Characterization of Piecewise-Smooth Surfaces using the 3D Continuous Shearlet Transform

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Abstract

One of the most striking features of the Continuous Shearlet Transform is its ability to precisely characterize the set of singularities of multivariable functions through its decay at fine scales. In dimension $n = 2$, it was previously shown that the continuous shearlet transform provides a precise geometrical characterization for the boundary curves of very general planar regions, and this property sets the groundwork for several successful image processing applications. The generalization of this result to dimension $n = 3$ is highly nontrivial, and so far it was known only for the special case of 3D bounded regions where the boundary set is a smooth 2-dimensional manifold with everywhere positive Gaussian curvature. In this paper, we extend this result to the general case of 3D bounded regions with piecewise-smooth boundaries, and show that also in this general situation the continuous shearlet transform precisely characterizes the geometry of the boundary set.

Key words: Analysis of singularities, continuous wavelets, curvelets, directional wavelets, edge detection, shearlets, wavelets.
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1 Introduction

The shearlet transform has emerged in recent years as one of the most successful extensions of the traditional wavelet transform. Indeed, while the continuous wavelet transform is able to identify the locations of singularities of functions and distributions through its asymptotic behavior at fine scales [15,18], it lacks the ability to capture additional information about the geometry of the singularity set. This is a major disadvantage in dealing with multidimensional data, which are frequently dominated by distributed singularities such as edges or surface boundaries. One manifestation of this fact is that wavelets are far from optimal with respect to their ability to approximate piecewise smooth multivariable functions [1,7,9].

The reason for the limitation of the traditional wavelet framework lies in its intrinsic isotropic nature. By contrast, the shearlet approach is designed to capture singularities defined along curves, surfaces and other anisotropic features with very high efficiency. This is achieved by mapping a function or distribution f into the elements

$$\mathcal{SH}_\psi f(a, s, t) = \langle f, \psi_{ast} \rangle,$$

where the analyzing functions ψ_{ast} (called *shearlets*) are well-localized waveforms obtained through the action of anisotropic dilations, shearing transformations and translations, parametrized by $a > 0$, $s \in \mathbb{R}$ and $t \in \mathbb{R}^2$, respectively, on a generator function ψ. This approach allows one to decompose f not only in terms of locations and scales, like the traditional wavelet approach, but also according to their directional information and taking advantage of the anisotropic features of f. [2]

It follows that the shearlet transform has a unique ability to capture the geometry of the set of singularities of functions and distributions, as was fully established in dimension $n = 2$. Specifically, let $B = \chi_S$, where $S \subset \mathbb{R}^2$, and its boundary ∂S is a piecewise smooth curve. It was shown in [8] (extending and refining previous results in [11,16]) that both the location and the orientation of the boundary curve ∂S can be precisely identified from the asymptotic decay of $\mathcal{SH}_\psi B(a, s, p)$ at fine scales (as $a \to 0$). In fact the following estimates hold:

- If $p \notin \partial S$, then $|\mathcal{SH}_\psi B(a, s, p)|$ *decays rapidly*, as $a \to 0$, for each $s \in \mathbb{R}$. By

[2] Notice that the continuous curvelet transform [2] also employs analyzing elements defined at various locations, scales and orientations, and it shares some of the properties of the continuous shearlet transform. However, the shearlet transform has the distinctive feature of being derived from the theory of affine systems, and this provides several advantages in terms of discretization and extensions to higher dimensions [3,4,9,13].

rapid decay, we mean that, given any $N \in \mathbb{N}$, there is a $C_N > 0$ such that $|\mathcal{SH}_\psi B(a,s,p)| \le C a^N$, as $a \to 0$.

- If $p \in \partial S$ and ∂S is smooth near p, then $|\mathcal{SH}_\psi B(a,s,p)|$ *decays rapidly*, as $a \to 0$, for each $s \in \mathbb{R}$ unless $s = s_0$ is the normal orientation to ∂S at p. In this last case, $|\mathcal{SH}_\psi B(a,s_0,p)| \sim a^{\frac{3}{4}}$, as $a \to 0$.
- If p is a corner point of ∂S and $s = s_0$, $s = s_1$ are the normal orientations to ∂S at p, then $|\mathcal{SH}_\psi B(a,s_0,p)|, |\mathcal{SH}_\psi B(a,s_1,p)| \sim a^{\frac{3}{4}}$, as $a \to 0$. For all other orientations, the asymptotic decay of $|\mathcal{SH}_\psi B(a,s,p)|$ is faster (even if not necessarily "rapid").

These results provide the theoretical justification and the groundwork for very competitive numerical algorithms for edge analysis and detection, such as those presented in [19,22], and this further demonstrates the benefits of the shearlet multiscale directional framework with respect to the traditional wavelet approach. Also recall that the localization properties of the continuous shearlet transform are related to the sparsity properties of the corresponding discrete shearlet transform [7,12,17].

The mathematical framework of the 2-dimensional shearlet transform extends naturally to higher dimensions. In fact, the shearlet transform is closely related to the square integrable representations of the shearlet group, and this group has several n-variate generalizations, as shown in [3,10]. The 3-dimensional case, in particular, is of great interest in applications such as medical and seismic imaging, where important phenomena are usually associated with surfaces of discontinuities.

As observed in [10], while it is straightforward to define a 3-dimensional shearlet transform $\mathcal{SH}_\psi$, many of the arguments introduced in the 2-dimensional setting for the analysis of curve singularities do not carry over to the 3D setting. This is due to the additional geometric complexity of dealing with singularity sets defined on surfaces rather than curves. Hence, to deal with the 3D problem, several new ideas were introduced by the authors in [10]. Using these estimates we were able to show that, similar to the 2-dimensional case, if $B = \chi_C$, where $C \subset \mathbb{R}^3$ is a convex region with positive Gaussian curvature, then the 3-dimensional continuous shearlet transform of B has rapid asymptotic decay at fine scales for all locations, except for the boundary surface ∂C when the orientation variable corresponds to the normal direction to the surface. However, the positive Gaussian curvature assumption required by the argument used in [10] is too restrictive to model the types of surfaces of discontinuities usually found in applications. Thus, the goal of this paper is to extend the 3D result to a much more general and realistic setting. This requires a new approach. In particular, a major new technical tool developed in this paper is based on a method to approximate any regular surface using a quadratic surface (see Lemma 4.5). This approach allows us to translate the complicated geometric properties of the surface (e.g., the curvature) into the "algebraic" properties of the coefficients of the quadratic form. The new characterization result presented in this paper includes the one in [10] as a special case (elliptic quadratic form) as well as many new important cases which fall into the setting of general piecewise smooth surfaces.

As in the 2D case, the theoretical estimates derived in this work have a direct impact in the development of numerical algorithms for the analysis of boundaries of 3D objects, as shown by the preliminary numerical results presented in [20].

The paper is organized as follows. The definition of the shearlet transform, including the special properties which are needed for the applications discussed in the paper, is given in Section 2. The main theorem is presented in Section 3. The proof of the main theorems the other results which are needed for its proof are given in Section 4.

2 The shearlet transform

We recall the definition of the continuous shearlet transform, which was originally introduced in [16] (see also related results in [3,5]) and extended to the 3D setting in [10]. Consider the subspace of $L^2(\mathbb{R}^3)$ given by $L^2(C_1)^\vee = \{f \in L^2(\mathbb{R}^3) : \operatorname{supp} \hat{f} \subset C_1\}$, where C_1 is the truncated pyramidal region in the frequency plane given by:

$$C_1 = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_1| \geq 2, \ |\tfrac{\xi_2}{\xi_1}| \leq 1 \text{ and } |\tfrac{\xi_3}{\xi_1}| \leq 1\}.$$

The following proposition, which is a simple generalization of a result from [16], provides sufficient conditions on the function ψ for obtaining a reproducing system of continuous shearlets on $L^2(C_1)^\vee$.

Proposition 2.1 *Consider the shearlet group* $\Lambda^{(1)} = \{(M_{as_1s_2}, p) : 0 \leq a \leq \frac{1}{4}, -\frac{3}{2} \leq s_1 \leq \frac{3}{2}, -\frac{3}{2} \leq s_2 \leq \frac{3}{2}, p \in \mathbb{R}^2\}$, *where* $M_{as_1s_2} = \begin{pmatrix} a & -a^{1/2}\,s_1 & -a^{1/2}\,s_2 \\ 0 & a^{1/2} & 0 \\ 0 & 0 & a^{1/2} \end{pmatrix}$. *For* $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $\xi_1 \neq 0$, *let* $\psi^{(1)}$ *be defined by*

$$\hat{\psi}^{(1)}(\xi) = \hat{\psi}^{(1)}(\xi_1, \xi_2, \xi_3) = \hat{\psi}_1(\xi_1)\, \hat{\psi}_2(\tfrac{\xi_2}{\xi_1})\, \hat{\psi}_2(\tfrac{\xi_3}{\xi_1}),$$

where:

(i) $\psi_1 \in L^2(\mathbb{R})$ *satisfies the (generalized) Calderòn condition*

$$\int_0^\infty |\hat{\psi}_1(a\xi)|^2 \, \frac{da}{a} = 1 \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{1}$$

and supp $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$;

(ii) $\|\psi_2\|_{L^2} = 1$ *and* $\text{supp}\,\hat{\psi}_2 \subset [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]$.

Let $\psi^{(1)}_{as_1s_2p}(x) = |\det M_{as_1s_2}|^{-\frac{1}{2}} \psi^{(1)}(M^{-1}_{as_1s_2}(x-p))$. *Then, for all* $f \in L^2(C_1)^\vee$,

$$f(x) = \int_{\mathbb{R}^3} \int_{-\frac{3}{2}}^{\frac{3}{2}} \int_{-\frac{3}{2}}^{\frac{3}{2}} \int_0^{\frac{1}{4}} \langle f, \psi^{(1)}_{as_1s_2p}\rangle\, \psi^{(1)}_{as_1s_2p}(x)\, \frac{da}{a^4}\, ds_1\, ds_2\, dp,$$

with convergence in the L^2 *sense.*

If the assumptions of Proposition 2.1 are satisfied, we say that the functions

$$\Psi^{(1)} = \{\psi^{(1)}_{as_1s_2p} : 0 \le a \le \tfrac{1}{4}, -\tfrac{3}{2} \le s_1 \le \tfrac{3}{2}, -\tfrac{3}{2} \le s_2 \le \tfrac{3}{2}, p \in \mathbb{R}^2\} \qquad (2)$$

are *continuous shearlets* for $L^2(C_1)^\vee$ and that the corresponding mapping

$$L^2(C_1)^\vee \ni f \mapsto \mathcal{SH}^{(1)} f(a, s_1, s_2, p) = \langle f, \psi^{(1)}_{as_1s_2p}\rangle$$

is the *continuous shearlet transform* on $L^2(C_1)^\vee$ with respect to $\Lambda^{(1)}$. The index (1) used above in the notation of the shearlet system (and of the corresponding shearlet transform) indicates that the system (2) has frequency support in the truncated pyramidal region C_1; similar shearlet systems which are defined in the two other complementary truncated pyramidal regions of $\mathbb{R}^3$ be defined below.

