_a \);
(ii) \( D_{a_1} D_{a_2} = D_{a_1 a_2} \), for each \( a_1, a_2 \in GL_n(\mathbb{R}) \);
(iii) for \( U = D_a T_y \), then \( \hat{U} = \hat{D}_a M_{-y} \), where \( \hat{D}_a \hat{f}(\xi) = |\det a|^{1/2} \hat{f}(\xi a) \);
(iv) for a measurable set \( S \subset \mathbb{R}^n \) and \( L^2(S) = \{ f \in L^2(\mathbb{R}^n) : \text{supp} f \subseteq S \} \), we have: \( \hat{D}_a L^2(S) = L^2(Sa^{-1}) \).

### 3 The admissibility condition

In Section 2, we have examined some special cases of **affine systems** associated with the lattice \( \mathbb{Z}^n \) in \( \mathbb{R}^n \), a countable collection \( \mathcal{C} \subset GL_n(\mathbb{R}) \) containing the \( n \times n \) identity matrix \( I_n \), and a set \( \Psi = \{ \psi^1, \ldots, \psi^L \} \subset L^2(\mathbb{R}^n) \), having the form

\[ \mathcal{A}_C(\Psi) = \{ D_c T_k \Psi : c \in \mathcal{C}, k \in \mathbb{Z}^n \}. \] (1)

Our main concern here is to establish conditions on \( \mathcal{C} \) that guarantee the existence of a finite set of functions \( \Psi \) such that \( \mathcal{A}_C(\Psi) \), given by (1), is either an orthonormal basis or a Parseval frame for \( L^2(\mathbb{R}^n) \). When this is the case, we say that \( \Psi \) is an **orthonormal (ON) \( \mathcal{C} \)-multiwavelet** or a **Parseval frame \( \mathcal{C} \)-multiwavelet**, respectively, for \( L^2(\mathbb{R}^n) \). More generally, when \( S \subset \mathbb{R}^n \) has positive Lebesgue measure and \( Sc = S \), for each \( c \in \mathcal{C} \), we say that \( \Psi \) is an ON or a Parseval frame \( \mathcal{C} \)-multiwavelet for \( L^2(S)^\wedge \), if \( \mathcal{A}_C(\Psi) \) is an ON basis or a Parseval frame, respectively, for \( L^2(S)^\wedge \). For example, in the construction of Section 2, we consider affine systems on \( L^2(S_i)^\wedge \), \( i \in \mathbb{Z} \), where the strip domains \( S_i \subset \mathbb{R}^2 \) are invariant with respect to the matrices \( b \in B \).

It is an open problem to give necessary and sufficient conditions on \( \mathcal{C} \) for
which \( C \) multiwavelets for \( L^2(S)^\vee \) exist. In all known cases where they exist, \( C \) satisfies a geometric condition that we call the tiling property. Namely, if there exist measurable subsets \( R_1, \ldots, R_L \) of \( S \) such that a.e. \( \xi \in S \subset \mathbb{R}^n \) uniquely determines an index \( 1 \leq i \leq L, \eta \in R_i, \) and a \( c \in C, \) for which \( \xi = \eta c^{-1}, \) we say that the sets \( \{ R_\ell : \ell = 1, \ldots, L \} \) are \( S \)-tiling sets for the dilation set \( C^{-1}. \) Equivalently, we have that

\[
S = \bigcup_{c \in C} \bigcup_{1 \leq \ell \leq L} R_\ell c^{-1}, \tag{2}
\]

where the union is disjoint in measure. If \( S = \mathbb{R}^n, \) we simply say that the sets \( \{ R_\ell : \ell = 1, \ldots, L \} \) are tiling sets for \( C^{-1}. \) The property (2) ensures that \( L^2(S)^\vee \) is the orthogonal direct sum

\[
L^2(S)^\vee = \bigoplus_{c \in C, 1 \leq \ell \leq L} L^2(R_\ell c^{-1})^\vee.
\]

Therefore, for \( \Psi = \{ \psi^1, \ldots, \psi^L \}, \) where \( \psi^\ell = (\chi_{R_\ell})^\vee, \) the system \( A_C(\Psi), \) given by (1), is a Parseval frame for \( L^2(S)^\vee \) if and only if, for each \( 1 \leq i \leq L, \) the collection

\[
\{(T_k \psi^i)^\wedge = e^{2\pi i k \cdot R_i} : k \in \mathbb{Z}^n\}
\]

is a Parseval frame for \( L^2(R_i)^\vee. \) By an elementary Fourier series argument, this occurs precisely if the sets \( R_1, \ldots, R_L \) satisfy

\[
(R_\ell + k) \cap R_\ell = 0, \quad \text{for} \ k \in \hat{\mathbb{Z}}^n \setminus \{0\}, 1 \leq \ell \leq L, \quad \text{up to sets of measure zero,}
\]

in which case we say that the sets \( \{ R_\ell : \ell = 1, \ldots, L \} \) are packing sets for \( \mathbb{Z}^n \) translations.\(^3\) Observe that this condition implies that the measure of each set \( R_\ell \) cannot be larger than one. Therefore we have the following:

**Proposition 3** Let \( \Psi = \{ \psi^1, \ldots, \psi^L \} \subset L^2(S)^\vee, \) where \( \psi^\ell = (\chi_{R_\ell})^\vee, \) for \( 1 \leq \ell \leq L. \) \( \Psi \) is a Parseval frame \( C \)-multiwavelet for \( L^2(S)^\vee \) if and only if (2) and (3) hold.

Whenever \( \Psi \) is of the form given by Proposition 3, we say that \( \Psi \) is a tiling \( C \)-multiwavelet of \( L^2(S)^\vee. \) In Section 5, we show how tiling \( C \)-multiwavelets can be smoothed off to obtain more general \( C \)-multiwavelets. It is an open problem whether any \( C \)-multiwavelet arises by modifying a tiling one.

Note that in the example of Parseval frame \( AB \)-wavelet from Section 2, we construct a set \( T \subset \hat{\mathbb{R}}^2 \) having the properties:

\(^3\) Recall that in Section 2 we introduced the notion of “fundamental domain”. Observe that a packing set for \( \mathbb{Z}^n \) translations is a subset of a fundamental domain for \( \mathbb{Z}^n. \)
\[ (i) \bigcup_{i,j} T (a^i b^j)^{-1} = \mathbb{R}^2 \setminus \{(0, \xi_2) : \xi_2 \in \mathbb{R}\} \]

\[ (ii) (T + k) \cap T = \emptyset \quad \text{for all } k \in \mathbb{Z}^2 \setminus \{0\} \]

This shows that equation (2) and (3) are satisfied, and so it follows that 
\[ \psi = (\chi_T)^\vee \] is a PF \( \mathcal{C} \)-wavelet for \( L^2(\mathbb{R}^2) \), where \( \mathcal{C} = \{a^i b^j : i, j \in \mathbb{Z}\} \).

The set \( \mathcal{C} \) is called \( S \)-admissible if tiling multiwavelets for \( L^2(S)^\vee \) exist. In case \( S = \mathbb{R}^n \), we will simply say admissible (rather than \( \hat{\mathbb{R}}^n \)-admissible). Observe that there are no known examples of dilations sets \( \mathcal{C} \) for which \( \mathcal{C} \)-wavelets exist and that are not \( S \)-admissible.

In the following, we will briefly examine the relationship between the notion of admissibility that we have just introduced, and the theory of continuous wavelets (Section 3.1). Next, in Section 3.2 we will show that the admissibility condition is closely related to a condition that we call local admissibility. In Section 3.3 we examine the admissibility for dilation sets of the form \( \mathcal{C} = a B \), and look at two types of examples unlike those in Section 2. In Section 3.4 we give a complete discussion of the theory that generalizes the examples in Section 2.

### 3.1 Connection to the theory of continuous multiwavelets.

For \( \mathcal{C} \) and \( S \) defined as in the previous section, we say that \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(S)^\vee \) is a continuous \( \mathcal{C} \)-multiwavelet if

\[ \|f\|^2 = \sum_{\ell=1}^L \sum_{c \in \mathcal{C}} \int_{\mathbb{R}^n} |\langle f, D_c T_y \psi^\ell \rangle|^2 dy \quad \text{(4)} \]

for all \( f \in L^2(S)^\vee \). By a trivial extension of an argument in [16], one shows that \( \Psi \) satisfies (4) if and only if it satisfies the Calderon equation:

\[ \sum_{\ell=1}^L \sum_{c \in \mathcal{C}} |\hat{\psi^\ell}(\xi c)|^2 = 1 \quad \text{for a.e. } \xi \in S. \quad \text{(5)} \]

It is easy to see that every tiling \( \mathcal{C} \)-multiwavelet is also a continuous \( \mathcal{C} \)-multiwavelet. In fact, if \( \Psi = \{\psi^1, \ldots, \psi^L\} \), where \( \psi^\ell = (\chi_{R_\ell})^\vee \) and the sets \( \{R_\ell : 1 \leq \ell \leq L\} \) satisfy equations (2) and (3), then (5) is immediately satisfied. More generally, it is shown in [13] that, when \( \mathcal{C} \) satisfies a technical property called the local integrability condition (LIC), then the Calderon equation is one of a family of equations characterizing \( \mathcal{C} \)-multiwavelets for \( L^2(\mathbb{R}^n) \). The LIC is satisfied, for example, when \( \mathcal{C} \) is of the form \( \mathcal{C} = \{a^i : i \in \mathbb{Z}\} \) where \( a \in GL_n(\mathbb{R}) \) is an expanding matrix (that is, all the eigenvalues \( \lambda \) of \( a \) satisfy \( |\lambda| > 1 \)). Finally, observe that there are no known examples of
C–multiwavelets which are not continuous C–multiwavelets, i.e., do not satisfy the Calderón equation.

3.2 The local admissibility condition.

As above, let $C \subset GL_n(\mathbb{R})$ be a countable set containing the identity matrix $I$. We say that $C$ is **locally admissible** if, for a.e. $\xi \in \mathbb{R}^n$, there is an open neighborhood $U$ of $\xi$ such that, for $c_1, c_2 \in C$ with $c_1 \neq c_2$, we have that $U c_1^{-1} c_2 \cap U = \emptyset$. In particular, this means that the set of points $\{\xi c^{-1} : c \in C\}$ is discrete in the topology of $\mathbb{R}^n$. Since any open neighborhood of $\xi$ has the same property, we can assume that $U$ is contained in a cube of side 1 centered at $\xi$. Then, for $S = \cup_{c \in C} U c^{-1}$, the set $C$ is $S$–admissible and $\chi_S^\vee$ is a tiling wavelet for $L^2(S)^\vee$.

Under certain assumptions on $C$, one can take $S = \mathbb{R}^n$. Consider for example the situation where $C = \{a^i : i \in \mathbb{Z}\}$ and $a \in GL_n(\mathbb{R})$ is an expanding matrix. $C$ is clearly locally admissible. Let $U \subset [-1/2, 1/2]^n \subset \mathbb{R}^n$ be an open neighborhood of the origin (this implies that $U$ is a packing set for $\mathbb{Z}^n$ translations). Since $a$ is expanding, it follows that $U a^{-1} \subset U$. Therefore, if we let $T = U \setminus (U a^{-1})$, then $T$ is a tiling set for the set of dilations $C$ (observe that $C$ is a group and so $C = C^{-1}$). This shows that a tiling $C$–wavelet for $L^2(\mathbb{R}^n)$ exists. The following section elaborates this situation further, by showing an example of a dilation set that is not locally admissible.

3.2.1 Example of a non-admissible dilation set.

Consider the set $C = \{2^i 3^j : i, j \in \mathbb{Z}\}$. This set is not locally admissible in view of the fact that $\ln 3/\ln 2$ is irrational and so $\{\ln(2^i 3^j) : i, j \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Using this fact, the following argument will show that no $C$–wavelets for $L^2(\mathbb{R})$ exist.

In fact, if such a wavelet $\psi$ exists, then it satisfies the Calderón condition

$$\sum_{i,j \in \mathbb{Z}} |\hat{\psi}(2^i 3^j \xi)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}.$$  

We claim that no such $\psi$ exists. If it did, then we could find an $n \in \mathbb{Z}$ and a measurable set $R \subset [n, n+1]$ of positive measure such that $|\hat{\psi}(\xi)| \geq \delta$ for some $\delta > 0$, for all $\xi \in R$. Fix such $n$ and $\delta$. Since $\|\psi\| \leq 1$, it follows that $\int_{n+1}^{n+1} |\hat{\psi}(\xi)|^2 d\xi \geq \delta^2 |R|$ and so

$$\delta^2 \leq \frac{1}{|R|} \leq \frac{|n| + 1}{|R|}.$$  

(6)
It is easy to see that there is a countably infinite set \( P \) of elements \( p \) of the form \( p = 2^i 3^j, i, j \in \mathbb{Z} \) such that

\[
1 < p < 1 + \frac{\delta^2 |R|}{2(|n| + 1)}. \tag{7}
\]

From (6) and (7) we have that \( p < 1 + |R|/(2|R|) = 3/2 \), and, thus, \( 1/p > 1/2 \).

Using this observation, (7) and the fact that \( \|\psi\| \leq 1 \), we have:

\[
\int_n^{n+1} |\hat{\psi}(p\xi)|^2 d\xi = \frac{1}{p} \int_{pn}^{p(n+1)} |\hat{\psi}(\xi)|^2 d\xi \\
\geq \frac{1}{2} \left( \int_n^{n+1} |\hat{\psi}(\xi)|^2 d\xi - \int_n^{pn} |\hat{\psi}(\xi)|^2 d\xi \right) \\
\geq \frac{1}{2} \left( \delta^2 |R| - n(p - 1) \right) \\
\geq \frac{1}{2} \left( \delta^2 |R| - n\delta^2 |R|/(2|n| + 1) \right) \geq \frac{\delta^2 |R|}{4}. \tag{8}
\]

Thus, using (8) and the Calderón condition we have:

\[
1 = \int_n^{n+1} \sum_{i,j \in \mathbb{Z}} |\hat{\psi}(2^i 3^j \xi)|^2 d\xi \\
\geq \int_n^{n+1} \sum_{p \in P} |\hat{\psi}(p\xi)|^2 d\xi \geq \frac{\delta^2 |R|}{4} \sum_{p \in P} 1 = \infty.
\]

This contradicts the Calderón condition. It is easy to see that the same argument applies to any \( A = \{a^i b^j : i, j \in \mathbb{Z} \} \), with \( a, b \in \mathbb{Z} \setminus \{0, 1\} \) relatively prime. The same argument also applies if one replaces \( L^2(\mathbb{R}) \) by \( L^2(S)^\vee \), where \( S \subset \hat{R} \) is a set of positive measure.

Consider, on the other hand, \( C' = \left\{ \begin{pmatrix} 2^i & 0 \\ 0 & 3^j \end{pmatrix} : i, j \in \mathbb{Z} \right\} \). This set is locally admissible, and an argument similar to the one described above, where \( C = \{a^i : i \in \mathbb{Z} \} \) and \( a \) is an expanding matrix, shows that tiling \( C' \)-wavelets for \( L^2(\mathbb{R}^2) \) exist.