Since, in the frequency domain, a shearlet element $\psi^{(1)}_{as_1s_2p} \in \Psi^{(1)}$ has the form:

$$\hat{\psi}^{(1)}_{as_1s_2p}(\xi_1, \xi_2, \xi_3) = a\, \hat{\psi}_1(a\,\xi_1)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_2}{\xi_1} - s_1))\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tfrac{\xi_3}{\xi_1} - s_2))\, e^{-2\pi i \xi \cdot p},$$

it follows that the functions $\hat{\psi}^{(1)}_{as_1s_2p}$ have supports in the sets:

$$\{(\xi_1, \xi_2, \xi_3) : \xi_1 \in [-\tfrac{2}{a}, -\tfrac{1}{2a}] \cup [\tfrac{1}{2a}, \tfrac{2}{a}],\, |\tfrac{\xi_2}{\xi_1} - s_1| \le \tfrac{\sqrt{2}}{4} a^{\frac{1}{2}},\, |\tfrac{\xi_3}{\xi_1} - s_2| \le \tfrac{\sqrt{2}}{4} a^{\frac{1}{2}}\}.$$

That is, the frequency support of each function is a pair of hyper-trapezoids, symmetric with respect to the origin, with orientation determined by the shearing variables s_1, s_2. The support regions become increasingly more elongated as $a \to 0$. Examples of these support regions are illustrated in Figure 1.

There is a variety of examples of functions ψ_1 and ψ_2 satisfying the assumptions of Proposition 2.1. In particular, one can find a number of such examples with the additional property that $\hat{\psi}_1, \hat{\psi}_2 \in C_0^\infty$ [6,16]. For the applications which are discussed in this paper, some additional properties are needed. Namely, in the following we will assume that

$$\hat{\psi}_1 \in C_0^\infty,\ \text{supp}\,\hat{\psi}_1 \subset [-2, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 2], \text{odd, nonnegative in } [\tfrac{1}{2}, 2] \text{ and it satisfies (1);} \qquad (3)$$

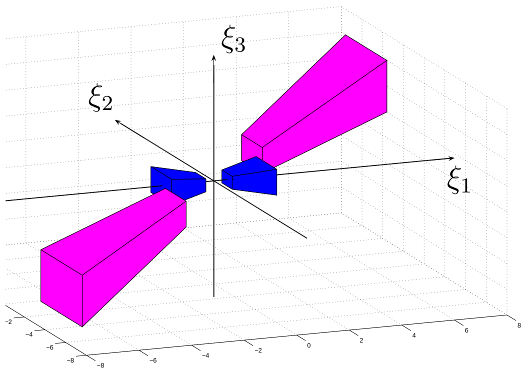

Fig. 1. Support of the shearlet $\hat{\psi}^{(1)}_{as_1s_2p}$, in the frequency domain, for $a = 1/4$, $s_1 = s_2 = 0$ (blue region) and for $a = 1/16$, $s_1 = 0.7, s_2 = 0.5$ (magenta region).

$$\hat{\psi}_2 \in C_0^\infty, \operatorname{supp} \hat{\psi}_2 \subset [-\tfrac{\sqrt{2}}{4}, \tfrac{\sqrt{2}}{4}], \text{ even, nononegative, decreasing in } [0, \tfrac{\sqrt{2}}{4}) \text{ and } \|\psi_2\| = 1. \tag{4}$$

According to Proposition 2.1, the shearlet system $\Psi^{(1)}$, given by (2), is a reproducing system for only a proper subspace of $L^2(\mathbb{R}^3)$. To extend this construction and the corresponding continuous shearlet transform to deal with the whole space $L^2(\mathbb{R}^3)$, one can introduce similar systems defined on the complementary truncated pyramidal regions. Namely, let

$$C^{(2)} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_2| \geq 2, |\tfrac{\xi_1}{\xi_2}| < 1, |\tfrac{\xi_3}{\xi_2}| \leq 1\}.$$

and

$$C^{(3)} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi_3| \geq 2, |\tfrac{\xi_2}{\xi_3}| < 1, |\tfrac{\xi_1}{\xi_3}| < 1\},$$

and, for $i = 2, 3$, define the sets

$$\Lambda^{(i)} = \{(M_{as_1s_2}, p)^{(i)} : 0 \leq a \leq \frac{1}{4}, -\frac{3}{2} \leq s_1 \leq \frac{3}{2}, -\frac{3}{2} \leq s_2 \leq \frac{3}{2}, p \in \mathbb{R}^2\},$$

where

$$M^{(2)}_{as_1s_2} = \begin{pmatrix} a^{1/2} & 0 & 0 \\ -a^{1/2}\, s_1 & a & -a^{1/2}\, s_2 \\ 0 & 0 & a^{1/2} \end{pmatrix}, \quad M^{(3)}_{as_1s_2} = \begin{pmatrix} a^{1/2} & 0 & 0 \\ 0 & a^{1/2} & 0 \\ -a^{1/2}\, s_1 & -a^{1/2}\, s_2 & a \end{pmatrix}.$$

Next, let

$$\hat{\psi}^{(2)}(\xi) = \hat{\psi}^{(2)}(\xi_1, \xi_2, \xi_3) = \hat{\psi}_1(\xi_2)\, \hat{\psi}_2(\tfrac{\xi_1}{\xi_2})\, \hat{\psi}_2(\tfrac{\xi_3}{\xi_2}),$$
$$\hat{\psi}^{(3)}(\xi) = \hat{\psi}^{(2)}(\xi_1, \xi_2, \xi_3) = \hat{\psi}_1(\xi_3)\, \hat{\psi}_2(\tfrac{\xi_1}{\xi_3})\, \hat{\psi}_2(\tfrac{\xi_2}{\xi_3}),$$

where $\hat{\psi}_1$, $\hat{\psi}_2$ satisfy the same assumptions as in Proposition 2.1, and denote

$$\psi^{(i)}_{as_1s_2p} = |\det M^{(i)}_{as_1s_2}|^{-\frac{1}{2}}\psi^{(i)}(M^{(i)}_{as_1s_2})^{-1}(x-p)), \quad \text{for } i = 2,3.$$

Hence, an argument similar to Proposition 2.1 shows that, for $i = 2,3$, the functions

$$\Psi^{(i)} = \{\psi^{(i)}_{as_12_2p} : 0 \le a \le \tfrac{1}{4}, -\tfrac{3}{2} \le s_1 \le \tfrac{3}{2}, -\tfrac{3}{2} \le s_1 \le \tfrac{3}{2}, p \in \mathbb{R}^2\}$$

are continuous shearlets for $L^2(C^{(i)})^\vee$. Accordingly, for $i = 2,3$, the mappings $f \to \mathcal{SH}^{(i)}_\psi f(a,s_1,s_2,p) = \langle f, \psi^{(i)}_{as_1s_2p}\rangle$ are the continuous shearlet transforms on $L^2(C^{(i)})^\vee$ with respect to $\Lambda^{(i)}$. Finally, by introducing an appropriate smooth, bandlimited window function W, the functions with frequency support on the set $[-2,2]^3$ can be expanded as

$$f = \int_{\mathbb{R}^3} \langle f, W_p\rangle W_p \, dp,$$

where $W_p(x) = W(x-p)$. As a result, any function $f \in L^2(\mathbb{R}^3)$ can be represented with respect to the full system of shearlets consisting of the systems $\bigcup_{i=1}^3 \Psi^{(i)}$ together with the coarse-scale isotropic functions W_p. The shearlet representation we have just described generalizes a similar representation originally introduced in [16] for dimension $n = 2$.

Notice that, for the purposes of this paper, it is only the behavior of the fine-scale shearlets that matters. Indeed, the continuous shearlet transforms $\mathcal{SH}^{(i)}_\psi$, $i = 1,2,3$, will be applied at fine scales ($a \to 0$) to resolve and precisely describe the boundaries of certain solid regions. Since the behavior of these transforms is essentially the same on each cone domain $C^{(i)}$, in the following sections, without of loss of generality, we will only consider the continuous shearlet transform $\mathcal{SH}^{(1)}_\psi$. For simplicity of notation, we will drop the superscript (1) in the following.

3 Main Results

As described above, the continuous shearlet transform has the ability to characterize very precisely the set of singularities of multivariable functions and distributions through its asymptotic decay properties as $a \to 0$. The situation in dimension $n = 2$ was completely solved in [11,8]. In higher dimensions, only the special case of boundary regions with nonvanishing Gaussian curvature was known so far [10]. In this paper, we are able to deal with the situation of general boundaries of 3D solid region, thanks to a new argument that also simplifies many of the results previously known.

Consider the functions $B = \chi_\Omega$, where Ω is a subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a 2-dimensional manifold. We say that $\partial\Omega$ is *piecewise smooth* if:

(i) $\partial\Omega$ is a C^∞ manifold except possibly for finite many separating C^3 curves on $\partial\Omega$;
(ii) at each point on a separating curve, $\partial\Omega$ has exactly two outer normal vectors which are not on the same line.

Let the outer normal vector of $\partial\Omega$ be $\vec{n}_p = \pm(\cos\theta_0 \sin\phi_0, \sin\theta_0 \sin\phi_0, \cos\phi_0)$ for some $\theta_0 \in [0, 2\pi]$, $\phi_0 \in [0, \pi]$. We say that $s = (s_1, s_2)$ corresponds to the normal direction $\vec{n}_p$ if $s_1 = a^{-\frac{1}{2}} \tan\theta_0$, $s_2 = a^{-\frac{1}{2}} \cot\phi_0 \sec\theta_0$.

The following theorem shows that for a bounded region in $\mathbb{R}^3$ whose boundary is a piecewise smooth 2-dimensional manifold, the continuous shearlet transform of B, denoted by $\mathcal{SH}_\psi B(a, s_1, s_2, p)$, has rapid asymptotic decay as $a \to 0$ for all locations $p \in \mathbb{R}^3$, except when p is on the boundary of Ω and the orientation variables s_1, s_2 correspond to normal direction of the boundary surface at p, or when p is on a separating curve and the shearing variables s_1, s_2 correspond to normal directions of the boundary surface at p (see Figure 2).

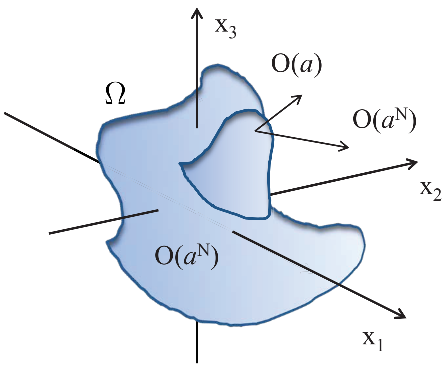

Fig. 2. The continuous shearlet transform of a bounded region Ω with piecewise smooth boundary has rapid decay everywhere, except when the location variable p is on the surface and the shearing variables correspond to the normal orientation, in which case it decays like $O(a)$, as $a \to 0$.

Theorem 3.1 *Let Ω be a bounded region in $\mathbb{R}^3$ and denote its boundary by $\partial\Omega$. Assume that $\partial\Omega$ is a piecewise smooth 2-dimensional manifold. Let γ_j, $j = 1, 2, \cdots, m$ be the separating curves of $\partial\Omega$. Then we have*

(i) If $p \notin \partial\Omega$ then

$$\lim_{a\to 0^+} a^{-N} \mathcal{SH}_\psi B(a, s_1, s_2, p) = 0, \qquad \textit{for all } N > 0.$$

(ii) If $p \in \partial\Omega \backslash \bigcup_{j=1}^m \gamma_j$ and (s_1, s_2) does not correspond to the normal direction

of $\partial\Omega$ at p, then

$$\lim_{a\to 0^+} a^{-N}\, \mathcal{SH}_\psi B(a,s_1,s_2,p)=0, \qquad \textit{for all } N>0.$$

(iii) If $p\in\partial\Omega\setminus\bigcup_{j=1}^m\gamma_j$ and (s_1,s_2) corresponds to the normal direction of $\partial\Omega$ at p or $p\in\bigcup_{j=1}^m\gamma_j$ and (s_1,s_2) corresponds to one of the two normal directions of $\partial\Omega$ at p, then

$$\lim_{a\to 0^+} a^{-1}\, \mathcal{SH}_\psi B(a,s_1,s_2,p)\neq 0.$$

(iv) If $p\in\gamma_j$ and (s_1,s_2) does not correspond to the normal directions of $\partial\Omega$ at p, then

$$|\mathcal{SH}_\psi B(a,s_1,s_2,p)|\le C a^{\frac{3}{2}}.$$

The proof of Theorem 3.1 is given in the next section.

Before presenting the proof, we mention that the result presented above is expected to extended to higher dimensions using a similar argument, at least in the situation of smooth boundaries. Similar to the result valid in dimensions $n=2$ and $n=3$, the continuous shearlet transform will exhibit rapid asymptotic decay for all locations and orientations, except when the location parameter is at the boundary and the shearing parameters correspond to the normal orientation, in which case the decay will be of the order of $a^{\frac{n+1}{4}}$.

4 Proof of The Theorems

The proof requires some construction.

4.1 Useful lemmata and constructions

Our first observation is that, by using the divergence theorem, one can make explicit the dependence of the shearlet transform of a compactly supported function f and the boundaries of the support of f. Notice that this property was also employed in [8] and follows a classical method from [14].

Let $\Omega\subset\mathbb{R}^3$ be a solid region whose boundary surface $S=\partial\Omega$ is a 2-dimensional manifold. Let $B=\chi_\Omega$. By the divergence theorem, the Fourier transform of B can be expressed as

$$\hat{B}(\xi)=\widehat{\chi}_\Omega(\xi)=-\frac{1}{2\pi i|\xi|^2}\int_S e^{-2\pi i\xi\cdot x}\,\xi\cdot\vec{n}(x)\,d\sigma(x), \tag{5}$$

where $\vec{n}$ is the outer normal vector to S at x (see [14]). By representing $\xi \in \mathbb{R}^3$ using spherical coordinates as $\xi = \rho\,\Theta$, where $\rho \in \mathbb{R}^+$ and $\Theta = \Theta(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$, expression (5) can be written as

$$\hat{B}(\rho,\theta,\phi) = -\frac{1}{2\pi i\rho}\int_S e^{-2\pi i\rho\,\Theta(\theta,\phi)\cdot x}\,\Theta(\theta,\phi)\cdot\vec{n}(x)\,d\sigma(x). \tag{6}$$

The second observation is that the shearlet transform is a local transform, in the sense that the shearlet transform of a function f decays rapidly (as $a \to 0$) away from the locations where f is discontinuous.

Let $p \in \mathbb{R}^3$. For $\epsilon > 0$, let $\beta_\epsilon(p)$ be the ball with radius ϵ and center p and let $P_\epsilon(p) = S \bigcap \beta_\epsilon(p)$. Using this notation, we break up the integral (6) into a component close to p and another component away from p as

$$\hat{B}(\rho,\theta,\phi) = T_1(\rho,\theta,\phi) + T_2(\rho,\phi,\theta),$$

where

$$T_1(\rho,\theta,\phi) = -\frac{1}{2\pi i\rho}\int_{P_\epsilon(p)} e^{-2\pi i\rho\,\Theta(\theta,\phi)\cdot x}\,\Theta(\theta,\phi)\cdot\vec{n}(x)\,d\sigma(x)$$
$$T_2(\rho,\theta,\phi) = -\frac{1}{2\pi i\rho}\int_{S\setminus P_\epsilon(p)} e^{-2\pi i\rho\,\Theta(\theta,\phi)\cdot x}\,\Theta(\theta,\phi)\cdot\vec{n}(x)\,d\sigma(x)$$

It follows that

$$\mathcal{SH}_\psi B(a,s_1,s_2,p) = \langle B, \psi_{as_1s_2p}\rangle = I_1(a,s_1,s_2,p) + I_2(a,s_1,s_2,p),$$

where

$$I_1(a,s_1,s_2,p) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} T_1(\rho,\theta,\phi)\,\overline{\hat{\psi}_{as_1s_2p}}(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \tag{7}$$
$$I_2(a,s_1,s_2,p) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} T_2(\rho,\theta,\phi)\,\overline{\hat{\psi}_{as_1s_2p}}(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

The following lemma from [10] shows that the asymptotic decay of the shearlet transform $\mathcal{SH}_\psi B(a,s_1,s_2,p)$, as $a \to 0$, is only determined by the values of the boundary surface S which are close to the location variable p.

Lemma 4.1 [10] *For any positive integer N, there is a constant $C_N > 0$ such that*

$$|I_2(a,s_1,s_2,p)| \le C_N\,a^N,$$

asymptotically as $a \to 0$, uniformly for all $s_1, s_2 \in [-\frac{3}{2}, \frac{3}{2}]$.

Since the proof of this lemma will be used below, we repeat the following argument from [10].

By direct computation, we have that:

$$
\begin{aligned}
&-2\pi i\, I_2(a, s_1, s_2, p) \\
&= \int_{S\setminus P_\epsilon(p)} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-2\pi i\rho\,\Theta\cdot x}\, \Theta\cdot\vec{n}(x)\, \overline{\hat{\psi}_{as_1s_2p}}(\rho,\phi,\theta)\, \rho\sin\phi\, d\rho\, d\phi\, d\theta\, d\sigma(x) \\
&= a\int_{S\setminus P_\epsilon(p)} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \hat{\psi}_1(a\rho\sin\phi\cos\theta)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s_1)) \\
&\quad \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi\sec\theta - s_2))\, e^{2\pi i\rho\,\Theta(\phi,\theta)\cdot(p-x)}\, \Theta\cdot\vec{n}(x)\, \rho\, \sin\phi\, d\rho\, d\phi\, d\theta\, d\sigma(x) \\
&= \frac{1}{a}\int_{S\setminus P_\epsilon(p)} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \hat{\psi}_1(\rho\sin\phi\cos\theta)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta - s_1)) \\
&\quad \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi\sec\theta - s_2))\, e^{2\pi i\frac{\rho}{a}\,\Theta(\phi,\theta)\cdot(p-x)}\, \Theta\cdot\vec{n}(x)\, \rho\, \sin\phi\, d\rho\, d\phi\, d\theta\, d\sigma(x).
\end{aligned}
$$

Notice that, by assumption, there exists an $\epsilon > 0$ such that $\|p - x\| \geq \epsilon$ for all $x \in S \setminus P_\epsilon(p)$. Let $s_1 = \tan\theta_0$ with $|\theta_0| < \frac{\pi}{2}$ and $s_2 = \cot\phi_0\sec\theta_0$ with $|\phi_0 - \frac{\pi}{2}| < \frac{\pi}{2}$. By the support condition of $\hat{\psi}_2$, it follows that, for a near 0, θ is away from $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and ϕ is away from 0 or π. Let J be the set of these θ and ϕ. It is easy to see that $\{\Theta(\phi,\theta),\ \Theta_\phi(\phi,\theta),\ \Theta_\theta(\phi,\theta)\}$ form a basis for $\mathbb{R}^3$ for $(\phi,\theta) \in J$. It follows that there is a constant $C_p > 0$ such that $|\Theta(\phi,\theta)\cdot(p-x)| + |\Theta_\phi(\phi,\theta)\cdot(p-x)| + |\Theta_\theta(\phi,\theta)\cdot(p-x)| \geq C_p$, where C_p is independent of $(\phi,\ \theta) \in J$, and $x \in S \setminus P_\epsilon(p)$.

Define

$$
\begin{aligned}
J_1 &= \{\phi,\theta) : \inf_{x\in S\setminus P_\epsilon(p)} |\Theta(\phi,\theta)\cdot(p-x)| \geq \frac{C_p}{3}\}, \\
J_2 &= \{\phi,\theta) : \inf_{x\in S\setminus P_\epsilon(p)} |\Theta_\phi(\phi,\theta)\cdot(p-x)| \geq \frac{C_p}{3}\}, \\
J_3 &= \{\phi,\theta) : \inf_{x\in S\setminus P_\epsilon(p)} |\Theta_\theta(\phi,\theta)\cdot(p-x)| \geq \frac{C_p}{3}\}.
\end{aligned}
$$

We can express integral I_2 as a sum of three integrals corresponding to J_1, J_2, and J_3 respectively. On J_1, we integrate by parts with respect to the variable ρ; on J_2 we integrate by parts with respect to the variable ϕ, and on J_3 we integrate by parts with respect to the variable θ. Doing this repeatedly, it yields that, for any positive integer n, $|I_2| \leq C_n\, a^{\frac{n}{2}}$. This finishes the proof. $\square$

The following lemma is a special case of Proposition 5 at page 342 in [21].