### 3.3 Admissibility condition. The AB case.

If \( B \subset GL_n(\mathbb{R}) \) is \( S \)-admissible and \( c \in GL_n(\mathbb{R}) \), then \( cB \) is \( Sc^{-1} \)-admissible since the unitary operator \( D_c \) maps the PF \( A_B(\psi) \) for \( L^2(S)^\vee \) onto the PF \( A_{cB}(\psi) \) for \( L^2(Sc^{-1})^\vee \), where \( \psi \) is a PF \( cb \)-wavelet for \( L^2(Sc^{-1})^\vee \). In particular, this holds for \( c = b^{-1} \), where \( b \in B \). In this case, \( b^{-1}B \) is still \( S \)-admissible, since \( Sb = S \), and thus there is no loss of generality in assuming \( I_n \in B \). We will be especially interested in the situation where \( B \) is \( S \)-admissible and there
is a countable set $A \subset GL_n(\mathbb{R})$ for which $S$ is a tiling set for $A$. Then

$$L^2(\mathbb{R}^n) = \bigoplus_{a \in A} L^2(S a^{-1})^\vee = \bigoplus_{a \in A} D_a \left( L^2(S) \right)^\vee,$$

and it follows that the set $C = AB = \{ a b : a \in A, b \in B \}$ is admissible, and $\psi$ is a PF $AB$–wavelet whenever $\psi$ is a $B$–wavelet for $L^2(S)^\vee$. It is clear that a similar approach holds for multiwavelets $\Psi \subset L^2(\mathbb{R}^n)$. A particular instance of this phenomenon was illustrated in Section 2, where $A = \{ a^i : i \in \mathbb{Z} \}$, with $a = \begin{pmatrix} 2 & 0 \\ 0 & \epsilon \end{pmatrix}$, and $B = \{ b^j : j \in \mathbb{Z} \}$, with $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Observe that in these examples the right action of $A$ doubles the first coordinate. On the other hand, the action of $A$ on the second coordinate is irrelevant, since the right action of $B$ leaves the first coordinate fixed and uses the first coordinate to control the second one.

In the following sections, will show that there are many possible choices for $A$ and $B$, and that they do not have to be subgroups of $GL_n(\mathbb{R})$. Unifying all these examples of admissible $AB$–multiwavelets that we are going to construct is a not necessarily linear change of coordinates map $\phi(t, s) a = \phi(t a', s')$, where $a \in A'$ and $A'$ is a set (or a group) of operators on $\mathbb{R}^k$ that admits tiling sets for the $A'$ dilations. The action of $A$ on the coordinate $s \in \mathbb{R}^{n-k}$ is irrelevant. On the other hand, the action of $B$ will leave the coordinate $t \in \mathbb{R}^k$ invariant: for $b \in B$ we have that $\phi(t, s) b = \phi(t, \sigma_i(s,b))$, for some transformation $\sigma_i(\cdot, b)$ on $\mathbb{R}^{n-k}$. As $t$ varies over a compact set $K \subset \mathbb{R}^k$, we will be able to construct a set $R$ which is an $S$–tiling set for the $B$ dilations, where $S$ is the strip domain $K \times \mathbb{R}^{n-k}$. This general procedure will be illustrated in Sections 3.3.1 and 3.3.2, for the case of spherical and hyperbolic coordinates, respectively. Next, in Section 3.4, we consider the linear coordinate systems, by generalizing the examples in Section 2.

### 3.3.1 Orthogonal $AB$–multiwavelets.

Perhaps the simplest class of admissible $AB$–multiwavelets is obtained when $B$ is a finite group. Since $B$ is conjugate to a subgroup of the orthogonal group $O_n(\mathbb{R})$ (i.e., given any finite group $B$, there is a $P \in GL_n(\mathbb{R})$ and a subgroup $\tilde{B} \subset O_n(\mathbb{R})$ such that $PBP^{-1} = \tilde{B}$), without loss of generality, we may assume that $B \subset O_n(\mathbb{R})$. Let $S_0 \subset \mathbb{R}^n$ be a compact region, starlike with respect to the origin, with the property that $B$ maps $S_0$ into itself. In many situations, one can find a lattice $L \subset \mathbb{R}^n$ and a region $U_0 \subset S_0$ such that $U_0$ is both a $S_0$–tiling set for the $B$ dilations and a packing set for the $\Lambda$ translations (i.e., $(U_0 + \lambda) \cap U_0 = 0$, for $\lambda \in \Lambda \setminus \{0\}$), where $\Lambda = \{ \lambda \in \mathbb{R}^n : \lambda l \in \mathbb{Z}, \forall l \in L \}$.
is the lattice dual to $L$. Then

$$\Phi_B = \{D_b T_l (\chi_{U_0})^\vee : b \in B, l \in L\}$$

is a PF for $L^2(S_0)^\vee$. Next suppose that $A = \{a^i : i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$ is expanding, $a B a^{-1} = B$ and $S_0 \subseteq S_0 a = S_1$. These assumptions imply that each region $S_i = S_0 a^i$, $i \in \mathbb{Z}$, is $B$-invariant and the family of disjoint regions $S_{i+1} \setminus S_i$, $i \in \mathbb{Z}$, tile $\mathbb{R}^n$. Thus, one can decompose $L^2(\mathbb{R}^n)$ as in (9). Since $B$ is finite, there exist many choices of a measurable set $R \subset S_1 \setminus S_0$ for which $R$ is a $(S_1 \setminus S_0)$-tiling set for the $B$ dilations. Since $a$ is expanding, we can always take $S_0$ to be contained in a small neighborhood of the origin, and thereby ensuring that $R$ is a packing set for the $\Lambda$ translations. Then

$$\Psi_{AB} = \{D_b D_b T_l (\chi_R)^\vee : b \in B, i \in \mathbb{Z}, l \in L\}$$

is a PF of $L^2(\mathbb{R}^n)$. On the other hand, if $U_0$ is a tiling region for the $\Lambda$ translations, that is, $\bigcup_{\lambda \in \Lambda} (U_0 + \lambda) = \mathbb{R}^n$ where the union is disjoint, every such tiling set has the same measure as $U_0$. If $|\det a| \in \mathbb{N}$, then $|S_1| = |\det a||S_0| = |\det a||\text{card}(B)||U_0|$ and it follows that no single subset $R$ of $S_1 \setminus S_0$ can be both a $(S_1 \setminus S_0)$-tiling set for the $B$ dilations and a tiling set for the $\Lambda$ translations. Instead, if $R$ is a $(S_1 \setminus S_0)$-tiling set for the $B$ dilations, then one can decompose $R$ into a disjoint union of subregions $R_1, \ldots, R_N$ (where $N = |\det a| - 1$) each of which is a tiling set for the $\Lambda$ translations. It follows that

$$\tilde{\Psi}_{AB} = \{D_b^j D_b T_l (\chi_{R_\ell})^\vee : i \in \mathbb{Z}, b \in B, l \in L, \ell = 1, \ldots, N\}$$

is an ON $AB$–multiwavelet for $L^2(\mathbb{R}^n)$. Moreover, in this case, the set $\Phi_B$ is an ON basis for $L^2(S_0)^\vee$. Some special examples of this construction can be found in [10] and [11, Sec.2.2].

### 3.3.2 Hyperbolic $AB$–wavelets.

By using a nonlinear system of coordinates, we can construct a variant of the system described in Section 2, where $B$ does not consist of shear matrices.

Fix $\lambda > 1$ and let

$$B = \{b_j = \begin{pmatrix} \lambda^j & 0 \\ 0 & \lambda^{-j} \end{pmatrix} : j \in \mathbb{Z}\}.$$ 

For $k > 0$, the set $H_k = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \xi_2 = k\}$ consists of four hyperbolas. Observe that, for any $\xi = (\xi_1, \xi_2) \in H_k$, every other point $\xi'$ on the same hyperbola has the unique representation $\xi' = (\xi_1 \lambda^t, \xi_2 \lambda^{-t})$ for some $t \in \mathbb{R}$. We can parametrize any $\xi = (\xi_1, \xi_2)$ in the first quadrant by $\xi(r, t) = (\sqrt{r} \lambda^t, \sqrt{r} \lambda^{-t})$, where $r \geq 0$, $t \in \mathbb{R}$. Then, for any $k_1 < k_2$, the set $T^1(k_1, k_2) = \{\xi(r, t) : k_1 \leq r < k_2, 0 \leq t < 1\}$ is an hyperbolic trapezoid.
Also observe that, for any $k \neq 0$, the right action of $B$ preserves the set $H_k$ since
\[ \xi b_j = (\xi_1, \xi_2) \begin{pmatrix} \lambda^j & 0 \\ 0 & \lambda^{-j} \end{pmatrix} = (\xi_1 \lambda^j, \xi_2 \lambda^{-j}) = (\eta_1, \eta_2), \]
and $\eta_1 \eta_2 = \xi_1 \xi_2$. Therefore, the set $T^1(k_1, k_2)$ is an $S^1(k_1, k_2)$–tiling set for the $B$ dilations, where $S^1(k_1, k_2)$ is the hyperbolic strip $\{\xi(r, t) : k_1 \leq r < k_2\}$. Proceeding similarly in the other quadrants, we obtain that the similarly defined hyperbolic trapezoids $T^\ell(k_1, k_2), \ell = 2, 3, 4$, are $S^\ell(k_1, k_2)$–tiling sets for the $B$ dilations. By taking unions, we have that $T(k_1, k_2) = \bigcup_{\ell=1}^4 T^\ell(k_1, k_2)$ is a $S(k_1, k_2)$–tiling set for the $B$ dilations, where $S(k_1, k_2) = \bigcup_{\ell=1}^4 S^\ell(k_1, k_2)$.

Now let $A = \{a^i : i \in \mathbb{Z}\} \subset GL_2(\mathbb{R})$, where $a$ is diagonal with $m = |\det a| > 1$. Then, for each $k > 0$, $H_k a = H_{mk}$. Thus, for any $k_0 > 0$, $S(k_0/m, k_0)$ is a tiling set for the $A$ dilations. By choosing $k_0$ small enough, the set $T = T(k_0/m, k_0)$ is contained in the fundamental domain $[1/2, 1/2)^2$ and, thus, $\psi = (\chi_T)^\vee$ is a PF $AB$–wavelet, where $AB = \{a^i b : i \in \mathbb{Z}, b \in B\}$.

### 3.4 The shear group.

We would like to find a general setting in which the systems $\{D_i^a D_j^b T_k \psi^\ell : i, j \in \mathbb{Z}, k \in \mathbb{Z}^2, \ell = 1, 2, 3\}$ described in Section 2 are included. Observe that the matrix $b$ satisfies $(b - I_2)^2 = 0$. Let us first characterize all such matrices in the $n$–dimensional case. We say that a matrix $b \in \mathbb{R}^{n \times n}$ is a shear matrix if
\[ (b - I_n)^2 = 0. \]

Each such $b$ has a Jordan form that consists of $k$ blocks of the form
\[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \]
with $k \leq n/2$, followed by an $(n-2k) \times (n-2k)$ identity matrix. That is, $b = p^J p^\top$, where $p \in GL_n(\mathbb{R})$, $J = I_n + \sum_{j=1}^k \hat{e}_{2j-1} \hat{e}_{2j}$, and $\{e_1, \ldots, e_k\}, \{\hat{e}_1, \ldots, \hat{e}_k\}$ are the canonical bases vectors of $\mathbb{R}^n$ and $\mathbb{R}^n$, respectively. This implies that a general shear matrix has the form
\[ b = I_n + \sum_{j=1}^k y^{(j)} \eta^{(j)}, \]  
(10)

where $\eta^{(j)} \eta^{(i)} = 0$, for each $1 \leq i, j \leq k$, and $y^{(j)} = p e_{2j-1}, \eta^{(j)} = \hat{e}_{2j} p^{-1}$ (observe that, for $y \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$, $\eta y$ is the $n \times n$ matrix with entries $(y_i \eta_j), 1 \leq i, j \leq n$, and $\eta y$ is the scalar $\sum_{i=1}^n \eta_i y_i$).

Let $y \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$. If $b = I_n + y \eta$, where $\eta y = 0$, then $(b - I_n)^2 = y(\eta y) \eta = 0$, and, thus, $b$ is a shear matrix. We will call an elementary shear matrix
Fig. 3. Shearing transformation. Vector field induced by the right action of the shear group $B = \{ b^j : j \in \mathbb{Z} \}$, where $b^j$ is given by (11).

any matrix of this form. Observe that, if $y\eta = 0$ and $b$ is an elementary shear matrix, then the mapping $\xi \mapsto \xi b = \xi + (\xi y)\eta$ has the property that $\xi \in \mathbb{R}^n$ is fixed by $b$ if and only if $\xi$ lies in the hyperplane $y^\perp = \{ z \in \mathbb{R}^n : z y = 0 \}$, otherwise $\xi$ is translated in the direction $\eta \in y^\perp$ (see Figure 3). In the examples from Section 2, $y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\eta = (0, 1)$ and so, for each $j \in \mathbb{Z}$,

$$b^j = I_n + jy\eta = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (11)

As we observed there, in this situation, $\xi \in \mathbb{R}^2$ is fixed under the right action of $b^j$ if and only if $\xi = (0, \xi_2)$, otherwise $\xi$ is translated in the vertical direction.

A direct computation shows that, when $b_1 = I_n + y^{(1)} \eta^{(1)}$ and $b_2 = I_n + y^{(2)} \eta^{(2)}$ are elementary shear matrices, then $b_1b_2$ is a shear matrix if and only if $b_1b_2 = b_2b_1$. This occurs precisely when $\eta^{(1)} y^{(2)} = \eta^{(2)} y^{(1)} = 0$, with $b_1b_2 = I_n + \sum_{i=1}^2 y^{(i)} \eta^{(i)}$. Similarly, it follows that a general shear matrix $b$, given by (10), is a shear matrix, where $b = b_1b_2 \ldots b_k$, and the matrices $b_i$, $1 \leq i \leq k$ are commuting elementary shear matrices.

We will say that a subgroup $B$ of $GL_n(\mathbb{R})$ is an admissible shear group if $B$ is locally admissible and is generated by finitely many commuting elementary shear matrices. In this case, $B$ is maximal if $B$ is not a proper subgroup of any other shear group in $GL_n(\mathbb{R})$. 

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3.4.1 Characterization of the maximal locally admissible shear groups.