Lemma 4.2 *Suppose ψ is smooth and is supported in the unit ball; also let ϕ be a real-valued function so that, for some $k \geq 1$ we have $|\phi^{(k)}| \geq 1$ throughout the support of ψ. Then*

$$\left|\int_{-\infty}^{\infty} e^{i\lambda\phi(x)}\psi(x)\,dx\right| \leq C_k\,\lambda^{-\frac{1}{k}}.$$

For the proof of Theorem 3.1, we also need the following Lemmata. The first one is a generalization of Lemma 4.4 in [8]; the second one is contained in the proof of Lemma 4.4 in [8]

Lemma 4.3 *For $\alpha \in [0, 2\pi), y > 0$, let*

$$h(\alpha, y) = \int_0^{\sqrt{2}} \hat{\psi}_2(r\cos\alpha)\,\hat{\psi}_2(r\sin\alpha)\,\sin(\pi y r^2) r\,dr,$$

where ψ_2 satisfies the assumptions given by (4). Then $h(\alpha, y) > 0$.

Lemma 4.4 *Let $\psi_2 \in L^2(\mathbb{R})$ be chosen so that it satisfies the assumptions given by (4). Then, for each $\rho > 0$,*

$$\int_{-1}^{1} \hat{\psi}_2(u)\sin(\pi\rho u^2)\,du > 0 \;\; \textit{and} \;\; \int_{-1}^{1} \hat{\psi}_2(u)\cos(\pi\rho u^2)\,du > 0.$$

The final observation which will be needed is that, in order to estimate the asymptotic decay of the shearlet transform of $B = \chi_\Omega$ as $a \to 0$, it is possible to locally approximate the smooth surface $S = \partial\Omega$ using a quadratic surface. This observation will play a crucial role in the the proof of Theorem 3.1.

Near the point $p \in \mathbb{R}^3$, let $S = (G(u), u)$, where $u \in U \subset \mathbb{R}^2$ and $G(u)$ is a smooth function on U. There exists $u_0 \in U$ such that $p = (G(u_0), u_0)$. Without loss of generality, we may assume that $p = (0,0,0)$ so that $u_0 = (0,0)$ and $G(0,0) = 0$. Hence we define the quadratic approximation of S near $p = (0,0,0)$ by

$$S_0 = (G_0(u), u),$$

where G_0 is the second order approximation of G at $(0,0)$, given by by $G_0(u) = G_{u_1}(0,0)u_1 + G_{u_2}(0,0)u_2 + \frac{1}{2}[G_{u_1^2}(0,0)u_1^2 + 2G_{u_1u_2}(0,0)u_1u_2 + G_{u_2^2}(0,0)u_2^2]$. We define the function $B_0 = \chi_{\Omega_0}$, where $\partial\Omega_0$ is obtained by replacing $S = \partial\Omega$ in $B = \chi_S$ with the surface S_0 near the point $p = (0,0,0)$. We can now state the following result.

Lemma 4.5 *For any $s = (s_1, s_2) \in \mathbb{R}_2$ with $|s_1| \leq \frac{3}{2}, |s_2| \leq \frac{3}{2}$, we have*

$$\lim_{a\to 0^+} a^{-1}\;|\mathcal{SH}_\psi B(a,s,0)) - \mathcal{SH}_\psi B_0(a,s,0)| = 0.$$

Proof. Without loss of generality, we may assume $s = (0,0)$. Let γ be chosen such that $\frac{2}{5} < \gamma < \frac{1}{2}$ and assume that a is sufficiently small, so that $a^\gamma < \epsilon$.

A direct calculation shows that

$$\begin{aligned} |\mathcal{SH}_\psi B(a,0,0) - \mathcal{SH}_\psi B_0(a,0,0)| &\le \int_{\mathbb{R}^3} |\psi_{a00}(x)|\, |\chi_\Omega(x) - \chi_{\Omega_0}(x)|\, dx \\ &= T_1(a) + T_2(a), \end{aligned}$$

where, for $x = (x_1, x_2, x_3)$, we have:

$$\begin{aligned} T_1(a) &= a^{-1} \int_{D(a^\gamma,(0,0,0))} |\psi(a^{-1}x_1, a^{-\frac{1}{2}}x_2, a^{-\frac{1}{2}}x_3)|\, |\chi_\Omega(x) - \chi_{\Omega_0}(x)|\, dx, \\ T_2(a) &= a^{-1} \int_{D^c(a^\gamma,(0,0,0))} |\psi(a^{-1}x_1, a^{-\frac{1}{2}}x_2, a^{-\frac{1}{2}}x_3)|\, |\chi_\Omega(x) - \chi_{\Omega_0}(x)|\, dx. \end{aligned}$$

Observe that:

$$T_1(a) \le C\, a^{-1} \int_{D(a^\gamma,(0,0,0))} |\chi_\Omega(x) - \chi_{\Omega_0}(x)|\, dx.$$

To estimate the above quantity, it is enough to compute the volume between the regions Ω and Ω_0. Since G_0 is the Taylor polynomial of G of degree 2, we have

$$T_1(a) \le C\, a^{-1} \int_{|x|<a^\gamma} |x|^3 dx \le C\, a^{-1} \int_{r<a^\gamma} r^4 dr \le C\, a^{5\gamma-1}.$$

Since $\gamma > \frac{2}{5}$, the above estimate shows that $T_1(a) = o(a)$.

The assumptions on ψ imply that, for each $N > 0$, there is a constant $C_N > 0$ such that $|\psi(x)| \le C_N\, (1+|x|^2)^{-N}$. Thus, for $a < 1$, we can estimate $T_2(a)$ as:

$$\begin{aligned} T_2(a) &\le C\, a^{-1} \int_{D^c(a^\gamma,(0,0,0))} |\psi(a^{-1}x_1, a^{-\frac{1}{2}}x_2, a^{-\frac{1}{2}}x_3)|\, dx \\ &\le C_N\, a^{-1} \int_{D^c(a^\gamma,(0,0,0))} \left(1 + (a^{-1}x_1)^2 + (a^{-\frac{1}{2}}x_2)^2 + (a^{-\frac{1}{2}}x_3)^2\right)^{-N} dx \\ &\le C_N\, a^{-1} \int_{D^c(a^\gamma,(0,0,0))} \left((a^{-1/2}x_1)^2 + (a^{-\frac{1}{2}}x_2)^2 + (a^{-\frac{1}{2}}x_3)^2\right)^{-N} dx \\ &= C_N\, a^{N-1} \int_{D^c(a^\gamma,(0,0,0))} \left(x_1^2 + x_2^2 + x_3^2\right)^{-N} dx \\ &= C_N\, a^{N-1} \int_{a^\gamma}^\infty r^{1-2N}\, dr \\ &= C_N\, a^{2N(\frac{1}{2}-\gamma)}\, a^{2\gamma-1}. \end{aligned}$$

Since $\gamma < \frac{1}{2}$ and N can be chosen arbitrarily large, it follows that $T_2(a) = o(a)$. $\square$

4.2 Proof of Theorem 3.1

The proof of statement (i) of the theorem follows directly from Lemma 4.1. The proof of statement (iii), which is the "hardest" part of the proof, is based on a completely new argument and will be presented first. Next, we shall present the proofs of statements (ii) and (iv), which are much simpler.

Proof of (iii). This is the situation where either $p \in \partial\Omega \setminus \bigcup_{j=1}^{m} \gamma_j$ and the shearing variables (s_1, s_2) correspond to the normal orientation or p is on a separating curve and the shearing variables (s_1, s_2) correspond to one of the two normal orientations. We discuss separately the situations where (I) $p \in \partial\Omega \setminus \bigcup_{j=1}^{m} \gamma_j$ and where (II) $p \in \gamma_j$ for some j. We will only examine the behavior of $I_1(a, s, p)$ for $|s_1|, |s_2| \le 1$ (in which case we use the shearlet transform on the truncated pyramidal region $C^{(1)}$). The other cases can be handled in a very similar way.

(iii) – Situation (I): Let $p \in \partial\Omega \setminus \bigcup_{j=1}^{m} \gamma_j$.

By Lemma 4.1 and Lemma 4.5, in order to prove statement (iii) it is sufficient to show that

$$\lim_{a \to 0^+} a^{-1}\, I_1(a, s_1, s_2, p) \neq 0,$$

where the integral I_1 is taken on S_0 rather than S.

For simplicity, let $p = (0, 0, 0)$, $\theta_0 = 0, \phi_0 = \frac{\pi}{2}$, so that $s_1 = s_2 = 0$. The general situation can be reduced to this special case, as shown in Remark 4.1 at the end of this section. Also, let $S, G(u), S_0, G_0(u)$ be given as in Lemma 4.5, with $S = (G(u), u)$ and $p = (G(0), 0)$. Using polar coordinates, the integral $I_1(a, 0, 0, 0)$, taken on S_0, can be written as

$$\begin{aligned} & I_1(a,0,0,0) \\ = & -\frac{1}{2\pi i a} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \hat{\psi}_1(\rho \sin\phi \cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi \sec\theta)) \\ & \times \int_U e^{2\pi i \frac{\rho}{a} H_{\theta,\phi}(u)}\, \Theta(\theta,\phi) \cdot (-1, \nabla G_0(u))\, du\, \rho \sin\phi\, d\rho\, d\phi\, d\theta. \end{aligned} \tag{8}$$

where

$$H_{\theta,\phi}(u) = -\Theta(\theta,\phi) \cdot (G_0(u), u) \tag{9}$$

Let $F(\phi, \theta, u) = (F_1(\phi, \theta, u), F_2(\phi, \theta, u))$, where

$$F_1(\phi,\theta,u) = (H_{\theta,\phi})_{u_1} = -\Theta(\theta,\phi) \cdot (G_{0u_1}(u), 1, 0)$$

$$F_2(\phi,\theta,u) = (H_{\theta,\phi})_{u_2} = -\Theta(\theta,\phi)\cdot(G_{0u_2}(u),0,1).$$

Since $\vec{n}(p) = (1,0,0)$ and $\Theta(\theta_0,\phi_0) = \pm\vec{n}(p)$, it follows that $\phi_0 = \frac{\pi}{2}$ and $\theta_0 = 0$ or π. We will only consider the case $\theta_0 = 0$ since the argument for the case $\theta_0 = \pi$ is similar. In this situation, we have

$$G_{0u_1}(u_0) = G_{0u_1}(0,0) = 0, \quad G_{0u_2}(u_0) = G_{u_2}(0,0) = 0$$

so that

$$G_0(u) = \frac{1}{2}\left(G_{u_1^2}(0,0)\,u_1^2 + 2G_{u_1u_2}(0,0)\,u_1u_2 + G_{u_2^2}(0,0)\,u_2^2\right).$$

Without loss of generality (if necessary, by changing the (u_1,u_2) coordinates), we may assume that $G_{u_1u_2}(0,0) = 0$. As a consequence, we have that

$$G_0(u) = \frac{1}{2}(G_{u_1^2}(0,0)u_1^2 + G_{u_2^2}(0,0)u_2^2).$$

For brevity, in the following we will use the notation:

$$A = G_{u_1^2}(0,0),\ B = G_{u_2^2}(0,0). \tag{10}$$

Hence we have:

$$\begin{aligned} F_1(\phi,\theta,u) &= -\Theta(\theta,\phi)\cdot(Au_1,1,0) = -\cos\theta\sin\phi\,Au_1 - \sin\theta\sin\phi \\ F_2(\phi,\theta,u) &= -\Theta(\theta,\phi)\cdot(Bu_2,0,1) = -\cos\theta\sin\phi\,Bu_2 - \cos\phi. \end{aligned}$$

If $A \neq 0$, let $u_{1,\theta,\phi} = \frac{-\sin\theta\sin\phi}{A\cos\theta\sin\phi} = -\frac{1}{A}\tan\theta$ so that $F_1(\phi,\theta,u_{1,\theta,\phi}) = 0$. Similarly, if $B \neq 0$, let $u_{2,\theta,\phi} = -\frac{1}{B}\sec\theta\cot\phi$ so that $F_2(\phi,\theta,u_{2,\theta,\phi}) = 0$. We have now four cases to discuss, corresponding on $A \neq 0, B \neq 0$ or $A = B = 0$ or $A = 0, B \neq 0$ or $A \neq 0, B = 0$. Notice that the last two cases are equivalent.

• *Case 1:* $A \neq 0, B \neq 0$. In this case, the phase term $H_{\theta,\phi}(u)$, given by (9), can be expressed as

$$\begin{aligned} H_{\theta,\phi}(u_1,u_2) = &-\frac{1}{2}\cos\theta\sin\phi\ A(u_1 - u_{1,\theta,\phi})^2 + \frac{\sin^2\theta\sin^2\phi}{2A\cos\theta\sin\phi} + \\ &-\frac{1}{2}\cos\theta\sin\phi\ B(u_2 - u_{2,\theta,\phi})^2 + \frac{\cos^2\phi}{2B\cos\theta\sin\phi}\ . \end{aligned}$$

Since $p = (0,0,0)$ is an interior point in S_0, from the proof of Lemma 4.1 it follows that we may choose the domain U of G_0 as $U = \{(u_1,u_2) : \alpha_1 \le u_1 \le \beta_1,\ \alpha_2 \le u_2 \le \beta_2\}$ with $\alpha_1,\ \alpha_2 < 0,\ \beta_1,\ \beta_2 > 0$. Hence, the integral over U from (8) becomes:

$$
\begin{aligned}
&\int_U e^{2\pi i \frac{\rho}{a} H_{\theta,\phi}(u)}\, \Theta(\theta,\phi)\cdot(-1,\nabla G_0(u))\; du \\
&= e^{\pi i \frac{\rho}{a}\left(\frac{\sin^2\theta\sin^2\phi}{A\cos\theta\sin\phi}+\frac{\cos^2\phi}{B\cos\theta\sin\phi}\right)} \\
&\times \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi A(u_1-u_{1,\theta,\phi})^2} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi B(u_2-u_{2,\theta,\phi})^2} \\
&\times(-\cos\theta\sin\phi + A\sin\theta\sin\phi\; u_1 + B\cos\phi\; u_2)\; du_2\, du_1 \\
&= K_0(\theta,\phi,a)+K_1(\theta,\phi)+K_2(\theta,\phi,a),
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
K_0(\theta,\phi,a) =& -\cos\theta\sin\phi\, e^{\pi i\frac{\rho}{a}\left(\frac{\sin^2\theta\sin^2\phi}{A\cos\theta\sin\phi}+\frac{\cos^2\phi}{B\cos\theta\sin\phi}\right)} \\
&\times\int_{\alpha_1}^{\beta_1} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi A(u_1-u_{1,\theta,\phi})^2} du_1 \int_{\alpha_2}^{\beta_2} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi B(u_2-u_{2,\theta,\phi})^2} du_2 \\
K_1(\theta,\phi,a) =& A\sin\theta\sin\phi\, e^{\pi i\frac{\rho}{a}\left(\frac{\sin^2\theta\sin^2\phi}{A\cos\theta\sin\phi}+\frac{\cos^2\phi}{B\cos\theta\sin\phi}\right)} \\
&\times\int_{\alpha_1}^{\beta_1} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi A(u_1-u_{1,\theta,\phi})^2} u_1 du_1 \int_{\alpha_2}^{\beta_2} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi B(u_2-u_{2,\theta,\phi})^2} du_2 \\
K_2(\theta,\phi,a) =& B\cos\phi\, e^{\pi i\frac{\rho}{a}\left(\frac{\sin^2\theta\sin^2\phi}{A\cos\theta\sin\phi}+\frac{\cos^2\phi}{B\cos\theta\sin\phi}\right)} \\
&\times\int_{\alpha_1}^{\beta_1} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi A(u_1-u_{1,\theta,\phi})^2} du_1 \int_{\alpha_2}^{\beta_2} e^{-\pi i\frac{\rho}{a}\cos\theta\sin\phi B(u_2-u_{2,\theta,\phi})^2} u_2 du_2
\end{aligned}
$$

In the expression (8) for I_1, the domain of integration with respect to θ can be broken up into the intervals $[-\frac{\pi}{2},\frac{\pi}{2}]$ and $[\frac{\pi}{2},\frac{3\pi}{2}]$. For the interval $[\frac{\pi}{2},\frac{3\pi}{2}]$, we apply the change of variable $\theta'=\theta-\pi$, so that $\theta'\in[-\frac{\pi}{2},\frac{\pi}{2}]$ and $\sin\theta=-\sin\theta'$, $\cos\theta=-\cos\theta'$. Using this observation and expression (11), it follows that

$$
I_1(a,0,0,0) = I_{10}(a,0,0,0)+I_{11}(a,0,0,0)+I_{12}(a,0,0,0),
$$

where, for $j=0,1,2$,

$$
\begin{aligned}
&I_{1j}(a,0,0,0) \\
=& -\frac{1}{2\pi i a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\infty}\hat\psi_1(\rho\sin\phi\cos\theta)\hat\psi_2(a^{-\frac12}(\tan\theta))\hat\psi_2(a^{-\frac12}(\cot\phi\sec\theta)) \\
&\times K_j(\theta,\phi,a)\,\rho\sin\phi\, d\rho\, d\phi\, d\theta + \\
&+\frac{1}{2\pi i a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\infty}\hat\psi_1(\rho\sin\phi\cos\theta)\hat\psi_2(a^{-\frac12}(\tan\theta))\hat\psi_2(a^{-\frac12}(\cot\phi\sec\theta)) \\
&\times K_j(\theta+\pi,\phi,a)\rho\sin\phi d\rho d\phi d\theta.
\end{aligned}
$$

For $\theta\in(-\frac{\pi}{2},\frac{\pi}{2}),\phi\in(0,\pi)$, let $t_1=a^{-\frac12}\tan\theta$ and $t_2=a^{-\frac12}\cot\phi\sec\theta$, so that $\cos^2\theta=\frac{1}{at_1^2+1}$, $\sin^2\phi=\frac{at_1^2+1}{at_1^2+at_2^2+1}$, and $J(\theta,\phi)=-\frac{1}{a\cos^3\theta\sin^2\phi}$. Under this change of variables, we have

$$
\begin{aligned}
&2\pi i\, I_{10}(a,0,0,0)\\
&=\int_0^\infty\int_{-1}^1\int_{-1}^1 \hat{\psi}_1\left(\tfrac{\rho}{\sqrt{at_1^2+at_2^2+1}}\right)\hat{\psi}_2(t_1)\hat{\psi}_2(t_2)\,K_0(t_1,t_2,a)\,\frac{\rho\,dt_1\,dt_2\,d\rho}{(at_1^2+at_2^2+1)^{3/2}}\\
&+\int_0^\infty\int_{-1}^1\int_{-1}^1 \hat{\psi}_1\left(\tfrac{\rho}{\sqrt{at_1^2+at_2^2+1}}\right)\hat{\psi}_2(t_1)\hat{\psi}_2(t_2)\,\bar{K}_0(t_1,t_2,a)\,\frac{\rho\,dt_1\,dt_2\,d\rho}{(at_1^2+at_2^2+1)^{3/2}}
\end{aligned}
$$

where, (with an abuse of notation, we let $K_0(t_1,t_2,a)$ denote the function $K_0(\theta,\phi,a)$ after the change of variables),

$$
\begin{aligned}
K_0(t_1,t_2,a) = &-\frac{e^{\pi i\frac{\rho}{\sqrt{at_1^2+at_2^2+1}}(\frac{1}{A}t_1^2+\frac{1}{B}t_2^2)}}{\sqrt{at_1^2+at_2^2+1}}\int_{\alpha_1}^{\beta_1} e^{-\pi i\frac{\rho}{\sqrt{at_1^2+at_2^2+1}}A(\frac{u_1}{\sqrt{a}}+\frac{t_1}{A})^2}\,du_1\\
&\times\int_{\alpha_2}^{\beta_2} e^{-\pi i\frac{\rho}{\sqrt{at_1^2+at_2^2+1}}B(\frac{u_2}{\sqrt{a}}+\frac{t_2}{B})^2}\,du_2.
\end{aligned}
$$

Notice that, due to the change of variable, $K_0(\theta+\pi,\phi,a)$ has become $-\bar{K}_0(t_1,t_2,a)$, where $\bar{K}_0$ denoted the complex conjugate of K_0.