As we will show below in Theorem 5, after a change of coordinates, the general maximal locally admissible shear group $B \subset GL_n(\mathbb{R})$ has the form:

$$\left\{ b_{(j_1, \ldots, j_{n-k})} = \begin{pmatrix} I_k & j_1 e_1 & \ldots & j_k e_k & j_{k+1} c_{k+1} & \ldots & j_{n-k} e_{n-k} \\ 0 & & & & & & I_{n-k} \end{pmatrix} : j_1, \ldots, j_{n-k} \in \mathbb{Z} \right\},$$

(12)

where $k \leq n/2$, $\{e_1, \ldots, e_k\}$ is the canonical basis of $\mathbb{R}^k$ and $\{c_{k+1}, \ldots, c_{k-n}\}$ are general non-zero column vectors in $\mathbb{R}^k$.

In the following we will illustrate some special cases of such $B$. Let $\{\hat{e}_1, \ldots, \hat{e}_n\}$ be the dual basis of $\hat{\mathbb{R}}^n$, and, for $i \neq j$, let $b^{i,j} = I_n + e_i \hat{e}_j$.

(i) For $k = 1$, the simplest $B$ of form (12) arises by choosing $c_i = e_1$, for $2 \leq i \leq n-1$. This yields the maximal admissible shear group

$$B = \left\{ b_{(j_1, \ldots, j_{n-1})} = \begin{pmatrix} 1 & j_1 & \ldots & j_{n-1} \\ 0 & & & I_{n-1} \end{pmatrix} : j_1, \ldots, j_{n-1} \in \mathbb{Z} \right\},$$

generated by $\{b^{i,j} : 2 \leq j \leq n\}$.

(ii) For $k = n/2$, then $n - k = k$ and the expression of $B$ given by (12) simplifies since there are no vectors $\{c_i\}$ to be chosen. Then

$$B = \left\{ b_{(j_1, \ldots, j_k)} = \begin{pmatrix} j_1 & \ldots & 0 \\ I_k & 0 & \ldots & 0 \\ 0 & 0 & j_k \\ 0 & & & I_k \end{pmatrix} : j_1, \ldots, j_k \in \mathbb{Z} \right\},$$

is the admissible shear group generated by $\{b^{j+k,j} : 1 \leq j \leq k\}$.

(iii) Suppose $k \geq 2$, $\ell = \ell_1 + \cdots + \ell_k$, where $\ell_j \in \mathbb{N}$, and $n = k + \ell_1 + \cdots + \ell_k$. For $1 \leq i \leq k$, let $B_i$ be the subgroup of $GL_{(\ell_i+1)}(\mathbb{R})$ of the form (i). In $GL_n(\mathbb{R})$ we can form the group

$$B = \left\{ b = \begin{pmatrix} \beta_1 & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \beta_k \end{pmatrix} : \beta_i \in B_i, \ 1 \leq i \leq k \right\},$$

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and regard $B$ as the outer direct product of the groups $B_1, \ldots, B_k$. By rearranging the order of the columns, we can recast $B$ as the the set of all matrices of the form (12) where $\ell_i - 1$ of the column vectors $\{c_{k+1}, \ldots, c_{n-k}\}$ are chosen to be equal to $e_i$, for $1 \leq i \leq k$.

In the following, we describe some examples of groups of shear matrices that are not locally admissible, but contain locally admissible subgroups or subsets.

(iv) For $n = 2$, the non-commuting elementary shear matrices $b^{1,2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b^{2,1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$. It is easy to verify that $SL_2(\mathbb{Z})$ is not locally admissible, although subgroups of $SL_2(\mathbb{Z})$ not generated by elementary shear matrices may be locally admissible. Consider, for example, the hyperbolic shear group in Section 3.3.2 or the finite group of the isometries of the square $[-1,1]^2$ (a special case of the finite groups in Section 3.3.1).

(v) For $n = 3$, the non-commuting elementary shear matrices $b^{1,2}$ and $b^{2,3}$ generate the integral Heisenberg group $H_3 = \left\{ b_{(i,j,k)} = \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} : i, j, k \in \mathbb{Z} \right\}$.

For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we have $\xi b_{(i,j,k)} = (\xi_1, \xi_2 + i\xi_1, \xi_3 + j\xi_2 + k\xi_1)$. If $\xi_1/\xi_2 \notin \mathbb{Q}$, then $\{j\xi_2 + k\xi_1 : j, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ and thus the $H_3$ orbit is not discrete in $\mathbb{R}^3$. Observe that $H_3$ is not a shear group. However, the subgroup $\{b_{(i,0,k)} : i, k \in \mathbb{Z}\}$ of $H_3$ is a maximal admissible shear group of the form given by the example (i), and the subset $\{b_{(i,j,0)} : i, j \in \mathbb{Z}\}$ is locally admissible. More generally, for $n \geq 3$, let $B_i$ be the shear group generated by $b^{i,i+1}$, for $1 \leq i \leq n$. Then the set product

$$B_{n-1}B_{n-2} \cdots B_1 = \left\{ b_{(j_1, \ldots, j_{n-1})} = \begin{pmatrix} 1 & j_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & j_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} : j_1, \ldots, j_{n-1} \in \mathbb{Z} \right\}$$

is locally admissible. Indeed, the set product is locally admissible for any ordering of the non-commuting groups $B_1, \ldots, B_{n-1}$.

The following proposition elaborates the above observations further.
Proposition 4 Let $\tilde{B}$ be a subset of $GL_n(\mathbb{R})$ containing the group $B$ generated by two non-commuting elementary shear matrices $b_1 = I_n + y^{(1)}\eta^{(1)}$ and $b_2 = I_n + y^{(2)}\eta^{(2)}$. Then $\tilde{B}$ is not admissible.

Proof. Since $b_1b_2 \neq b_2b_1$, then either $\eta^{(1)}y^{(2)}$ or $\eta^{(2)}y^{(1)}$ is nonzero. In the case when $\eta^{(1)}y^{(2)} = 0$ and $\eta^{(2)}y^{(1)} \neq 0$, then $B$ is isomorphic to the integral Heisenberg group $H_3$ and is not locally admissible for the same reason discussed in Example (iv). When both $\eta^{(1)}y^{(2)}$ and $\eta^{(2)}y^{(1)}$ are nonzero, we can assume that their product is positive by replacing $b_1$ with $b_1^{-1}$ if needed. Using the rescaling $y\eta = (ky)(\eta/k)$ for $k > 0$, we may assume that $\eta^{(1)}y^{(2)} = c^{-1}$ and $\eta^{(2)}y^{(1)} = c$, for some $c > 0$. Then $B$ is isomorphic to $SL_2(\mathbb{Z})$ if $c = 1$ and, in general, $B$ is conjugate to the subgroup $B_c$ of $GL_n(\mathbb{R})$ generated by

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ c^{-1} & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & I_{n-2} \\ 0 & 1 \end{pmatrix}), \quad \begin{pmatrix} 0 & I_{n-2} \\ 0 & 1 \end{pmatrix}.$$ 

It is easy to see that $B_c$ is not locally admissible for any $c$. Thus, in all cases, the group $B$ generated by $b_1$ and $b_2$ is not locally admissible and, so, any subset $B$ of $GL_n(\mathbb{R})$ containing $B$ is not locally admissible. \( \Box \)

Observe that Proposition 4 does not apply to the locally admissible subgroups mentioned in Example (iii) (that are not generated by elementary shear matrices), and does not apply to the locally admissible sets of Example (v), obtained as products of non-commuting elementary shear matrices.

We can now state the main result of this section.

Theorem 5 Let $B \subset GL_n(\mathbb{R})$ be a maximal locally admissible shear group.

(a) There is a unique index $k \leq n/2$ and a change of basis matrix $P$ such that $B = P^{-1}BP$ is of the form given by example (ii).

(b) If $a \in GL_n(\mathbb{R})$ is such that $P^{-1}aP = \begin{pmatrix} c \ast \\ 0 & d \end{pmatrix}$, where $c \in GL_k(\mathbb{R})$ is expanding and $d \in GL_{n-k}(\mathbb{R})$, then $AB = \{ a'b : i \in \mathbb{Z}, b \in B \}$ is admissible.

Proof. (a) Let $\ell$ be the minimal numbers of elementary shear matrix generators for $B$ and $\{ b_i = I_n + y^{(i)}\eta^{(i)} : 1 \leq i \leq \ell \}$ a particular set of such generators. For $V = \text{span} \{ y^{(i)} : 1 \leq i \leq \ell \}$ and $k = \text{dim} V$, we will show that $\ell = n - k$ and $\{ \eta^{(1)}, \ldots, \eta^{(\ell)} \}$ is a basis for the $V^\perp$, the annihilator of $V$, given by $\{ v \in \mathbb{R}^n : \nu v = 0, \forall v \in V \}$. Let $W = \text{span} \{ \eta^{(i)} : 1 \leq i \leq \ell \}$. Since $\eta^{(i)}y^{(j)} = 0$ for $1 \leq i, j \leq \ell$, then $W \subseteq V^\perp$. For any $v \in V$, $\eta \in V^\perp \setminus W$, the elementary shear matrix $\tilde{B} = I_n + v\eta$ commutes with every member of $B$. Let $\tilde{B}$ be the shear group generated by $B$ and $\tilde{b}$. Since $B$ is locally admissible and
\( \eta \notin W \), then \( \tilde{B} \) is locally admissible and this contradicts the maximality of \( B \). Hence \( V^\perp = W \) and \( \ell \geq n - k \).

In order to prove that \( \ell = n - k \), we argue by contradiction and assume that \( \eta^{(1)}, \ldots, \eta^{(\ell)} \) are linearly dependent. Let \( m < \ell \) be the largest index for which \( \eta^{(1)}, \ldots, \eta^{(m)} \) are linearly independent, \( B_{m+1} \) be the subgroup of \( B \) generated by \( \{ b_i : 1 \leq i \leq m + 1 \} \) and \( W_m \) be the \( m \)-dimensional subspace of \( V^\perp \) spanned by \( \{ \eta^{(i)} : 1 \leq i \leq m + 1 \} \). By assumption, \( \eta^{(m+1)} = \sum_{i=1}^{m} c_i \eta^{(i)} \) for some scalars \( c_1, \ldots, c_m \). Since \( B \) is locally admissible, so is \( B_{m+1} \), that is, the orbit \( \Gamma_\xi = \{ \xi - \xi b : b \in B_{m+1} \} \) is discrete in \( \mathbb{R}^n \) for a.e. \( \xi \in \mathbb{R}^n \).

Since \( B_{m+1} = \{ I_n + \sum_{i=1}^{m+1} j_i v^{(i)} \eta^{(i)} : (j_1, \ldots, j_{m+1}) \in \mathbb{Z}^{m+1} \}, \) then \( \Gamma_\xi \) is the additive subgroup of \( W_m \) generated by the linear dependent vectors \( (\xi y^{(i)}) \eta^{(i)}, \) \( 1 \leq i \leq m + 1 \), and \( \Gamma_\xi \) is discrete in \( W_m \) if and only if these vectors are linearly dependent over the rational numbers \( \mathbb{Q} \). It follows that, for a.e. \( \xi \in \mathbb{R}^n \), \( \xi y^{(i)} \neq 0 \) for \( 1 \leq i \leq m + 1 \), and \( c_i \xi y^{(m+1)} \) is a rational multiple of \( \xi y^{(i)} \) for \( 1 \leq i \leq m \). By suppressing all indices for which \( c_i = 0 \) and renaming the remaining indices, we can assume that \( c_i \neq 0 \) for each \( i \). Since the quotient \( q(\xi) \) of two linear functions over \( \mathbb{R}^n \) can take values in \( \mathbb{Q} \) for a.e. \( \xi \in \mathbb{R}^n \) if and only if \( q \) is constant, it follows that \( y^{(m+1)} \) and \( y^{(i)} \) are linearly dependent for each \( 1 \leq i \leq m \). By re-scaling \( y = (ky) \), we may then assume that \( y^{(1)} = y^{(2)} = \ldots = y^{(m+1)} = y \) for some \( y \in V \). Then, for all \( \xi \in \mathbb{R}^n \), we have \( \Gamma_\xi = (\xi y) \Gamma \), where \( \Gamma = \mathbb{Z} \eta^{(1)} + \cdots + \mathbb{Z} \eta^{(m+1)} \). Since \( \Gamma \) is a lattice in \( W \), we can replace \( \eta^{(1)}, \ldots, \eta^{(m+1)} \) by a lattice basis \( \nu^{(1)}, \ldots, \nu^{(m+1)} \). This means that the elementary shear matrices \( b_i' = I_n + y \nu^{(i)} : 1 \leq i \leq m \) are an alternative set of generators for \( B_{m+1} \) and \( b_1', \ldots, b_m', b_{m+2}, \ldots, b_\ell \) is a generating set for \( B \) with \( \ell - 1 \) members. This contradicts the assumption that \( \ell \) is the minimal number of elementary shear matrix generators for \( B \). Thus we conclude that \( \{ \eta^{(i)} : 1 \leq i \leq \ell \} \) is a linearly independent set, hence \( \ell = n - k \) and \( \{ \eta^{(i)} : 1 \leq i \leq n - k \} \) is a basis for \( W = V^\perp \).

By reordering the \( \{ b_i \} \), we may assume that \( \{ y^{(i)} : 1 \leq i \leq k \} \) is a basis for \( V \) and choose a set of vectors \( v^{(k+1)}, \ldots, v^{(n)} \) in \( \mathbb{R}^n \) for which \( B = \{ y^{(1)}, \ldots, y^{(k)}, v^{(k+1)}, \ldots, v^{(n)} \} \) is a basis for \( \mathbb{R}^n \) with \( v^{(i)} v^{(k+j)} = 0 \) for \( 1 \leq i, j \leq n - k \). Let \( P \) be the change of basis matrix mapping \( B \) to the standard basis \( \{ e_1, \ldots, e_n \} \) for \( \mathbb{R}^n \). Then \( \tilde{B} = P^{-1}BP \) is of the form given by (12).

(b) By our proof of (a), there is no loss of generality in assuming \( B \) to be of the form (12), and (using the hypotheses) in letting \( a = \begin{pmatrix} c \ast & \ast \\ 0 & d \end{pmatrix} \), where \( c \in GL_k(\mathbb{R}) \) is expanding and \( d \in GL_{n-k}(\mathbb{R}) \). For notational convenience, let \( c_i = e_i \) for \( 1 \leq i \leq k \). We can regard \( \mathbb{R}^n \) as \( \mathbb{R}^k \times \mathbb{R}^{n-k} \) and select a small annular set \( K \), about the origin in \( \mathbb{R}^k \), so that \( \mathbb{R}^k = \bigcup_{c \in \mathbb{Z}} K c^{-1} \) is a disjoint union. For \( \xi = (\nu, \eta) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \), with \( \nu \in \mathbb{R}^k \setminus \{0\} \), there is a unique index \( i \in \mathbb{Z} \) for which \( \xi' = \xi a^i = (\nu', \eta') \), with \( \nu' = \nu c^i \in K \),
Fig. 4. $I$ is an $AB$ tiling set of the cone $H$ and $\psi = (\chi_I)^\vee$ is a PF $\tilde{AB}$–wavelet for $L^2(\mathbb{R}^2 \setminus [-1/2, 1/2]^2)^\vee$ (see Section 3.5).