Taking the limit for $a\to 0$, we have

$$
\begin{aligned}
\lim_{a\to 0}\frac{1}{a}K_0(t_1,t_2,a) &= -e^{\pi i\rho(\frac{1}{A}t_1^2+\frac{1}{B}t_2^2)}\int_{-\infty}^{\infty} e^{-\pi i\rho A(u_1+\frac{t_1}{A})^2}\,du_1\int_{-\infty}^{\infty} e^{-\pi i\rho B(u_2+\frac{t_2}{B})^2}\,du_2\\
&= -C_\rho\, e^{\pi i\rho(\frac{1}{A}t_1^2+\frac{1}{B}t_2^2)},
\end{aligned}
$$

where

$$C_\rho = \int_{-\infty}^{\infty} e^{-\pi i\rho A u_1^2}\,du_1\int_{-\infty}^{\infty} e^{-\pi i\rho B u_2^2}\,du_2.$$

Recalling the Fresnel integrals

$$\int_{-\infty}^{\infty}\cos(\pi x^2)\,dx = \int_{-\infty}^{\infty}\sin(\pi x^2)\,dx = \frac{\sqrt{2}}{2},$$

it follows that

$$C_\rho = \frac{-i}{\rho\sqrt{AB}}. \tag{12}$$

Notice that C_ρ is a real or imaginary quantity, depending on the signs of A and B. Similarly, we have

$$\lim_{a\to 0}\frac{1}{a}\bar{K}_0(t_1,t_2,a) = \bar{C}_\rho e^{-\pi i\rho(\frac{1}{A}t_1^2+\frac{1}{B}t_2^2)}.$$

Due to the presence of the linear term u_1 and u_2 in the integrals of K_1 and K_2, respectively, a similar calculation to the one above shows that $K_1(\theta,\phi,a)$, $K_1(\theta+\pi,\phi,a)$, $K_2(\theta,\phi,a)$ $K_2(\theta+\pi,\phi,a)$ are all $O(a^{\frac{3}{2}})$. That is, as $a\to 0$, all those terms are dominated by the term K_0 and, thus,

$$\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0) = \lim_{a\to 0}\frac{2\pi i}{a}I_{10}(a,0,0,0).$$

It follows that

$$
\begin{aligned}
&\lim_{a\to 0}\frac{2\pi i}{a} I_1(a,0,0,0)\\
&= \int_0^\infty C_\rho\, \hat{\psi}_1(\rho) \int_{-1}^{1} e^{\pi i\rho \frac{1}{A} t_1^2} \hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1} e^{\pi i\rho\frac{1}{B}t_2^2}\hat{\psi}_2(t_2)\, dt_2\, \rho d\rho\\
&+ \int_0^\infty \bar{C}_\rho\, \hat{\psi}_1(\rho) \int_{-1}^{1} e^{-\pi i\rho \frac{1}{A} t_1^2} \hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1} e^{-\pi i\rho\frac{1}{B}t_2^2}\hat{\psi}_2(t_2)\, dt_2\, \rho d\rho\\
&= 2\,\Re\left\{\int_0^\infty C_\rho\, \hat{\psi}_1(\rho) \int_{-1}^{1} e^{-\pi i\rho \frac{1}{A} t_1^2} \hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1} e^{-\pi i\rho\frac{1}{B}t_2^2}\hat{\psi}_2(t_2)\, dt_2\, \rho d\rho\right\}, \quad (13)
\end{aligned}
$$

where C_ρ is given by (12). The value of the above limit depends on the various combinations of signs of A and B, which determines whether C_ρ is real or imaginary. Hence we have the following two possible subcases.

Subcase 1.1: $A > 0, B > 0$ or $A < 0, B < 0$. In this case, $C_\rho = \frac{-i}{\rho\sqrt{|AB|}}$ is an imaginary number. It follows from (13) that

$$
\begin{aligned}
&\lim_{a\to 0}\frac{2\pi i}{a} I_1(a,0,0,0)\\
&= \frac{2}{\sqrt{|AB|}}\,\Re\left\{-i\int_0^\infty \hat{\psi}_1(\rho) \int_{-1}^{1} e^{-\pi i\rho \frac{1}{A} t_1^2} \hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1} e^{-\pi i\rho\frac{1}{B}t_2^2}\hat{\psi}_2(t_2)\, dt_2\, d\rho\right\}\\
&= \frac{2}{\sqrt{|AB|}}\,\Re\left\{-\int_0^\infty \hat{\psi}_1(\rho) \int_{-1}^{1}\int_{-1}^{1} i\, e^{-\pi i\rho(\frac{1}{A}t_1^2+\frac{1}{B}t_2^2)}\ \hat{\psi}_2(t_1)\ \hat{\psi}_2(t_2) dt_1\ dt_2 d\rho\right\}\\
&= \frac{-2}{\sqrt{|AB|}}\int_0^\infty \hat{\psi}_1(\rho)\int_0^{2\pi}\int_0^{\sqrt{2}} \sin\left(\pi\rho(\frac{1}{A}\sin^2\alpha + \frac{1}{B}\sin^2\alpha)r^2\right)\\
&\times \hat{\psi}_2(\cos\alpha r)\,\hat{\psi}_2(\sin\alpha r)\, r dr\, d\alpha\, d\rho.
\end{aligned}
$$

Using Lemma 4.3, with $y = \rho(\frac{1}{A}\sin^2\alpha + \frac{1}{B}\sin^2\alpha)$, and the assumptions on $\hat{\psi}_1$, it follows that the last expression is a strictly negative quantity if $A > 0, B > 0$ and a strictly positive quantity if $A < 0, B < 0$.

Subcase 1.2: $A > 0, B < 0$ or $A < 0, B > 0$. In this case, $C_\rho = \frac{-1}{\rho\sqrt{|AB|}}$ is a real number. It follows that

$$
\begin{aligned}
&\lim_{a\to 0}\frac{2\pi i}{a} I_1(a,0,0,0)\\
&= \frac{-2}{\sqrt{|AB|}}\,\Re\left\{\int_0^\infty \hat{\psi}_1(\rho) \int_{-1}^{1} e^{-\pi i\rho \frac{1}{A} t_1^2} \hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1} e^{-\pi i\rho\frac{1}{B}t_2^2}\hat{\psi}_2(t_2)\, dt_2\, d\rho\right\}\\
&= \frac{-2}{\sqrt{|AB|}}\int_0^\infty \hat{\psi}_1(\rho)\Big(\int_{-1}^{1}\cos(\pi\rho\frac{1}{A}t_1^2)\,\hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1}\cos(\pi\rho\frac{1}{B}t_2^2)\,\hat{\psi}_2(t_2)\, dt_2 +
\end{aligned}
$$

$$- \int_{-1}^{1} \sin(\pi\rho \frac{1}{A} t_1^2)\, \hat{\psi}_2(t_1)\, dt_1 \int_{-1}^{1} \sin(\pi\rho \frac{1}{B} t_2^2)\, \hat{\psi}_2(t_2)\, dt_2 \Big)\, d\rho.$$

The expression in parenthesis in the last equation is strictly positive by Lemma 4.4 (notice in particular that exactly one of A and B is positive, the other is negative). Hence, using the properties of $\hat{\psi}_1$ it follows that the last expression is strictly negative.

• *Case 2:* $A = 0, B = 0$. Since $G_0(u) = 0$, the phase term $H_{\theta,\phi}(u)$, given by (9), vanishes. Hence, and choosing again the domain U of G_0 as $U = \{(u_1, u_2) : \alpha_1 \le u_1 \le \beta_1,\ \alpha_2 \le u_2 \le \beta_2\}$ with $\alpha_1, \alpha_2 < 0$, $\beta_1, \beta_2 > 0$, as in Case 1, the integral over U from (8) becomes:

$$\begin{aligned}
&\int_U e^{-2\pi i \frac{\rho}{a} H_{\theta,\phi}(u)}\, \Theta(\theta, \phi) \cdot (-1, \nabla G_0(u)) du \\
=& - \cos\theta \sin\phi \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} e^{-2\pi i \frac{\rho}{a} (\sin\theta \sin\phi\, u_1 + \cos\phi\, u_2)}\, du_2\, du_1.
\end{aligned}$$

It follows that

$$\begin{aligned}
&2\pi i a\, I_1(a, 0, 0, 0) \\
=& \int_0^\infty \int_0^{2\pi} \int_0^\pi \hat{\psi}_1(\rho \sin\phi \cos\theta)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\, \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi \sec\theta)) \\
&\times \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} e^{-2\pi i \frac{\rho}{a} (\sin\theta \sin\phi\, u_1 + \cos\phi\, u_2)}\, du_2\, du_1 \right) \rho \cos\theta\, \sin^2\phi\, d\phi\, d\theta\, d\rho \\
=& \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \Big(\int_0^\infty \int_0^{2\pi} \int_0^\pi \hat{\psi}_1(\rho \sin\phi \cos\theta)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\, \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi \sec\theta)) \\
&\times e^{-2\pi i \frac{\rho}{a} (\sin\theta \sin\phi\, u_1 + \cos\phi\, u_2)} \rho \cos\theta \sin^2\phi\, d\phi\, d\theta\, d\rho \Big)\, du_2\, du_1 \\
=& \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \Big(\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \hat{\psi}_1(\rho \sin\phi \cos\theta)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\, \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi \sec\theta)) \\
&\times e^{-2\pi i \frac{\rho}{a} (\sin\theta \sin\phi\, u_1 + \cos\phi\, u_2)} \rho \cos\theta \sin^2\phi\, d\phi\, d\theta\, d\rho \Big)\, du_2\, du_1 \\
&+ \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \Big(\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\pi \hat{\psi}_1(\rho \sin\phi \cos\theta)\, \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\, \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi \sec\theta)) \\
&\times e^{-2\pi i \frac{\rho}{a} (\sin\theta \sin\phi\, u_1 - \cos\phi\, u_2)} \rho \cos\theta \sin^2\phi\, d\phi\, d\theta\, d\rho \Big)\, du_2\, du_1.
\end{aligned}$$

Notice that, in the last step, we have split the integral over $[0, 2\pi]$ with respect to θ into two integrals over $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \frac{3\pi}{2}]$. For the second integral, we have applied the change of variable $\theta' = \theta - \pi$, so that $\theta' \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin\theta = -\sin\theta'$, $\cos\theta = -\cos\theta'$, and used the fact that $\hat{\psi}_2$ is even and $\hat{\psi}_1$ is odd.

From the last expression, using the change of variables $t_1 = a^{-\frac{1}{2}} \tan\theta$ and $t_2 = a^{-\frac{1}{2}} \cot\phi \sec\theta$, as in Case 1, and taking the limit as $a \to 0$, we have:

$$
\begin{aligned}
&\lim_{a\to 0} \frac{2\pi i}{a} I_1(a,0,0,0)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho\, \hat\psi_1(\rho)\int_{-1}^{1}\int_{-1}^{1}\hat\psi_2(t_1)\hat\psi_2(t_2)\, e^{-2\pi i \rho t_1 u_1} e^{-2\pi i\rho t_2 u_2} dt_1 dt_2 d\rho\, du_1 du_2\\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\rho\, \hat\psi_1(\rho)\int_{-1}^{1}\int_{-1}^{1}\hat\psi_2(t_1)\hat\psi_2(t_2)\, e^{-2\pi i \rho t_1 u_1} e^{2\pi i\rho t_2 u_2} dt_1 dt_2 d\rho\, du_1 du_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \hat\psi_1(\rho)\, \psi_2(\rho u_1)\, \psi_2(\rho u_2)\, \rho\, d\rho\, du_1\, du_2\\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \hat\psi_1(\rho)\, \psi_2(\rho u_1)\, \psi_2(-\rho u_2)\, \rho\, d\rho\, du_1\, du_2\\
&= 2\, (\hat\psi_2(0))^2 \int_0^\infty \frac{\hat\psi_1(\rho)}{\rho}\, d\rho.
\end{aligned}
$$

This quantity is positive, due to the assumptions on $\hat\psi_1$. Notice that, in the last step, we have used the property that $\hat\psi_2$ is even.

• *Case 3:* $A = 0,\ B \neq 0$ (the case $A \neq 0, B = 0$ is similar and is omitted). Without loss of generality, we may assume that $B > 0$. In this case, choosing again the domain U of G_0 as $U = \{(u_1, u_2) : \alpha_1 \le u_1 \le \beta_1, \alpha_2 \le u_2 \le \beta_2\}$ with $\alpha_1,\ \alpha_2 < 0,\ \beta_1,\ \beta_2 > 0$, as in Case 1, the integral over U from (8) becomes:

$$
\int_U e^{-2\pi i \frac{\rho}{a} H_{\theta,\phi}(u)}\, \Theta(\theta,\phi)\cdot(-1, \nabla G_0(u))du = K_0(\theta,\phi,a) + K_1(\theta,\phi,a),
$$

where

$$
\begin{aligned}
K_0(\theta,\phi,a) =& -\cos\theta\, \sin\phi\, e^{\pi i \frac{\rho}{a}\frac{\cos^2\phi}{B\cos\theta\sin\phi}}\\
&\times \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} e^{-\pi i \frac{\rho}{a} B\, \cos\theta \sin\phi (u_2 - u_{2,\theta,\phi})^2}\, e^{-2\pi i \frac{\rho}{a}\sin\theta\sin\phi\, u_1}\, du_2\, du_1\\
K_1(\theta,\phi,a) =& \, B\, \cos\phi\, e^{\pi i \frac{\cos^2\phi}{B\cos\theta\sin\phi}}\\
&\times \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} e^{-\pi i \frac{\rho}{a} B\, \cos\theta \sin\phi\, (u_2 - u_{2,\theta,\phi})^2}\, e^{-2\pi i \frac{\rho}{a}\sin\theta\sin\phi\, u_1}\, u_2\, du_2\, du_1.
\end{aligned}
$$

Hence, after splitting the integral with respect to θ in (8) into two integrals over $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \frac{3\pi}{2}]$, and applying the change of variable $\theta' = \theta - \pi$, as it was done in Case 1 and case 2, we can write

$$
I_1(a,0,0,0) = I_{10}(a,0,0,0) + I_{11}(a,0,0,0),
$$

where, for $j = 0, 1$,

$$
\begin{aligned}
&2\pi i a\, I_{1j}(a,0,0,0)\\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\pi\int_0^\infty \hat\psi_1(\rho\sin\phi\cos\theta)\, \hat\psi_2(a^{-\frac{1}{2}}(\tan\theta))\, \hat\psi_2(a^{-\frac{1}{2}}(\cot\phi\sec\theta)) \times
\end{aligned}
$$

$$\times K_j(\theta,\phi,a)\,\rho\,\sin\phi\,d\rho\,d\phi\,d\theta$$
$$+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\infty}\hat{\psi}_1(\rho\sin\phi\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\,\hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi\sec\theta))$$
$$\times K_j(\theta+\pi,\phi,a)\,\rho\,\sin\phi\,d\rho\,d\phi\,d\theta.$$

As in Case 1, we apply the change of variables $t_1 = a^{-\frac{1}{2}}\tan\theta$ and $t_2 = a^{-\frac{1}{2}}\cot\phi\sec\theta$ so that

$$\begin{aligned}
&2\pi i\, I_{10}(a,0,0,0)\\
&=\int_0^{\infty}\!\int_{-1}^{1}\!\int_{-1}^{1}\hat{\psi}_1\Big(\tfrac{\rho}{\sqrt{at_1^2+at_2^2+1}}\Big)\,\hat{\psi}_2(t_1)\hat{\psi}_2(t_2)\,K_0(t_1,t_2,a)\,\frac{\rho\,dt_1\,dt_2\,d\rho}{(at_1^2+at_2^2+1)^{3/2}}\\
&+\int_0^{\infty}\!\int_{-1}^{1}\!\int_{-1}^{1}\hat{\psi}_1\Big(\tfrac{\rho}{\sqrt{at_1^2+at_2^2+1}}\Big)\,\hat{\psi}_2(t_1)\hat{\psi}_2(t_2)\,\bar{K}_0(t_1,t_2,a)\,\frac{\rho\,dt_1\,dt_2\,d\rho}{(at_1^2+at_2^2+1)^{3/2}}
\end{aligned}$$

where (again, with abuse of notation, $K_0(t_1,t_2,a)$ denotes the function $K_0(\theta,\phi,a)$ after the change of variables)

$$\begin{aligned}
K_0(t_1,t_2,a) &= \frac{-1}{\sqrt{at_1^2+at_2^2+1}}\,e^{\pi i\rho\frac{t_2^2}{B}\frac{1}{\sqrt{1+at_1^2+at_2^2}}}\\
&\quad\times\int_{\alpha_1}^{\beta_1}\!\int_{\alpha_2}^{\beta_2} e^{-\pi i\rho B\frac{1}{\sqrt{at_1^2+at_2^2+1}}(\frac{u_2}{\sqrt{a}}+\frac{\sqrt{a}t_2}{B})^2}\, e^{-2\pi i\rho\frac{t_1}{\sqrt{at_1^2+at_2^2+1}}\frac{u_1}{\sqrt{a}}}\,du_2\,du_1.
\end{aligned}$$

As in Case 1, the term K_1 is dominated by K_0, as $a\to 0$, so that

$$\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0) = \lim_{a\to 0}\frac{2\pi i}{a}I_{10}(a,0,0,0).$$

Thus

$$\begin{aligned}
&\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0)\\
&=\hat{\psi}_2(0)\int_0^{\infty}\hat{\psi}_1(\rho)\Big(\int_{-\infty}^{\infty}e^{-i\pi B\rho u_2^2}du_2\int_{-1}^{1}e^{i\pi\frac{1}{B}\rho t_2^2}\hat{\psi}_2(t_2)\,dt_2\Big)\,d\rho\\
&\quad+\hat{\psi}_2(0)\int_0^{\infty}\hat{\psi}_1(\rho)\Big(\int_{-\infty}^{\infty}e^{i\pi B\rho u_2^2}du_2\int_{-1}^{1}e^{-i\pi\frac{1}{B}t_2^2}\hat{\psi}_2(t_2)\,dt_2\Big)\,d\rho\\
&=2\,\hat{\psi}_2(0)\int_0^{\infty}\hat{\psi}_1(\rho)\,\Re\Big\{\int_{-\infty}^{\infty}e^{-i\pi B\rho u_2^2}du_2\int_{-1}^{1}e^{i\pi\frac{1}{B}\rho t_2^2}\hat{\psi}_2(t_2)\,dt_2\Big\}\,d\rho\\
&=\frac{\sqrt{2}\,\hat{\psi}_2(0)}{\sqrt{B}}\int_0^{\infty}\frac{\hat{\psi}_1(\rho)}{\sqrt{\rho}}\Big(\int_{-1}^{1}\cos(\pi\rho\frac{1}{B}t_2^2)\,\hat{\psi}_2(t_2)\,dt_2\\
&\quad+\int_{-1}^{1}\sin(\pi\rho\frac{1}{B}t_2^2)\,\hat{\psi}_2(t_2)\,dt_2\Big)\,d\rho.
\end{aligned}$$

The last quantity is strictly positive by Lemma 4.4 and the properties of

$\hat{\psi}_1$. This completes the proof of (iii) in Situation (I). Let us now consider Situation (II).

(iii) – Situation (II): Let $p \in \gamma_j$ for some j.

Without loss of generality, we may assume that there is only one separating curve γ. Also in this case, as above, it is sufficient to show that

$$\lim_{a \to 0^+} a^{-1}\, I_1(a, s_1, s_2, p) \neq 0,$$

where the integral I_1 is given by (8). We still adopt the same notation introduced above with $S_0 = (G_0(u), u)$, where $u \in U$.

If $p = (0, 0, 0)$ is on the curve γ, then locally we may choose the domain U of G_0 as $U = \{(u_1, u_2) : \alpha_1 \leq u_1 \leq \beta_1, g(u_1) \leq u_2 \leq g(u_1) + \beta_2\}$ with $\alpha_1 < 0$, $\beta_1, \beta_2 > 0$, where g is a C^3 smooth function (this follows from the assumption that the separating curve γ is C^3 smooth) and $g(0) = 0$. Notice that $\{(G(u_1, g(u_1)),\ \alpha_1 \leq u_1 \leq \beta_1\} \subset \gamma$. We can write $g(u_1) = g'(0)u_1 + O(u_1^2)$. As described in the following, the proof is a simple modification of the argument for the smooth boundaries (Situation (I)).

Also in this case, we need to consider 3 cases, depending on the signs of A and B in (10). In the following, for brevity, we only describe below how the argument above need to be modified for each one of these cases.