$\eta' \in \hat{\mathbb{R}}^{n-k}$. For $b = b(j_1, \ldots, j_{n-k})$ as in (12), we have that $\xi'' = \xi' b = (\nu', \eta'')$, where, for each $1 \leq i \leq n - k$, $\eta''_i = j_i(\nu' c_i) + \eta'_i$ is the $i$–component of $\eta''$, and, $\eta'_i$ is the the $i$–component of $\eta'$. Observe that, for each $i$, we have $\nu' c_i \neq 0$ on a set of full measure in $K$. Therefore there is a unique choice of $j_1, \ldots, j_{n-k}$ for which $0 \leq \eta''_i/(\nu' c_i) < 1$, for each $i$. Finally, let $T_{\nu'}$ be the set of all elements $\eta'' \in \hat{\mathbb{R}}^{n-k}$ satisfying these inequalities. It follows that the set $R = \bigcup_{\nu' \in K} \{\nu'\} \times T_{\nu'}$ is a tiling set for the $(AB)^{-1}$–dilations. By taking $K$ small enough, we can ensure that $R$ is also a packing set for the $\hat{\mathbb{Z}}^n$ translations. Thus $\{D_{a,b} T_k (\chi_R)^\vee : i \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^n\}$ is a PF $AB$ wavelet and the dilation set $AB$ is admissible.

3.5 The contourlets.

In this section, we describe a variation of the example given in Section 2, that is similar to the contourlets have been recently introduced by Do and Vetterli [7]. The contourlet construction, that is inspired by the curvelets, uses a multiresolution analysis framework, with the decomposition:

$$L^2(\mathbb{R}^2) = V_0 \oplus \bigoplus_{i < i_0} W_i,$$

where $V_i = L^2(S_i)^\vee$, $S_i = \{\xi \in \hat{\mathbb{R}}^2 : \|\xi\|_{\ell^1} \leq 2^{-i}\}$, $W_i = V_{i-1} \cap V_i^\perp$. In addition, for each $i < i_0$, each subspace $W_i$ is subdivided into the ‘directional’ components:

$$W_i = \bigoplus_{j=0}^{2^i} W_{i,j}^{(l_j)}.$$
We will obtain a very similar construction using the general setting of the $AB$–wavelets.

Let $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $T(\alpha, \beta) = T^+(\alpha, \beta) \cup T^-(\alpha, \beta)$, where $T^+(\alpha, \beta)$ is the trapezoid with vertices $(\alpha, 0)$, $(\alpha, \alpha)$, $(\beta, 0)$ and $(\beta, \beta)$, and $T^-(\alpha, \beta) = \{ \xi \in \hat{\mathbb{R}}^2 : -\xi \in T^+(\alpha, \beta) \}$. We denote by $H$ the truncated cone

$$H = \{ (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : |\xi_1| \geq 1/2, 0 \leq |\xi_2/\xi_1| \leq 1 \},$$

and let $I = T(1/2, 1)$. These sets are illustrated in Figure 4. Then a simple computation shows that the sets $\{ I^{a_i b_j} : i \geq 0, -2^i \leq j \leq 2^i - 1 \}$, form a tiling for $H$. Thus, for

$$AB = \{ a^{-i} b^{-j} = (b^j a^i)^{-1} : i \geq 0, -2^i \leq j < 2^i - 1 \},$$

the function $\psi = (\chi_I)^{\wedge}$ is a PF $AB$-wavelet for $L^2(H)^{\vee}$, and the set $AB$ is $H$–admissible.

Next, let $\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since this matrix produces a rotation by $\pi/2$, then

$$V = H \rho^{-1} = \rho H$$

is the truncated cone:

$$V = \{ (\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : |\xi_2| \geq 1/2, 0 \leq |\xi_1/\xi_2| \leq 1 \}.$$

Observe that $(D_\rho \psi)^\wedge(\xi) = \hat{\psi}(\xi \rho) = \chi_{I\rho^{-1}}(\xi)$ (the set $I\rho^{-1}$ is illustrated in Figure 4) and, thus, by the properties of $T$, the sets

$$I^{b_j a_i \rho^{-1}} = (I\rho^{-1}) \rho b_j a_i \rho^{-1}, \quad \text{for } i \geq 0, -2^i \leq j < 2^i,$$

form a tiling for $V$. This shows that $D_\rho \psi$ is a PF $(\rho AB\rho^{-1})$–wavelet for $L^2(V)^{\vee}$. Moreover, since $H \cup V = \hat{\mathbb{R}}^2 \setminus [-1/2, 1/2]^2$, it follows that $\psi$ is a PF $AB$–wavelet for $L^2(\hat{\mathbb{R}}^2 \setminus [-1/2, 1/2]^2)^{\vee}$, where $AB = AB \cup \rho AB\rho^{-1}$.

The expression (13) shows that when the scale index $i$ is increased by 1, the number of directions $j$ is doubled. Observe that, in the contourlet construction of Do and Vetterli, as well as in the case of curvelets, the number of directions doubles every time $i$ is increased by 2, and this ensures that the elements of the systems satisfy a parabolic scaling, that is, the essential support of these systems obeys approximately the relationship

$$\text{length} \approx 2^{-i} \quad \text{width} \approx 2^{-2i}.$$

As shown in [4,7], this property is needed to obtain representations that are optimally sparse for functions in a certain class. In the construction above,
we made a different choice of the width-to-length ration, in order use to same matrices as in the example from Section 2. However, we can easily modify this construction by choosing $a = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, and letting $I = T(1/4, 1)$. By doing so, we obtain a Parseval frame of elements satisfying a parabolic scaling relation.

Finally, let us observe that the system we have obtained disregards the low-frequency region $[-1/2, 1/2]^2 \subset \mathbb{R}^2$, where standard (non-directional) wavelets are used (this is similar to the curvelets and contourlets construction).

4 AB multiresolution analysis. Part I.

As we already observed in Section 2, there are examples of AB multiwavelets that can be constructed within a framework very similar to the classical multiresolution analysis (MRA). In this section and in the following one we are going to develop a generalization of this theory, that will be useful to construct more examples of AB multiwavelets, as well as examples with properties that are of great potential in applications.

Let $B$ be a countable subset of $\widetilde{SL}_n(\mathbb{Z}) = \{ b \in GL_n(\mathbb{Z}) : |\det b| = 1 \}$ and $A = \{ a^i : i \in \mathbb{Z} \}$, where $a \in GL_n(\mathbb{Z})$ (notice that $a$ is an integral matrix). Also assume that $a$ normalizes $B$, that is, $a b a^{-1} \in B$ for every $b \in B$. We say that a sequence $\{V_i\}_{i \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is an AB multiresolution analysis (AB-MRA) if the following holds:

(i) $D_b T_k V_0 = V_0$, for any $b \in B$, $k \in \mathbb{Z}^n$;
(ii) for each $i \in \mathbb{Z}$, $V_i \subset V_{i+1}$, where $V_i = D_{a^{-i}} V_0$;
(iii) $\bigcap V_i = \{0\}$ and $\bigcup V_i = L^2(\mathbb{R}^n)$;
(iv) there exists $\phi \in L^2(\mathbb{R}^n)$ such that $\Phi_B = \{ D_b T_k \phi : b \in B, k \in \mathbb{Z}^n \}$ is a semi-orthogonal Parseval frame for $V_0$; that is, $\Phi_B$ is a Parseval frame for $V_0$ and, in addition, $D_b T_k \phi \perp D_{b'} T_{k'} \phi$ for any $b \neq b'$, $b, b' \in B$, $k, k' \in \mathbb{Z}^n$.

The space $V_0$ is called an AB scaling space and the function $\phi$ is an AB scaling function for $V_0$. If, in addition, $\Phi_B$ is an orthonormal basis, then we say that $\phi$ is an ON AB scaling function.

Observe that one could consider a more general definition, where $A$ is not necessarily a group, but simply a countable collection, that is, $A = \{ a_i : i \in \mathbb{Z} \}$. Furthermore, one could consider the situation where the set $\Phi_B$ is simply a Parseval frame for $V_0$ (not necessarily semi-orthogonal). The assumptions that we made in the above definition are the ‘simplest’, and they ensure that the
properties of the $AB$–MRA are very similar to those of the classical MRA. Also observe that there is a basic difference in the definition of $AB$–MRA that we just gave, from the definition of the classical MRA. In fact, in our definition, the space $V_0$ is invariant with respect to the integer translations \textit{and} with respect to the $B$–dilations. On the other hand, in the classical MRA, the space $V_0$ is only invariant with respect to the integer translation.

Therefore, in order to examine in detail the main features of the $AB$–MRA, it will be useful to study the properties of the subspaces of $L^2(\mathbb{R}^n)$ that are invariant with respect to the integer translations and with respect to the $B$–dilations. This will be done in Section 4.2. Before doing this, in Section 4.1, we will briefly recall some basic results from the theory of shift–invariant spaces.

\subsection*{4.1 Shift–invariant spaces}

A $\mathbb{Z}^n$–invariant space (or a shift–invariant space) of $L^2(\mathbb{R}^n)$ is a closed subspace $V \subset L^2(\mathbb{R}^n)$ for which $T_k V = V$ for each $k \in \mathbb{Z}^n$. For $\phi \in L^2(\mathbb{R}^n) \setminus \{0\}$, we denote by $\langle \phi \rangle$ the shift–invariant space generated by $\phi$, that is,

$$\langle \phi \rangle = \text{span}\{T_k \phi : k \in \mathbb{Z}^n\}. $$

Given $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$, their bracket product is defined as

$$[\phi_1, \phi_2](x) = \sum_{k \in \mathbb{Z}^n} \phi_1(x - k) \overline{\phi_2(x - k)}. \quad (1) $$

Let $\mathbb{T}^n$ be the $n$–torus $\mathbb{R}^n/\mathbb{Z}^n \simeq [0, 1]^n$ and regard $L^2(\mathbb{T}^n)$ as the space of the measurable $\mathbb{Z}^n$–periodic functions $t$ for which $\|t\|_{L^2(\mathbb{T}^n)} = \int_{[0,1]^n} |t(x)|^2 \, dx < \infty$. As usual, $\hat{\mathbb{T}}^n$ denotes the corresponding space of row vectors. The following properties of the bracket product are easy to verify, and they can be found, for example, in [21, Sec. 3].

\textbf{Proposition 6} Let $\phi, \phi_1, \phi_2 \in L^2(\mathbb{R}^n)$.

(i) The series (1) converges absolutely a.e. to a function in $L^1(\mathbb{T}^n)$.

(ii) The spaces $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$ are orthogonal if and only if $[\hat{\phi}_1, \hat{\phi}_2](\xi) = 0$ a.e.

(iii) Let $V(\phi) = \{T_k \phi : k \in \mathbb{Z}^n\}$. Then $V(\phi)$ is an orthonormal basis for $\langle \phi \rangle$ if and only if $[\hat{\phi}, \hat{\phi}](\xi) = 1$ a.e., and $V(\phi)$ is a Parseval frame for $\langle \phi \rangle$ if and only if $[\hat{\phi}, \hat{\phi}](\xi) = \chi_{\Omega_\phi}(\xi)$ a.e., where $\Omega_\phi = \{\xi \in \hat{\mathbb{T}}^n : \hat{\phi}(\xi + \hat{k}) \neq 0 \text{ for some } \hat{k} \in \hat{\mathbb{Z}}^n\}$.\,

(iv) Let $[\hat{\phi}, \hat{\phi}](\xi) = \chi_{\Omega_\phi}(\xi)$. Then $f \in \langle \phi \rangle$ if and only if $\hat{f} = m \hat{\phi}$, for some $m \in L^2(\Omega_\phi)$ satisfying $\|f\| = \|m\|_{L^2(\Omega_\phi)}$.

Let $U \subseteq \hat{\mathbb{R}}^n$ be measurable and $\Omega_U = \Omega_{\chi_U} = \bigcup_{k \in \hat{\mathbb{Z}}^n} (U + \hat{k})$. If this is a disjoint union (modulo a null set), then we say that $U$ is a $\Omega_U$–tiling set for $\hat{\mathbb{Z}}^n$.\,

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translations. It is clear that this is the case if and only if \( \langle \chi_U, \chi_U \rangle (\xi) = \chi_{\Omega_U}(\xi) \) a.e., or, equivalently, if and only if \( V((\chi_U)^\vee) = \{ T_k (\chi_U)^\vee : k \in \mathbb{Z}^n \} \) is a Parseval frame for \((\chi_U)^\vee\). Observe that, for any \( \Omega \subseteq \mathbb{R}^n \), every \( \Omega \)-tiling set for \( \hat{Z}_n \) translations is contained in a \( \mathbb{R}^n \)-tiling set for \( \hat{Z}_n \) translations, and all such tiling sets have measure one. Thus, when \( \phi = (\chi_U)^\vee \) and \( V(\phi) \) is a Parseval frame for \( \langle \phi \rangle \), then \( |U| \leq 1 \), with equality if and only if \( V(\phi) \) is an orthonormal basis for \( \langle \phi \rangle \). Also observe that when \( U \) is contained in a tiling set for \( \hat{Z}_n \) translations, then \( \langle (\chi_U)^\vee \rangle = L^2(U)^\vee \subseteq L^2(\mathbb{R}^n) \) since any \( \hat{f} \in L^2(U) \) extends uniquely to \( m \in L^2(\Omega_U) \) with \( \hat{f} = m \chi_U \).

Let \( V \) be a shift–invariant space of \( L^2(\mathbb{R}^n) \). \( \Phi = \{ \phi^1, \ldots, \phi^N \} \), with \( N \in \mathbb{N} \cup \{ \infty \} \), is a \( \mathbb{Z}^n \)-orthonormal set of generators for \( V \) if, for each \( 1 \leq j \leq N \), the set \( \{ T_k \phi^j : k \in \mathbb{Z}^n \} \) is an orthonormal basis for \( \langle \phi^j \rangle \). Equivalently, we have that \( [\hat{\phi}^i, \hat{\phi}^j] = \delta_{i,j} \) a.e. In addition, if this is the case, we have that \( V = \bigoplus_{j=1}^N \langle \phi^j \rangle \) and we can show that, for each \( f \in V \),

\[
\hat{f} = \sum_{j=1}^N [\hat{f}, \hat{\phi}^j] \hat{\phi}^j,
\]

with pointwise a.e. convergence if \( N < \infty \) and \( L^2 \)-convergence if \( N = \infty \). In fact, if \( f \in V \), then

\[
f = \sum_{j=1}^N \sum_{k \in \mathbb{Z}^n} c_{j,k} T_k \phi^j,
\]

where \( c_{j,k} = \langle f, T_k \phi^j \rangle \) with pointwise a.e. convergence if \( N < \infty \) and \( L^2 \)-convergence if \( N = \infty \). Next, by taking the Fourier transform on both sides and using a periodization argument, we obtain that

\[
\hat{f} = \sum_{j=1}^N \sum_{k \in \mathbb{Z}^n} \langle \hat{f}, M_{-k} \phi^j \rangle M_{-k} \hat{\phi}^j
\]

\[
\begin{align*}
&= \sum_{j=1}^N \sum_{k \in \mathbb{Z}^n} \left( \int_{T^n} [\hat{f}, \hat{\phi}^j](\xi) e^{2\pi i k \xi} d\xi \right) M_{-k} \hat{\phi}^j \\
&= \sum_{j=1}^N \sum_{k \in \mathbb{Z}^n} [\hat{f}, \hat{\phi}^j] \phi^j.
\end{align*}
\]

Observe that, by an application of Proposition 1 with \( G = \mathbb{Z}^n \), any two \( \mathbb{Z}^n \)-orthonormal sets of generators for the same shift–invariant spaces \( V \) must have the same number of generators.

Also observe that, while not every shift–invariant space \( V \) admits a set of generators that is \( \mathbb{Z}^n \)-orthonormal, one can always find a semi–orthogonal
set of generators $\Phi = \{\phi^1, \ldots, \phi^N\}$ for $V$, in the sense that

$$V = \bigoplus_{i=1}^{N} \langle \phi^i \rangle,$$

with $[\hat{\phi}^i, \hat{\phi}^j] = \chi_{\Omega_i}, 1 \leq i \leq N$, where $\Omega_i = \Omega_{\phi^i}$. In this situation, $N$ is not uniquely determined by $V$. However, an extension of the argument in Proposition 1 shows that the multiplicity function

$$m_V = \sum_{i=1}^{N} \chi_{\Omega_i} : \mathbb{T}^n \mapsto \mathbb{N} \cup \{0, \infty\}$$

is independent (a.e.) of the choice of $\Phi$.

4.2 $B \ltimes \mathbb{Z}^n$-invariant spaces

Let $\tilde{SL}_n(\mathbb{Z}) = \{b \in GL_n(\mathbb{Z}) : |\det b| = 1\}$. If $B$ is a subgroup of $\tilde{SL}_n(\mathbb{Z})$, then $B \ltimes \mathbb{Z}^n$ is a subgroup of the integral affine group $\tilde{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n (= \text{the semidirect product of } \tilde{SL}_n(\mathbb{Z}) \text{ and } \mathbb{Z}^n$). We define the $B \ltimes \mathbb{Z}^n$ invariant spaces as those closed subspaces $V \subseteq L^2(\mathbb{R}^n)$ for which $D_b T_k V = V$, for each $(b, k) \in B \ltimes \mathbb{Z}^n$. We will show that these spaces share many properties with the classical shift–invariant spaces.

For $\phi \in L^2(\mathbb{R}^n)$, denote by $\langle \langle \phi \rangle \rangle$ the $B \ltimes \mathbb{Z}^n$ invariant spaces generated by $\phi$, that is

$$\langle \langle \phi \rangle \rangle = \text{span} \{D_b T_k \phi : b \in B, k \in \mathbb{Z}^n\}.$$ 

For $b \in \tilde{SL}_n(\mathbb{Z})$, we have

$$\{D_b T_k : k \in \mathbb{Z}^n\} = \{T_{k'} D_b : k' \in \mathbb{Z}^n\},$$

and, as a consequence, $D_b \langle \phi \rangle = \langle D_b \phi \rangle$ for each $\phi \in L^2(\mathbb{R}^n)$. We also have that $\mathbb{Z}^n b = \mathbb{Z}^n$ and, thus,

$$[\hat{D}_b \hat{\phi}_1, \hat{\phi}_2](\xi) = [\hat{\phi}_1, \hat{D}_{b^{-1}} \hat{\phi}_2](\xi b), \quad (3)$$

for each $\phi_1, \phi_2 \in L^2(\mathbb{R}^n), \xi \in \hat{\mathbb{R}}^n$.

The following simple observations follow easily from Proposition 6.

Proposition 7 Let $\phi \in L^2(\mathbb{R}^n)$.

(i) The spaces $\langle D_b \phi \rangle$ and $\langle \phi \rangle$ are orthogonal if and only if $[\hat{D}_b \hat{\phi}, \hat{\phi}](\xi) = 0$ a.e.
(ii) Let \( V_B(\phi) = \{ D_b T_k \phi : b \in B, k \in \mathbb{Z}^n \} \). Then \( V_B(\phi) \) is an orthonormal basis for \( \langle \langle \phi \rangle \rangle \) if and only if \( [D_b \phi, \hat{\phi}] (\xi) = 0 \) a.e. for each \( b \in B \setminus \{ I \} \) and \( [\hat{\phi}, \hat{\phi}](\xi) = 1 \) a.e.

(iii) If \( V_B(\phi) \) is an orthonormal basis for \( \langle \langle \phi \rangle \rangle \), then the map \( f \mapsto \left( [\hat{f}, D_b \hat{\phi}] \right) \), where \( b \in B \), is an isometry from \( \langle \langle \phi \rangle \rangle \) onto the Hilbert space \( L^2(\mathbb{R}^n) = \{ m = (m_b)_{b \in B} : m_b \in L^2(\mathbb{R}^n) \text{ and } \| m \|^2 = \sum_{b \in B} \| m_b \|^2 < \infty \} \). In particular,

\[
\hat{f} = \sum_{b \in B} [\hat{f}, D_b \hat{\phi}] D_b \hat{\phi},
\]

for each \( f \in \langle \langle \phi \rangle \rangle \), with convergence in \( L^2(\mathbb{R}^n) \).

The set \( V_B(\phi) = \{ D_b T_k \phi : b \in B, k \in \mathbb{Z}^n \} \) is called a **semi-orthogonal Parseval frame** for the \( B \ltimes \mathbb{Z}^n \)-invariant space \( \langle \langle \phi \rangle \rangle \) if

\[
\langle \langle \phi \rangle \rangle = \bigoplus_{b \in B} D_b \langle \phi \rangle
\]

and \( \{ T_k : k \in \mathbb{Z}^n \} \) is a Parseval frame for \( \langle \phi \rangle \). A simple extension of Proposition 7(ii) gives that \( V_B(\phi) \) is a semi-orthogonal Parseval frame for \( \langle \langle \phi \rangle \rangle \) if and only if \( [D_b \phi, \hat{\phi}](\xi) = 0 \) for each \( b \in B \setminus \{ I \} \) and \( [\hat{\phi}, \hat{\phi}](\xi) = \chi_{\Omega_\phi} \) a.e., where \( \Omega_\phi = \{ \xi \in \mathbb{R}^n : \hat{\phi}(\xi + k) \neq 0 \text{ for some } k \in \mathbb{Z}^n \} \).

As a special case, consider \( \phi \in L^2(\mathbb{R}^n) \) defined by \( \hat{\phi} = \chi_U \) where \( U \subseteq \mathbb{R}^n \) is measurable and \( 0 < |U| < \infty \). In this case, \( (D_b \phi)^\wedge = \chi_{Ub^{-1}} \) and we have that \( [\hat{\phi}, D_b \hat{\phi}] = 0 \) a.e. if and only if \( |U \cap Ub^{-1}| = 0 \). Also, \( \Omega_\phi = \Omega_U = \bigcup_{k \in \mathbb{Z}^n} (U + k) \) and, therefore, \( [\hat{\phi}, \hat{\phi}] = \chi_{\Omega_U} \) a.e. if and only if \( U \) is a \( \Omega_U \)-tiling set for the \( \mathbb{Z}^n \) translations. It follows that \( D_b \phi : b \in B \) is a semi-orthogonal Parseval frame generator for \( \langle \langle \phi \rangle \rangle \) if and only if \( U \) is both an \( S \)-tiling set for \( B \) dilations, where \( S = \bigcup_{b \in B} Ub \), and a \( \Omega_U \)-tiling set for \( \mathbb{Z}^n \) translations. In this case, \( |U| \leq 1 \) with equality if and only if \( [\hat{\phi}, \hat{\phi}] = 1 \) a.e., \( \langle \phi \rangle = (L^2(U))' \) and \( \langle \langle \phi \rangle \rangle = (L^2(S))' = \bigoplus_{b \in B} (L^2(Ub^{-1}))' \).

Let \( V \) be a \( B \ltimes \mathbb{Z}^n \)-invariant space of \( L^2(\mathbb{R}^n) \). The set \( \Phi = \{ \phi^1, \ldots, \phi^N \} \), with \( N \in \mathbb{N} \cup \{ \infty \} \), is a \( B \ltimes \mathbb{Z}^n \)-orthonormal set of generators for \( V \) if the set \( \{ D_b T_k \phi^i : (b,k) \in B \ltimes \mathbb{Z}^n, 1 \leq i \leq N \} \) is an orthonormal basis for \( V \). Equivalently, we have that \( [D_b \phi^i, \hat{\phi}^j] = \delta_{ij} \delta_{bI_n} \) a.e. We make the following observation.

**Proposition 8** Let \( \Phi = \{ \phi^1, \ldots, \phi^N \} \) and \( \Psi = \{ \psi^1, \ldots, \psi^M \} \) be two \( B \ltimes \mathbb{Z}^n \)-orthonormal sets of generators for the \( B \ltimes \mathbb{Z}^n \)-invariant spaces \( V \) and \( W \), respectively. If \( W \subseteq V \), then \( M \leq N \) with \( M = N \) if and only if \( W = V \).
Proof. We first observe that

\[ M = \sum_{i=1}^{M} \| \psi_i \|^2 = \sum_{i=1}^{M} \sum_{b \in B} \| [\hat{\psi}_i, \hat{D}_b \hat{\phi}_j] \|^2_{L^2(T^n)}. \] (4)

On the other hand, for each \(1 \leq j \leq N\), the function \(\sum_{i=1}^{M} \sum_{b \in B} [\hat{\phi}_j, \hat{D}_b \hat{\psi}_i] \hat{\psi}_i\) is the orthogonal projection of \(\hat{\phi}_j\) into \(W\). Thus,

\[ 1 = \| \phi_j \|^2 \geq \sum_{i=1}^{M} \sum_{b \in B} \| [\hat{\phi}_j, \hat{D}_b \hat{\psi}_i] \|^2_{L^2(T^n)}. \] (5)

By (3) and the fact that \(b \in \overline{SL_n(\mathbb{Z})}\) (this implies that the map \(\xi \mapsto \xi b\) on \(\hat{\mathbb{R}}^n\) is a measure preserving map from \(T^n\) onto \(T^n\)) it follows that

\[ \| [\hat{D}_b \hat{\psi}_i, \hat{\phi}_j] \|^2_{L^2(T^n)} = \| [\hat{\psi}_i, \hat{D}_{b^{-1}} \hat{\phi}_j] \|^2_{L^2(T^n)}, \]

Using this observation, from (4) and (5) we obtain:

\[ M = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \| [\hat{\psi}_i, \hat{D}_b \hat{\phi}_j] \|^2_{L^2(T^n)} \]

\[ = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{b \in B} \| [\hat{\phi}_j, \hat{D}_{b^{-1}} \hat{\psi}_i] \|^2_{L^2(T^n)} \]

\[ \leq \sum_{j=1}^{N} \| \phi_j \|^2 = N, \]

with \(N = M\) if and only if \(\phi_j \in W\) for each \(1 \leq j \leq N\) (which is equivalent to \(W = V\)). \(\square\)

Recall that \(a \in GL_n(\mathbb{Z})\) normalizes \(B\) if \(aba^{-1} \in B\) for every \(b \in B\). Since \(B\) is a group, then \(aba^{-1}\) is a subgroup of \(B\). We have the following result.

**Proposition 9** Suppose that \(a \in GL_n(\mathbb{Z})\) normalizes \(B\) and that the quotient space \(B/(aba^{-1})\) has finite order \(N\). If \(\phi \in L^2(\mathbb{R}^n)\) satisfies the relation \([\hat{D}_b \hat{\phi}, \hat{\phi}] = \delta_{b,b_n}\) a.e. for each \(b \in B\), then there exists a \(B \rtimes \mathbb{Z}^n\)-orthonormal set of generators \(\Phi\) for the space \(D_{a^{-1}}^{\phi}\) with cardinality \(N |\det a|\).

Before proving this Proposition, we need to make some observations. Recall that for \(a \in GL_n(\mathbb{Z})\), \(a\mathbb{Z}^n\) is a subgroup of \(\mathbb{Z}^n\) and the quotient group \(\mathbb{Z}^n/(a\mathbb{Z}^n)\) has order \(M = |\det a|\). Thus, we can choose a complete set of representatives of \(\mathbb{Z}^n/(a\mathbb{Z}^n)\), i.e., a set \(\alpha_0, \ldots, \alpha_{M-1} \in \mathbb{Z}^n/(a\mathbb{Z}^n)\) so that each element \(k \in \mathbb{Z}^n\) can be uniquely expressed in the form

\[ k = a k' + \alpha_i, \]

where \(k' \in \mathbb{Z}^n\) and \(\alpha_i \in \alpha_i + (a\mathbb{Z}^n)\).
with \( k' \in \mathbb{Z}^n \) and \( 0 \leq i \leq M - 1 \). This shows that, for each \( k \in \mathbb{Z}^n \), we have

\[
D_a^{-1}T_k = T_{a^{-1}k}D_a^{-1} = T_{k'}T_{a^{-1}a_i}D_a^{-1} = T_{k'}D_a^{-1}T_{a_i},
\]

(6)

with \( k' \in \mathbb{Z} \) and \( 0 \leq i \leq M - 1 \). For any \( \phi \in L^2(\mathbb{R}^n) \setminus \{0\} \), the space \( D_a^{-1}\langle \phi \rangle \) is then the shift–invariant space generated by \( \Phi = \{ \phi' = D_a^{-1}T_{a_i}\phi : 0 \leq i \leq M - 1 \} \). Since \( D_a^{-1} \) is unitary, then \( \Phi \) is a \( \mathbb{Z}^n \)-orthonormal generating set for \( D_a^{-1}\langle \phi \rangle \) if and only if \( \phi \) is a \( \mathbb{Z}^n \)-orthonormal generating set for \( \langle \phi \rangle \). Thus, if \( \Phi \) is a \( \mathbb{Z}^n \)-orthonormal generating set for \( D_a^{-1}\langle \phi \rangle \), we have

\[
D_a^{-1}\langle \phi \rangle = \bigoplus_{i=0}^{M-1}\langle \phi^i \rangle,
\]

(7)

with \( [\hat{\phi}^i, \hat{\phi}^j] = 1 \) a.e. for \( 0 \leq i \leq M - 1 \).

We can now prove Proposition 9

**Proof of Proposition 9.** Take a complete collection of distinct representatives \( \beta_0, \ldots, \beta_{N-1} \) for \( B/(a B a^{-1}) \). Thus, each \( b \in B \) uniquely determines \( b' \in B \) and \( j \in \{0, \ldots, N-1 \} \) for which \( b = (a b' a^{-1}) \beta_j \). Then

\[
D_a^{-1}D_b\langle \phi \rangle = D_{a^{-1}b}\langle \phi \rangle = D_{b'}D_{a^{-1}}D_{\beta_j}\langle \phi \rangle = D_{b'}D_{a^{-1}}\langle D_{\beta_j}\phi \rangle.
\]

(8)

Take a complete collection of distinct representatives \( \alpha_0, \ldots, \alpha_{M-1} \) for the quotient space \( \mathbb{Z}^n/(a \mathbb{Z}^n) \), where \( M = |\det a| \). By equation (7), we have:

\[
D_a^{-1}\langle D_{\beta_j}\phi \rangle = \bigoplus_{i=0}^{N-1}\langle \phi_{i,j} \rangle,
\]

where \( \phi_{i,j} = D_a^{-1}D_{\alpha_i}D_{\beta_j}\phi \), with \( 0 \leq i \leq M - 1, 0 \leq j \leq N - 1 \). We also have:

\[
D_a^{-1}\langle \langle \phi \rangle \rangle = D_a^{-1}\bigg( \bigoplus_{b \in B} D_b\langle \phi \rangle \bigg) = \bigoplus_{b \in B} D_a^{-1}D_b\langle \phi \rangle.
\]

Thus, using (8), from the last expression we obtain:

\[
D_a^{-1}\langle \langle \phi \rangle \rangle = \bigoplus_{b' \in B} \bigoplus_{j} D_{b'}D_{a^{-1}}\langle D_{\beta_j}\phi \rangle = \bigoplus_{b' \in B} D_{b'}\bigg( \bigoplus_{i,j} \langle \phi_{i,j} \rangle \bigg) = \bigoplus_{i,j} \langle \phi_{i,j} \rangle.
\]

Since the unitary operator \( D_a^{-1} \) maps an orthonormal basis for \( \langle \langle \phi \rangle \rangle \) to an orthonormal basis for \( D_a^{-1}\langle \langle \phi \rangle \rangle \), it follows that the set \( \Phi = \{ \phi_{i,j} : 0 \leq i \leq M - 1, 0 \leq j \leq N - 1 \} \) is a \( B \times \mathbb{Z}^n \)-orthonormal set of generators for \( D_a^{-1}\langle \langle \phi \rangle \rangle \). \( \Box \)
5 \textit{AB} multiresolution Analysis. Part II.

In this section, we apply the techniques developed in Section 4 to obtain a number of basic results about \textit{AB} multiresolution analyses.

5.1 Basic results

Let \( \{V_i\}_{i \in \mathbb{Z}} \) be an \textit{AB}-MRA, as defined in Section 4. As in the classical multiresolution analysis, let \( W_0 \) be the orthogonal complement of \( V_0 \) in \( V_1 \), that is, \( W_0 = V_1 \cap (V_0)^\perp \). Then, \( V_1 = V_0 \oplus W_0 \). We have the following elementary result:

**Proposition 10**

(i) Let \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n) \) be such that \( \{D_b T_k \psi^\ell : b \in B, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\} \) is a PF for \( W_0 \). Then \( \Psi \) is a PF \textit{AB}-multiwavelet.

(ii) Let \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n) \) be such that \( \{D_b T_k \psi^\ell : b \in B, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\} \) is an orthonormal basis for \( W_0 \). Then \( \Psi \) is an ON \textit{AB}-multiwavelet.

**Proof.** Define the spaces \( W_j \) as \( W_j = V_{j+1} \cap (V_j)^\perp \), \( j \in \mathbb{Z} \). It follows from the definition of \textit{AB}-MRA that \( L^2(\mathbb{R}^n) = \bigoplus_{j \in \mathbb{Z}} W_j \). Since \( \{D_b T_k \psi^\ell : b \in B, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\} \) is a PF for \( W_0 \), then \( \{D_a D_b T_k \psi^\ell : b \in B, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\} \) is a PF for \( W_i \). Thus \( \{D_a D_b T_k \psi^\ell : b \in B, a \in A, \ell = 1, \ldots, L, k \in \mathbb{Z}^n\} \) is a PF for \( L^2(\mathbb{R}^n) \).

The proof for the orthonormal case is similar. \( \square \)

In the situation described by the hypotheses of Proposition 10 (where \( \Psi \) is not only a PF for \( L^2(\mathbb{R}^n) \), but it is also derived from an \textit{AB}-MRA), we say that \( \Psi \) is a \textbf{PF MRA} \textit{AB-multiwavelet} or an \textbf{ON MRA} \textit{AB-multiwavelet}, respectively.

We say that the PF MRA \textit{AB}-wavelet \( \psi \) is of \textbf{finite filter (FF) type} if there exists an \textit{AB} scaling function \( \phi \) for \( V_0 \) and a finite set \( \{b_1, \ldots, b_k\} \subset B \) such that

\[
\hat{\phi}(\xi a) = \sum_{j=1}^{k} m_0^{(j)}(\xi) \hat{\phi}(\xi b_j), \quad \hat{\psi}(\xi a) = \sum_{j=1}^{k} m_1^{(j)}(\xi) \hat{\phi}(\xi b_j),
\]

where \( m_0^{(j)}, m_1^{(j)}, 1 \leq j \leq k \), are periodic functions. Similarly, the ON MRA \textit{AB-multiwavelet} \( \Psi \) is of finite filter (FF) type if there exists an \textit{AB} scaling
function $\phi$ for $V_0$ and a finite set $\{b_1, \ldots, b_k\} \subset B$ such that

$$\hat{\phi}(\xi a) = \sum_{j=1}^{k} m_0^{(j)}(\xi) \hat{\phi}(\xi b_j), \quad \hat{\psi}^\ell(\xi a) = \sum_{j=1}^{k} m_1^{(j)}(\xi) \hat{\phi}(\xi b_j), \quad \ell = 1, \ldots, L,$$

where $m_0^{(j)}, m_1^{(j)}$, $1 \leq j \leq k$, are periodic functions. The reader can easily check that the examples of $AB$ multiwavelets presented in Section 2 are indeed MRA $AB$ multiwavelets of finite filter type.

It turns out that, while it is possible to construct a PF $AB$-wavelet using a single generator, that is, $\Psi = \{\psi\}$, in the case of orthonormal MRA $AB$-multiwavelets, multiple generators are needed, that is, $\Psi = \{\psi^1, \ldots, \psi^L\}$, where $L > 1$. This situation is similar to the classical MRA case (cf., for example, [21]). The following result establishes the number of generators needed to obtain an ON MRA $AB$–wavelet.

**Theorem 11** Let $\Psi = \{\psi^1, \ldots, \psi^L\}$ be an ON MRA $AB$-multiwavelet for $L^2(\mathbb{R}^n)$, and let $N = |B/aBa^{-1}|$ (the order of the quotient group $B/aBa^{-1}$). Assume that $|\det a| \in \mathbb{N}$. Then $L = N|\det a| - 1$.

**Proof.** Let $V_0 = \langle \psi^0 \rangle$ be the $AB$ scaling space for the $AB$–MRA, and $\psi^0$ be the corresponding ON AB scaling function. Then $V_1 = D_a^{-1}V_0 = V_0 \oplus W_0$, where $W_0 = \oplus_{\ell=1}^{L}\langle \psi^\ell \rangle$. Hence $\{\psi^0, \psi^1, \ldots, \psi^L\}$ is an ON $B \ltimes \mathbb{Z}^n$ generating set for $V_1$. By Proposition 9, $1 + L = N|\det a|$ and so $L = N|\det a| - 1$. \hfill \Box

In the case of the examples of ON $AB$ multiwavelets given in Section 2, where $B = \{b^j : j \in \mathbb{Z}\}$, with $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A = \{a^i : i \in \mathbb{Z}\}$, with $a = \begin{pmatrix} 2 & 0 \\ 0 & a_{2,2} \end{pmatrix} \in GL_2(\mathbb{Z})$ (we need to assume $a_{2,2} = 1$ or 2 to apply Theorem 11), we have used 3 generators. This number is confirmed by the formula given by Theorem 11. In fact, a calculation shows that $|B/aBa^{-1}| = 2|a_{2,2}|^{-1}$ and, thus, by Theorem 11, the number of generators must be $L = 2|a_{2,2}|^{-1}2|a_{2,2}| - 1 = 3$.

Observe that the condition on the number of generators described by this theorem is not needed if the $AB$ affine system does not come from an ON $AB$–MRA. In Section 6 we present an example of an $AB$ ON wavelet $\psi$ (a single generator) where $A = \{a^i : i \in \mathbb{Z}\}$, $|\det a| = 2$ and $N = |B/aBa^{-1}| = 2$. It is clear, by Theorem 11, that this example of $AB$–wavelet is not of MRA type.

The following theorem describes how to construct tiling ON $AB$ multiwavelets arising from an $AB$–MRA.

**Theorem 12** Let $B \subset \text{SL}_n(\mathbb{Z})$, $a \in GL_n(\mathbb{Z})$ with $aBa^{-1} \subset B$, and $L = N M - 1$, where $N = |B/aBa^{-1}|$ and $M = |\det a| > 1$. Suppose that $U \subset \mathbb{R}^n$
is a measurable set and \( \phi = (\chi_U)\hat{\psi} \in L^2(\mathbb{R}^n) \) is an ON AB scaling function for \( V_0 = \overline{\text{span}} \{ D_b T_k \phi : k \in \mathbb{Z}^n, b \in B \} \), with \( V_0 \subseteq D_{a^{-1}}^* V_0 \). Then there are sets \( T_\ell \subset \mathbb{R}^n, \ell = 1, \ldots, L \), for which \( \Psi = \{ \psi^\ell = (\chi_{T_\ell})\hat{\psi} : \ell = 1, \ldots, L \} \) is an ON MRA AB–multiwavelet, and \( \Psi \) is of FF type.

**Proof.** By hypothesis, \( \{ D_b T_k \phi : b \in B, k \in \mathbb{Z}^n \} \) is an ON basis of \( V_0 \), and, since \( \phi = \chi_U \), then \( V_0 = L^2(S_0)\hat{\psi} \), where \( S_0 = \bigcup_{b \in B} \overline{U} b \) and the union is disjoint. For \( i \in \mathbb{Z} \), let \( V_i = D_{a^{-i}}^* V_0 = L^2(S_i)\hat{\psi} \). Then \( V_i \subseteq V_{i+1} \), and \( S_i = S_0 a^i \subset S_{i+1} \). It follows easily that \( \cap_{i \leq 0} S_i \) and \( \mathbb{R}^n \backslash \cup_{i \geq 0} S_i \) are null sets.

Next let \( W_0 = V_1 \cap (V_0)\hat{\psi} = L^2(S_1 \backslash S_0)\hat{\psi} \). We will show that there are sets \( T_\ell \), \( 1 \leq \ell \leq L \), such that each \( T_\ell \) is a tiling set for \( \mathbb{Z}^n \) translations and the disjoint union \( U_{\ell=1}^\ell T_\ell \) is a \( (S_1 \backslash S_0) \)–tiling set for \( B \) dilations. In order to do that, let \( \beta_0, \ldots, \beta_{N-1} \) be a complete collection of coset representatives of \( B/aB a^{-1} \), with \( \beta_0 = I_n \), and let \( U_1 = \cup_{j=0}^{N-1} U \beta_j a \). Since each \( b \in B \) uniquely determines a \( b' \in B \) and a \( j \) for which \( b = \beta_j(a b' a^{-1}) \), we have:

\[
S_1 = S_0 a = \bigcup_{b \in B} U b a = \bigcup_{b' \in B} U_1 b'.
\]

Thus \( U_1 \) is an \( S_1 \)–tiling set for \( B \) dilations and, as a consequence, \( \tilde{U} = U_1 \cap S_0 \) is an \( S_0 \)–tiling set for \( B \) dilations and \( T = U_1 \backslash \tilde{U} \) is an \( (S_1 \backslash S_0) \)–tiling set for \( B \) dilations. Note that \( |\tilde{U}| = |U| = 1 \) since \( |\text{det } b| = 1 \) for each \( b \in B \). Also, \( |U_1| = N |\text{det } a| = N M \) and so \( |T| = N M - 1 = L \). By an easy calculation, \((\chi_{U_1}, \chi_U) = N M \) a.e. Thus, for a.e. \( \xi \in \mathbb{R}^n \), there are precisely \( N M \) points in \( (\xi + \mathbb{Z}^n) \cap T \) and exactly one of these points lies in \( \tilde{U} \). This implies that \((\chi_T, \chi_T) = L \) a.e. Now one can decompose \( T \) into disjoint subsets \( T_\ell, 1 \leq \ell \leq L \), with \((\chi_{T_\ell}, \chi_{T_\ell}) = 1 \) a.e. for each \( \ell \). The sets \( T_\ell \) have precisely the properties we were looking for, and, as a consequence, \( \Psi = \{ \psi^\ell = (\chi_{T_\ell})\hat{\psi} : \ell = 1, \ldots, L \} \) is an ON MRA AB–multiwavelet.

In order to prove the final statement, observe that \( T_\ell a^{-1} \subseteq U_1 a^{-1} = \cup_{j=0}^{N-1} U \beta_j \) and \( U a^{-1} \subseteq U_1 a^{-1} \). This implies that, for all \( 0 \leq \ell \leq L \), using the notation \( \psi^0 = \phi \) and \( T_0 = U \), we have

\[
\hat{\psi}^\ell(\xi a) = \chi_{T_\ell}(\xi a) = \chi_{T_\ell a^{-1}}(\xi) = \sum_{j=0}^N \chi(T_\ell a^{-1} \cap U \beta_j)(\xi) = \sum_{j=0}^N \chi(T_\ell a^{-1} \cap U \beta_j)(\xi) \chi(\beta_j^{-1})(\xi) = \sum_{j=0}^N m_j^\ell(\xi) \hat{\phi}(\xi \beta_j^{-1}),
\]

where \( m_j^\ell(\xi) \) is the \( \mathbb{Z}^n \) periodic extension of \( \chi(T_\ell a^{-1} \cap U \beta_j)(\xi) \). \( \square \)
5.2 Well-localized AB wavelets

Up to this point, our construction of AB multiwavelets has been limited to systems arising from compact tiling sets in the frequency domain \( \hat{\mathbb{R}}^n \). Such AB multiwavelets are smooth in \( \mathbb{R}^n \) but have slow decay. In this section, we will give an explicit construction of smooth AB wavelets with fast decay both in \( \mathbb{R}^n \) and \( \hat{\mathbb{R}}^n \). Systems with these properties are very important for applications since fast decay is essential for their numerical implementation. In the previous section we have seen how filters arise naturally in AB MRA systems. As is the case with classical MRA wavelets, the filters’ role will be even more prominent in the constructions of this section.

5.2.1 Example 1.

Let \( \psi_1 \in L^2(\mathbb{R}) \) be a (one-dimensional) dyadic band-limited wavelet with \( \text{supp} \hat{\psi}_1 \subset [-\Omega, \Omega], \Omega > 0 \), and \( \psi_2 \in L^2(\mathbb{R}) \) be another band-limited function with \( \text{supp} \hat{\psi}_2 \subset [-1, 1] \) and satisfying

\[
\sum_{j \in \mathbb{Z}} |\hat{\psi}_2(\xi + j)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}.
\]

Recall that, since \( \psi_1 \) is a dyadic wavelet, then it satisfies the Calderón equation (cf. Sec.3.1):

\[
\sum_{j \in \mathbb{Z}} |\hat{\psi}_1(2^j \xi)|^2 = 1 \quad \text{a.e. } \xi \in \mathbb{R}.
\]

As we will show later on, there are several choices of functions \( \psi_1 \) and \( \psi_2 \) satisfying these properties.

For any \( \omega = (\omega_1, \omega_2) \in \mathbb{R}^2, \omega_1 \neq 0 \), define \( \psi \in L^2(\mathbb{R}^2) \) by

\[
\hat{\psi}(\omega) = \hat{\psi}_1(2^s \omega_1) \hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right),
\]

where \( s \in \mathbb{Z} \) satisfies \( 2^s \geq 2\Omega \). This assumption ensures that \( \text{supp} \hat{\psi} \subset [-1/2, 1/2]^2 \). In fact, since \( \text{supp} \hat{\psi}_1 \subset [-\Omega, \Omega] \) and \( \text{supp} \hat{\psi}_2 \subset [-1, 1] \), it follows from (3) that \( \hat{\psi}(\omega_1, \omega_2) = 0 \) for \( |\omega_1| > 1/2 \) and \( |\omega_2| > 1/2 \). It is now simple to show that \( \psi \) is a PF AB-wavelet, where \( A = \{a^k = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}\} \) and \( B = \{b^j = \begin{pmatrix} 1 \\ j \\ 0 \end{pmatrix} : j \in \mathbb{Z}\} \). Indeed, observing that \( \omega a^k b^j = (2^k \omega_1, j 2^k \omega_1 + \omega_2) \), and using (1), (2) and (3), we have that
\[ \sum_{j,k \in \mathbb{Z}} |\hat{\psi}(\omega a^k b^j)|^2 = \sum_{j,k \in \mathbb{Z}} |\hat{\psi}_1(2^{s+k} \omega_1)|^2 |\hat{\psi}_2(2^{-k} \frac{\omega_2}{\omega_1} + j)|^2 \]
\[ = \sum_{k \in \mathbb{Z}} |\hat{\psi}_1(2^{s+k} \omega_1)|^2 \sum_{j \in \mathbb{Z}} |\hat{\psi}_2(2^{-k} \frac{\omega_2}{\omega_1} + j)|^2 = 1 \text{ a.e.} \]

The fact that \( \psi \) is a PF \( AB \)-wavelet now follows from the following general observation.

**Proposition 13** Let \( \psi \in L^2(\mathbb{R}^n) \) be such that \( \text{supp} \hat{\psi} \subset Q = [-1/2, 1/2]^n \), and
\[ \sum_{j,k \in \mathbb{Z}} |\hat{\psi}(\omega a^k b^j)|^2 = 1 \text{ a.e. } \omega \in \mathbb{R}^n, \]
where \( a, b \in \text{GL}_n(\mathbb{R}) \). Then \( \psi \) is a PF \( AB \)-wavelet, where \( A = \{a^i : i \in \mathbb{Z}\} \) and \( B = \{b^j : j \in \mathbb{Z}\} \).

**Proof.** For, \( i, k \in \mathbb{Z}, k \in \mathbb{Z}^n \), let \( \psi_{i,j,k} = D^i_a D^j_b T_k \psi \). Using the hypotheses on \( \psi \), the change of variable \( \eta = \xi a^i b^j \) and Plancherel theorem, then, for each \( f \in L^2(\mathbb{R}^n) \), we have that:

\[
\sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \left| \langle f, \psi_{i,j,k} \rangle \right|^2 = \sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \left| \int_{\mathbb{R}^n} \hat{f} (\omega) \overline{\hat{\psi}}(\omega a^i b^j) e^{2\pi i \omega a^i b^j k} \det a^{i/2} \det b^{j/2} d\omega \right|^2
\]
\[ = \sum_{i,j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \left| \int_Q \hat{f}(\eta b^{-j} a^{-i}) \overline{\hat{\psi}}(\eta) e^{2\pi i \eta k} \det a^{-i/2} \det b^{-j/2} d\eta \right|^2
\]
\[ = \sum_{i,j \in \mathbb{Z}} \int_Q \left| \hat{f}(\eta b^{-j} a^{-i}) \right|^2 |\hat{\psi}(\eta)|^2 \det a^{-i} \det b^{-j} d\eta
\]
\[ = \sum_{i,j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |\hat{\psi}(\xi a^i b^j)|^2 d\xi
\]
\[ = \|f\|^2. \quad \Box \]

As we mentioned before, there are many choices for the functions \( \psi_1 \) and \( \psi_2 \) that satisfy the assumptions we have described above. For example, we can choose \( \psi_1 \) to be the Lemarié-Meyer wavelet (see [14, Sec.1.4]), defined by \( \hat{\psi}_1(\xi) = e^{i\pi \xi} b(\xi) \), where

\[
b(\xi) = \begin{cases} \sin\left(\frac{\pi}{2} (3|\xi| - 1)\right) & \frac{1}{3} \leq |\xi| \leq \frac{2}{3} \\
\sin\left(\frac{3\pi}{4} (\frac{2}{3} - |\xi|)\right) & \frac{2}{3} < |\xi| \leq \frac{4}{3} \\
0 & \text{otherwise}. \end{cases}
\]

In order to construct \( \psi_2 \), let \( \phi \) be a compactly supported \( C^\infty \) bump function, with \( \text{supp} \phi \subset [-1, 1] \) (examples can be found in [19, Sec. 3.3] or [15, Sec. 1.4]),
and define \( \psi_2 \) by
\[
\hat{\psi}_2(\xi) = \frac{\phi(\xi)}{\sqrt{\sum_{k \in \mathbb{Z}} |\phi(\xi + k)|^2}}.
\]
It is clear that \( \psi_2 \in C^\infty(\mathbb{R}) \) and satisfies (1). It follows that \( \hat{\psi} \), given by (3), is in \( C^\infty(\mathbb{R}^2) \) and this implies that \( |\psi(x)| \leq K_N (1 + |x|)^{-N}, K_N > 0 \), for any \( N \in \mathbb{N} \).

Finally, let us observe that it is easy to generalize this construction for \( n > 2 \). For example, let \( \psi_1, \psi_2 \in L^2(\mathbb{R}) \) be defined as above, and, for any \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n, \omega_1 \neq 0 \), define \( \psi \in L^2(\mathbb{R}^n) \) by
\[
\hat{\psi}(\omega) = \hat{\psi}_1(2^s \omega_1) \hat{\psi}_2\left(\frac{\omega_2}{\omega_1}\right) \ldots \hat{\psi}_2\left(\frac{\omega_n}{\omega_1}\right),
\]
where \( s \in \mathbb{Z} \) satisfies \( 2^s \geq 2 \Omega \). It turns out that \( \psi \) is a PF \( AB \)-wavelet, where
\[
A = \{ a^i = \begin{pmatrix} 2^i & 0 \\ 0 & I_{n-1} \end{pmatrix} : i \in \mathbb{Z} \} \quad \text{and} \quad B = \{ b_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \} : j \in \mathbb{Z}^{n-1} \},
\]
where \( I_{n-1} \) is the \((n-1) \times (n-1)\) identity matrix. The proof is exactly as in the case \( n = 2 \), once is observed that, for \( j = (j_1, \ldots, j_{n-1}) \), with \( j_1, \ldots, j_{n-1} \in \mathbb{Z} \), we have:
\[
\hat{\psi}(\omega a^k b_j) = \hat{\psi}_1(2^{s+k} \omega_1) \hat{\psi}_2\left(\frac{\omega_2}{\omega_1} + j_1\right) \ldots \hat{\psi}_2\left(\frac{\omega_n}{\omega_1} + j_{n-1}\right).
\]
A similar idea can be applied to more general shear groups \( B \).

The next example shows how to construct \( AB \)-wavelets for \( L^2(\mathbb{R}^2) \) of MRA type, that are well–localized both in \( \mathbb{R}^n \) and \( \hat{\mathbb{R}}^n \).

5.2.2 Example 2.

Let \( \psi_1 \in L^2(\mathbb{R}) \) be a (one-dimensional) dyadic band-limited MRA wavelet with \( \text{supp} \hat{\psi}_1 \subset [-\Omega, \Omega], \Omega > 0 \), and \( \phi_1 \) be its associated scaling function. Let \( m_0 \) and \( m_1 \) be the low pass and high pass filters, respectively, associated with \( \phi_1 \) and \( \psi_1 \), that is, \( m_0 \) and \( m_1 \) are the periodic functions satisfying the equations
\[
\hat{\phi}_1(\omega_1) = m_0\left(\frac{\omega_1}{2}\right) \hat{\phi}_1\left(\frac{\omega_1}{2}\right) \quad \text{and} \quad \hat{\psi}_1(\omega_1) = m_1\left(\frac{\omega_1}{2}\right) \hat{\phi}_1\left(\frac{\omega_1}{2}\right).
\]
Let \( \psi_2 \in L^2(\mathbb{R}) \) be defined by
\[
\psi_2(x) = e^{i(N+1)\pi x} \left(\frac{\sin \pi x}{\pi x}\right)^{N+1},
\]
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where \( N \in \mathbb{N} \). That is, \( \hat{\psi}_2 \) is a basic spline of order \( N \) (cf. [14, Sec. 4.2]). This implies that \( \text{supp} \hat{\psi}_2 \subset [0, N + 1] \) and \( \hat{\psi}_2 \) satisfies the so-called two scale equation:

\[
\hat{\psi}_2(\xi) = \sum_{k=0}^{N+1} d_k^{(N)} \hat{\psi}_2(2\xi - k),
\]

(5)

where \( d_k^{(N)} = 2^{-N} \binom{N+1}{k} \).

For \( \omega = (\omega_1, \omega_2) \in \mathbb{R}^2 \), \( \omega_1 \neq 0 \), let \( \hat{\phi} \in L^2(\mathbb{R}^2) \) be defined by

\[
\hat{\phi}(\omega) = \hat{\phi}_1(2^s \omega_1) \frac{\hat{\psi}_2(\frac{\omega_2}{\omega_1})}{\sqrt{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(\frac{\omega_2}{\omega_1} + m)|^2}},
\]

where \( s \in \mathbb{Z} \) satisfies \( 2^s \geq 4 \Omega (\frac{N}{2} + 1) \). This assumption on \( s \) ensures that

\[
\text{supp} \hat{\phi} \subset \{ (\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_1| < \frac{1}{4} (N/2 + 1)^{-1}, |\omega_2| < \frac{1}{4} \}.
\]

(6)

Also, let \( \psi \in L^2(\mathbb{R}^2) \) be defined by:

\[
\hat{\psi}(\omega) = \sum_{k=0}^{N+1} d_k^{(N)} m_1(2^{s-1} \omega_1) M_0(a^{-1} \omega) \hat{\phi}(\omega a^{-1} b^{-k}),
\]

where the matrices \( a \) and \( b \) are as in Section 5.2.1, the coefficients \( d_k^{(N)} \) are those in (5), and \( M_0(\omega) \) is the \( \mathbb{Z}^2 \)-periodic function which, restricted to the fundamental region \( [-\frac{1}{2}, \frac{1}{2}]^2 \), is given by

\[
M_0(\omega) = \left( \frac{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(\frac{\omega_2}{\omega_1} + m)|^2}{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(\frac{\omega_2}{\omega_1} + m)|^2} \right)^{1/2}, \quad \omega \in [-\frac{1}{2}, \frac{1}{2}]^2.
\]

Using (5), we have that:

\[
\sum_{k=0}^{N+1} d_k^{(N)} M_0(\omega a^{-1}) \hat{\phi}(\omega a^{-1} b^{-k}) =
\]

\[
= \hat{\phi}_1(2^{s-1} \omega_1) \sum_{k=0}^{N+1} d_k^{(N)} \frac{\sqrt{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(\frac{\omega_2}{\omega_1} + m)|^2}}{\sqrt{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(2 \frac{\omega_2}{\omega_1} + m)|^2}}\frac{\hat{\psi}_2(2 \frac{\omega_2}{\omega_1} - k)}{\sqrt{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(2 \frac{\omega_2}{\omega_1} + m)|^2}} =
\]

\[
= \hat{\phi}_1(2^{s-1} \omega_1) \sum_{k=0}^{N+1} d_k^{(N)} \frac{\hat{\psi}_2(2 \frac{\omega_2}{\omega_1} - k)}{\sqrt{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(\frac{\omega_2}{\omega_1} + m)|^2}} =
\]

\[
= \hat{\phi}_1(2^{s-1} \omega_1) \frac{\hat{\psi}_2(\frac{\omega_2}{\omega_1})}{\sqrt{\sum_{m \in \mathbb{Z}} |\hat{\psi}_2(\frac{\omega_2}{\omega_1} + m)|^2}}.
\]
Theorem 14, is a PF

ization of all functions \(\Psi = \{\psi^1, \ldots, \psi^L\}\) such that the system \(\Psi_{AB}\), given by (1), is a PF \(AB\)-multiwavelet.

**Theorem 14** Let \(A = \{a^k \mid k \in \mathbb{Z}\} \subset GL_n(\mathbb{Z})\), \(B \subset SL_n(\mathbb{Z}) = \{b \in\)
\[ \text{GL}_n(\mathbb{R}) : |\det b| = 1 \] and \( \Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n) \). Suppose that
\[ \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}} \sum_{b \in B} \sum_{m \in \mathbb{Z}^n} \int_{\text{supp } \hat{f}} |\hat{\psi}^\ell(\xi m b a^k)|^2 |\hat{\psi}^\ell(\xi a^{-k} b^{-1})|^2 \, d\xi < \infty, \tag{7} \]
for all \( f \in \mathcal{D} \) where \( \mathcal{D} \) is a dense subspace of \( L^2(\mathbb{R}^n) \) contained in the set
\[ \{ f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \text{supp } \hat{f} \text{ is compact} \}. \]

Then \( \Psi_{AB} \), given by (1), is a PF for \( L^2(\mathbb{R}^n) \) if and only if
\[ \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}} \sum_{b \in B} |\hat{\psi}^\ell(\xi a^k b)|^2 = 1 \tag{8} \]
\[ \sum_{\ell=1}^L \sum_{k \geq 0} \sum_{b \in B} \hat{\psi}^\ell(\xi a^k b) \overline{\hat{\psi}^\ell((\xi + q) a^k b)} = 0 \quad \text{if } q \in \hat{\mathbb{Z}}^n \setminus (\hat{\mathbb{Z}}^n a) \tag{9} \]
\[ \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}} \sum_{b \in B} \hat{\psi}^\ell(\xi a^k b) \overline{\hat{\psi}^\ell((\xi + q) a^k b)} = 0 \quad \text{if } q \in \bigcap_{k \in \mathbb{Z}} (\hat{\mathbb{Z}}^n a^k) \setminus \{0\}. \tag{10} \]

The hypothesis (7) is the LIC referred to in Section 3.1. For all examples of \( AB \) multiwavelets discussed in this paper, one can show by lengthy computations that (7) is satisfied. Note that (8) is the Calderón equation to which we have often referred above. Equation (9) is the analogue of the so-called \( t_q \) equation for classic dyadic wavelets (cf.[14]). However, (10) has a different character. The striking differences between equations (9)–(10) and characterization equations for the classical dyadic wavelets were part of the motivation that led us to formulate our first examples of \( AB \) multiwavelets and subsequently develop the theory presented in this paper.

### 6 \( AB \)-wavelet sets

In this section, we will show how to construct singly generated ON \( AB \) wavelets. When \( A \) and \( B \) satisfy the hypotheses of Theorem 11, with \( L > 1 \), then these singly generated ON \( AB \) wavelets cannot be of MRA type. Below, we will carry out the demanding technical details for the example of Section 2, where \( a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) and \( b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). A much easier construction applies when \( a \) is replaced by \( a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \), and is presented in [11]. In both cases, the \( AB \)
wavelets are inverse Fourier transform of characteristic functions of fractal-like sets. Our point of view is that such ON non–MRA AB wavelets are “pathological” and far less useful than the Parseval frame wavelets such as those in Section 5.

Let $C \subset GL_n(\mathbb{R})$ be an admissible dilation set (cf. Section 3). A $C$–wavelet set is a measurable set $W \subset \mathbb{R}^n$ such that $\psi = (\chi_W)^\vee$ is an orthonormal $C$–wavelet.

It is easy to verify (cf. [12]) that $W$ is a wavelet set if and only if $W$ is both a tiling set for $\mathbb{Z}^n$ translations and a tiling set for $C^{-1}$ dilations. There are several examples of $C$–wavelet sets in the literature for $C = \{a^i : i \in \mathbb{Z}\}$, where $a \in GL_n(\mathbb{R})$ [1,2,12,13,17,18]. Many such constructions use a technique introduced in [6] that modifies a set $T$ for which $(\chi_T)^\vee$ is a Parseval frame $C$–wavelet to produce a wavelet set $W$ of the form $W = (T \setminus P) \cup Q$, where the union is disjoint, $P \subset T$ and $Q \subset \mathbb{R}^n$ are measurable. For a general $C \in GL_n(\mathbb{R})$, the conditions on $P$ and $Q$ are the following:

(i) $Q = \bigcup_{\xi \in P} Q_{\xi}$ is a disjoint union, where $Q_{\xi}$ is chosen so that $\xi C^{-1} = \bigcup_{\eta \in Q_{\xi}} \eta C^{-1}$;

(ii) let $\pi$ be the projection $\pi(\xi) = \xi + \mathbb{Z}^n$ from $\hat{\mathbb{R}}^n$ into $\mathbb{T}^n$; then $\pi_{|Q}$, i.e., the restriction of $\pi$ to $Q$, is one-to-one with image $\pi(P) \cup \left(\mathbb{T}^n \setminus \pi(T)\right)$.

In fact, since $(\chi_T)^\vee$ is a Parseval frame $C$–wavelet, then $T$ is a tiling set for $C^{-1}$ dilations, and

$$\hat{\mathbb{R}}^n = \bigcup_{\xi \in T} \xi C^{-1} = \left(\bigcup_{\xi \in T \setminus P} \xi C^{-1}\right) \cup \left(\bigcup_{\xi \in P} \xi C^{-1}\right),$$

where the union is disjoint. By (i), $\bigcup_{\xi \in P} \xi C^{-1} = \bigcup_{\eta \in Q_{\xi}} \eta C^{-1}$. Thus (i) implies that $W$ is a tiling set for $C^{-1}$ dilations. Next, since $(\chi_T)^\vee$ is a Parseval frame $C$–wavelet, then $T$ is a packing set for $\mathbb{Z}^n$ translations and so $\pi_{|T}$ is one-to-one. Also,

$$\mathbb{T}^n = \pi(T) \cup \left(\mathbb{T}^n \setminus \pi(T)\right) = \pi(T \setminus P) \cup \pi(P) \cup \left(\mathbb{T}^n \setminus \pi(T)\right),$$

where the union is disjoint. Thus, $W$ is a tiling set for $\mathbb{Z}^n$ translations iff $\pi$ maps $Q$ one-to-one onto $\pi(P) \cup \left(\mathbb{T}^n \setminus \pi(T)\right)$.

In [6], $C$ is assumed to contain an expanding matrix $a \in GL_n(\mathbb{R})$ for which $a C^{-1} = C^{-1}$. Since $a$ is expanding, then there is a tiling wavelet $(\chi_T)^\vee$ where $T \subset \mathbb{R}^n$ is measurable and bounded (cf. Section 3), and a measurable set $U \subset \mathbb{R}^n$ such that $T \subseteq U$, $U$ is a tiling set for $\mathbb{Z}^n$ translations and $U a \cap U = \emptyset$. Since $\xi C^{-1} = \xi a C^{-1}$, for all $\xi$, then for any $P \subseteq T$, condition (i) is satisfied by $Q = Pa$. Using the fact that $|\det a| > 1$, one can obtain a set $P \subseteq T$.
for which (ii) is also satisfied, and so $W = (T \setminus P) \cup Pa$ is a wavelet set.

This construction applies, for example, to $C = AB = \{a^i b : i \in \mathbb{Z}, b \in B\}$, where $a \in GL_n(\mathbb{R})$ is expanding, and $B \subset GL_n(\mathbb{R})$ satisfies $aBa^{-1} = B$. The orthogonal and the hyperbolic $AB$-wavelets described in Sections 3.3.1 and 3.3.2, respectively, are in this class.

More generally, let us consider the case $C = AB = \{a^i b : i \in \mathbb{Z}, b \in B\}$, where $B$ is a subgroup of $SL_n(\mathbb{Z})$, $a \in GL_n(\mathbb{R})$ is not necessarily expanding, and $aBa^{-1} \not\subset B$. These assumptions imply that $aC^{-1} = aBA = (aBa^{-1})(aA) \not\subset BA = C^{-1}$. Let $N = \text{card} \left( B / (aBa^{-1}) \right)$ and $\{b_1, \ldots, b_N\}$ be a complete set of coset representatives of $B / (aBa^{-1})$. Then we have that $BA = \bigcup_{j=1}^{N} b_j aBa^{-1} A = \bigcup_{j=1}^{N} b_j aBA$. Let $U_0$ be a tiling set for $\mathbb{Z}^n$ translations for which $S_0 = \bigcup_{b \in B} U_0 b$ is contained in $S_1 = S_0 a$, and let $T_0 \subset U_0$ be a tiling set for $(AB)^{-1}$ dilations. Thus, given $P \subset T_0$, we can satisfy condition (i) by setting $Q = \bigcup_{\xi \in P} Q_\xi$, where $Q_\xi = \{ \xi b_j(\xi) a : 1 \leq j \leq N \}$, and $\{b_1(\xi), \ldots, b_N(\xi)\}$ is a complete set of coset representatives of $B / (aBa^{-1})$. The dependence of the coset representatives $b_j(\xi)$ on $\xi$ will be clarified in the proof of the following theorem, where we will show the details for this construction for the example of Section 2. A similar construction holds for more general shear group matrices $B$. In these constructions, the coset representatives $b_j(\xi)$ are not bounded and, as a consequence, the wavelet set is unbounded.

**Theorem 15** Let $A = \{a^i : i \in \mathbb{Z}\}$, $B = \{b^j : j \in \mathbb{Z}\}$ where $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $AB$ wavelets exist.

**Proof.** The set $U_0 = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : 0 < |\xi_1| \leq 1 \text{ and } 0 \leq \xi_2 / \xi_1 \leq 1\}$ is both a tiling set for $\mathbb{Z}^2$ translations and an $S_0$-tiling set for $B$ dilations, where $S_0 = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \leq 1\}$. Let $T_0 = \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \frac{1}{2} \leq |\xi_1| \leq 1\} \subset U_0$. Then $T_0$ is a tiling set for $BA$ dilations and, thus, $(\chi_{T_0})^\vee$ is a PF $AB$ wavelet. Let $T_1 = T_0 a$, $U_1 = U_0 a$.

As in the general construction outlined before, we will construct a wavelet set of the form $W = (T_0 \setminus P) \cup Q$. As we did in Section 2, we shall denote $T_0 = T_0^- \cup T_0^+$, where $T_0^-$ and $T_0^+$ denote the intersection of $T$ with the half-planes $\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 0 \}$ and $\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 < 0 \}$, respectively. We will use a similar notation for any other set in $\mathbb{R}^2$. Since the construction is symmetric with respect to reflection through the origin, it will be sufficient to construct the set $W^+$.

Let $P_k^+ = U_k^+ \setminus T_0^+$ and, for each $k \geq 1$, let $P_k^+ = 2^{-k} P_0^+ + (r_k, 0)$, where $r_k = \sum_{i=1}^{k} 2^{-i} = 1 - 2^{-k}$, and $P^+ = \bigcup_{k \geq 1} P_k^+$. The triangles $P_k^+$ are illustrated
We shall now construct a measurable map $\pi$ maps $P_k^+$ onto $M_{k-1}^+$, for each $k \geq 1$, modulo null sets. Note that, for each $j \in \mathbb{Z}$ and $\xi = (\xi_1, \xi_2) \in P_k^+$, the map $\pi(\xi^{b^j(\xi)+1}a)$ has the form $(2\xi_1, \xi_2 + (2j+1)\xi_1 - m)$ for some $m \in \mathbb{Z}$. Once we construct such a map, then it follows that $\pi(Q^+) = \pi(\cup_{k \geq 1} Q_k^+) = (U_0^+ \setminus T_0^+) \cup P^+$, and, as a consequence (ii) is satisfied. This fact, together with the previous part of the
proof, implies that $U_0 = P_0 \cup T_0$ is a disjoint union and $W = (T_0 \setminus P) \cup Q$ is an $AB$–wavelet set. Thus, it only remains to construct the measurable map that we have described.

Fix $k \geq 1$. For $j, m \in \mathbb{Z}$, let $\pi_{j,m}(\xi_1, \xi_2) = (2\xi_1 - 1, \xi_2 + (2j + 1)\xi_1 - m)$ and let $T_{j,m} = \{\xi \in \text{interior of } P_k^+ : \pi_{j,m}(\xi) \in \text{interior of } M_{k-1}^+\}$. Let $J = \{(j, m) \in \mathbb{Z}^2 : T_{j,m} \neq \emptyset\}$. For $(j, m) \in J$, the set $T_{j,m}$ is an open triangle or an open quadrilateral in $P_k^+$ and $S_{j,m} = \pi_{j,m}(T_{j,m})$ is an open subset of similar shape in $M_{k-1}^+$, with $\text{Area}(S_{j,m}) = 2 \text{Area}(T_{j,m})$ since $\pi_{j,k}$ has Jacobian 2. For $\xi \notin \mathbb{Q}$, the set $\{(2j + 1)\xi_1 - m : j, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. It follows that the open set $\bigcup_{(j,m) \in J} T_{j,m}$ is dense in $P_k^+$ and similarly the set $\bigcup_{(j,m) \in J} S_{j,m}$ is dense in $M_{k-1}^+$. It is clear that, for $\xi \in T_{j,m}$, we have that $\pi(\xi b^{2(j+1)a}) = \pi_{j,m}(\xi)$. Let $\{\eta_i, m_i \} : i \geq 1$ be an enumeration of the countable set $J$ and let $T_1 = T_{j_1,m_1}$, with $j(\eta) = j_1$ on $T_1$. Then let $T_2 = T_1 \cup \{\xi \in T_{j_2,m_2} \setminus T_1 : \pi_{j_2,m_2}(\xi) \notin \pi_{j_1,m_1}(T_1)\}$ and $j(\eta) = j_2$ on $T_2 \setminus T_1$. We proceed inductively, with $T_n$ constructed so that $T_n = \bigcup_{i=1}^n T_n \cap T_{j_i,m_i}$ and $S_n = \bigcup_{i=1}^n \pi_{j_i,m_i}(T_n \cap T_{j_i,m_i})$ have disjoint unions in $P_k^+$ and $M_{k-1}^+$, respectively. Then we define $T_{n+1} = T_n \cup \{\xi \in T_{j_{n+1},m_{n+1}} \setminus T_n : \pi_{j_{n+1},m_{n+1}}(\xi) \notin S_n\}$ and let $j(\eta) = j_i$ on $T_{n+1} \cap T_{j_i,m_i}$. The sets $T_n$ and $S_n$ are unions of open polygons with $\text{Area}(S_n) = 2 \text{Area}(T_n)$. For each $c \in (r_k, r_{k+1})$, each of the maps $\pi_{j,m}$ sends the vertical line $\xi_1 = c$ to the vertical line $\eta_1 = 2c - 1$. Hence for $T = \bigcup_{n=1}^\infty T_n = \lim_{n \to \infty} T_n$ and $S = \bigcup_{n=1}^\infty S_n = \lim_{n \to \infty} S_n$, the segment $T \cap \{(\xi, \xi_2) : \xi_1 = c\}$ is a union of open intervals whose total length $\ell(c)$ coincides with the length of the segment $S \cap \{(\xi_1, \xi_2) : \xi_1 = 2c - 1\}$. If $\ell(c) = c - r_k$ and thus is equal to the length of the segment $P_k^+ \cap \{(\xi_1, \xi_2) : \xi_1 = c\}$, for a.e. $c \in (r_k, r_{k+1})$, then clearly $T$ has full measure in $P_k^+$. Otherwise, arguing by contradiction, let us suppose that $\ell(c) < c - r_k$ for some $c \notin \mathbb{Q}$. Then $P_k^+ \cap \{(\xi_1, \xi_2) : \xi_1 = c\}$ contains an open interval $I_c$ of points $(c, \xi_2)$ not in $T$ and, as a consequence, $M_{k-1}^+ \cap \{(\xi_1, \xi_2) : \xi_1 = 2c - 1\}$ contains an open interval $I_c$ of points $(2c - 1, \xi_2)$ not in $S$. By our comments before, it follows that $\pi_{j,m}(I_c) \subset J_c$ for some $(j, m) \in J$. However this contradicts the definition of $T$ since $(j, m) = (j_i, m_i)$ for some $i$, and $I_c$ would have been included in $T_i$. It follows that $T$ has full measure in $P_k^+$ and necessarily $S$ has full measure in $M_{k-1}^+$. Observe that the map $\xi \mapsto j(\xi)$ defined in the construction of $T$ is constant on polygonal sets and hence is measurable. This completes the proof that condition (ii) is satisfied. □
References