For Case 1 ($A, B \neq 0$), as in Situation (I), we have that the I_1 integral, as $a \to 0$, is dominated by the term involving K_0. For this term, we have that

$$\begin{aligned}
&\lim_{a \to 0} \frac{1}{a} K_0(\theta, \phi, a) \\
&= e^{\pi i \rho (\frac{t_1^2}{A} + \frac{t_2^2}{B})} \int_{-\infty}^{\infty} e^{-\pi i \rho A (u_1 + \frac{t_1}{A})^2} \int_{g_1'(0)u_1}^{\infty} e^{-\pi i \rho B (u_2 + \frac{t_2}{B})^2} du_2 du_1 \\
&= e^{\pi i \rho (\frac{t_1^2}{A} + \frac{t_2^2}{B})} \int_{-\infty}^{\infty} e^{-\pi i \rho A u_1^2} \int_{g_1'(0)(u_1 - \frac{t_1}{A}) + \frac{t_2}{B}}^{\infty} e^{-\pi i \rho B u_2^2} du_2 du_1 \\
&= e^{\pi i \rho (\frac{t_1^2}{A} + \frac{t_2^2}{B})} \int_{-\infty}^{\infty} e^{-\pi i \rho A u_1^2} \left(\int_{g_1'(0)(u_1 - \frac{t_1}{A}) + \frac{t_2}{B}}^{0} e^{-\pi i \rho B u_2^2} du_2 + \int_0^{\infty} e^{-\pi i \rho B u_2^2} du_2 \right) du_1.
\end{aligned}$$

Let

$$V(t_1, t_2) = \int_{-\infty}^{\infty} e^{-\pi i \rho A u_1^2} \int_{g_1'(0)(u_1 - \frac{t_1}{A}) + \frac{t_2}{B}}^{0} e^{-\pi i \rho B u_2^2}\, du_2\, du_1.$$

It can be verified that, for each (t_1, t_2), the improper integral V is convergent. Notice that $V(-t_1, -t_2) = -V(t_1, t_2)$, that is, V is an odd function of (t_1, t_2). Since $\hat{\psi}_2$ is even, it follows that this term will give no contribution in the integral I_1. Thus, using the same argument as in Situation I (Case 1), we have that

$$\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0)$$
$$= \Re\{\int_0^\infty \hat{\psi}_1(\rho)C_\rho \int_{-1}^1 e^{-\pi i\rho\frac{1}{A}t_1^2}\ \hat{\psi}_2(t_1)\ dt_1 \int_{-1}^1 e^{-\pi i\rho\frac{1}{B}t_2^2}\ \hat{\psi}_2(t_2)\ dt_2 d\rho\},$$

where C_ρ is given by (12). Notice that this expression is the same as (13), and it is strictly negative, as proven above.

For Case 2 ($A = B = 0$), the same argument as in Situation I (Case 2) gives that

$$\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0) = 2\int_{-\infty}^\infty \int_{g_1'(0)u_1}^\infty \int_0^\infty \hat{\psi}_1(\rho)\,\psi_2(\rho u_2)\,\psi_2(\rho u_1)\rho\, d\rho\, du_2\, du_1.$$

Notice that the function $W(u_1) = \int_0^{g_1'(0)u_1} \psi_2(\rho u_2)du_2$ is an odd function (recall that ψ_2 is even since $\hat{\psi}_2$ is even). It follows that the integral

$$\int_{-\infty}^\infty \int_0^{g_1'(0)u_1} \psi_2(\rho u_2)\psi_2(\rho u_1)\, du_2\, du_1 = 0,$$

for any $\rho > 0$. This implies that

$$\begin{aligned}
&\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0)\\
&= 2\int_{-\infty}^\infty \int_{g_1'(0)u_1}^\infty \int_0^\infty \hat{\psi}_1(\rho)\,\psi_2(\rho u_2)\,\psi_2(\rho u_1)\,\rho\, d\rho\, du_2\, du_1\\
&= 2\int_{-\infty}^\infty \int_0^\infty \int_0^\infty \hat{\psi}_1(\rho)\,\psi_2(\rho u_2)\,\psi_2(\rho u_1)\,\rho\, d\rho\, du_2\, du_1\\
&= \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\,\psi_2(u_2)\,\psi_2(u_2)\, d\rho\, du_2\, du_1\\
&= (\hat{\psi}_2(0))^2 \int_0^\infty \frac{\hat{\psi}_1(\rho)}{\rho}\, d\rho > 0.
\end{aligned}$$

For Case 3 ($A = 0, B > 0$), again adapting the argument from Situation (I), we have:

$$\begin{aligned}
&\lim_{a\to 0}\frac{2\pi i}{a}I_1(a,0,0,0)\\
&= \int_{-1}^1\int_{-1}^1\int_0^\infty \hat{\psi}_1(\rho)\,\hat{\psi}_2(t_1)\,\hat{\psi}_2(t_2)\left(\int_{-\infty}^\infty\int_{g_1'(0)u_1+\frac{t_2}{B}}^\infty e^{-i\pi\rho B u_2^2}du_2\, e^{-2\pi i\rho t_1 u_1}du_1\right)\\
&\times e^{i\pi\frac{1}{B}\rho t_2^2}\rho\, d\rho\, dt_1\, dt_2\\
&+ \int_{-1}^1\int_{-1}^1\int_0^\infty \hat{\psi}_1(\rho)\,\hat{\psi}_2(t_1)\,\hat{\psi}_2(t_2)\left(\int_{-\infty}^\infty\int_{g_1'(0)u_1+\frac{t_2}{B}}^\infty e^{i\pi\rho B u_2^2}du_2\, e^{2\pi i\rho t_1 u_1}du_1\right)\\
&\times e^{-i\pi\frac{1}{B}\rho t_2^2}\rho\, d\rho\, dt_1\, dt_2 =
\end{aligned}$$

$$= \int_{-1}^{1}\int_0^\infty \hat\psi_1(\rho)\hat\psi_2(t_2)\int_{-\infty}^\infty \int_{g_1'(0)u_1+\frac{t_2}{B}}^\infty e^{-i\pi\rho B u_2^2} du_2 \psi_2(\rho u_1) du_1 e^{i\pi\frac{1}{B}\rho t_2^2}\rho d\rho dt_2$$
$$+ \int_{-1}^{1}\int_0^\infty \hat\psi_1(\rho)\hat\psi_2(t_2)\int_{-\infty}^\infty \int_{g_1'(0)u_1+\frac{t_2}{B}}^\infty e^{i\pi\rho B u_2^2} du_2 \psi_2(\rho u_1) du_1 e^{-i\pi\frac{1}{B}\rho t_2^2}\rho d\rho dt_2.$$

Notice that

$$V(t_2,\rho) = \int_{-\infty}^\infty \int_{g_1'(0)u_1+\frac{t_2}{B}}^0 e^{-i\pi\rho B u_2^2}\, du_2\, \psi_2(\rho u_1)\, du_1$$

is an odd function of t_2 (this is shown by setting $(u_1,u_2) = -(u_1',u_2')$ and using the fact that ψ_2 is even). Hence, this gives no contribution to the integral above and we can replace the integral $\int_{g_1'(0)u_1-\frac{t_2}{B}}^\infty$ with $\int_0^\infty$. It follows that

$$\lim_{a\to 0}\frac{2\pi i}{a} I_1(a,0,0,0)$$
$$= \int_{-1}^{1}\int_0^\infty \hat\psi_1(\rho)\hat\psi_2(t_2)\left(\int_{-\infty}^\infty\int_0^\infty e^{-i\pi\rho B u_2^2} du_2\, \psi_2(\rho u_1)\, du_1\right) e^{i\pi\frac{1}{B}\rho t_2^2}\rho\, d\rho\, dt_2$$
$$+ \int_{-1}^{1}\int_0^\infty \hat\psi_1(\rho)\hat\psi_2(t_2)\left(\int_{-\infty}^\infty\int_0^\infty e^{i\pi\rho B u_2^2} du_2\, \psi_2(\rho u_1)\, du_1\right) e^{-i\pi\frac{1}{B}\rho t_2^2}\rho\, d\rho\, dt_2.$$

Thus, using the same argument as in Situation (I) (Case 3), we conclude that

$$\lim_{a\to 0}\frac{2\pi i}{a} I_1(a,0,0,0)$$
$$= \frac{\hat\psi_2(0)}{\sqrt{2B}}\int_0^\infty \frac{\hat\psi_1(\rho)}{\sqrt{\rho}}\left(\int_{-1}^{1}\cos(\pi\rho\frac{1}{B}t_2^2)\,\hat\psi_2(t_2)\,dt_2\right.$$
$$\left. + \int_{-1}^{1}\sin(\pi\rho\frac{1}{B}t_2^2)\,\hat\psi_2(t_2)\,dt_2\right) d\rho > 0.$$

This completes the proof of (iii).

Proof of (iv). In this case, the unit vector $\Theta(\theta_0,\phi_0)$ associated with the shearing variables (s_1,s_2) does not correspond to the normal orientations of Ω at p.

Again, without loss of generality, we may assume $p=(0,0,0)$, $\theta_0 = 0$, $\phi = \frac{\pi}{2}$ which yields $s_1 = s_2 = 0$. In this case, we can express the domain U of G as $U = U_1 \bigcup U_2$ with $U_1 = \{(u_1,u_2), \alpha \le u_1 \le \beta, g(u_1) \le u_2 \le g(u_1)+\epsilon\}$ and $U_2 = \{(u_1,u_2), \alpha \le u_1 \le \beta, g(u_1)-\epsilon \le u_2 \le g(u_1)\}$, where $\alpha<0$, $\beta>0$. Correspondingly, we write the boundary region near p as $S = S_1\bigcup S_2$ with $S_1 = \{(G_1(u),u):\ u\in U_1\}$ and $S_2 = \{(G_2(u),u):\ u\in U_2\}$. For $j=1,2$, let $\vec n_j(p)$ be the outer normal vector of S_j at p. Also we write I_1 as $I_{11}\bigcup I_{12}$, where I_{1j} corresponds to S_j, for $j=1,2$.

We will only examine the term I_{11} since the argument for the term I_{12} is similar.

Since $\vec{n}_1(p) \neq \pm(1,0,0)$, it follows that $\nabla G_1((0,0)) \neq (0,0)$. Without loss of generality, we may assume that $G_{1u_2}(0,0) \neq 0$. Since $(\phi,\theta) \to (\phi_0,\theta_0) = (1,0,0)$ as $a \to 0$, we can choose ϵ sufficiently small so that $\Theta(\phi,\theta)\cdot(G_{1u_2}(u),0,1) \neq 0$ for all ϕ, θ and all $u \in U$.

Using polar coordinates, we can express $I_{11}(a,0,0,0)$ as

$$\begin{aligned}
&I_{11}(a,0,0,0)\\
&= -\frac{1}{2\pi i a}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\hat{\psi}_1(\rho\sin\phi\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta))\hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi\sec\theta))\\
&\times\int_{U_1} e^{-2\pi i\frac{\rho}{a}\Theta(\phi,\theta)\cdot(G_1(u),u)}\,\Theta(\phi,\theta)\cdot(-1,\nabla G_1(u))du\,\rho\sin\phi\,d\rho\,d\phi\,d\theta.
\end{aligned}$$

Using the definition of U_1, we have that

$$\begin{aligned}
&\int_{U_1} e^{-2\pi i\frac{\rho}{a}\Theta(\phi,\theta)\cdot(G_1(u),u)}\,\Theta(\phi,\theta)\cdot(-1,\nabla G_1(u))du\\
&=\frac{-a}{2\pi i\rho}\int_\alpha^\beta\int_{g(u_1)}^{g(u_1)+\epsilon}\frac{\partial}{\partial u_2}(e^{-2\pi i\frac{\rho}{a}\Theta(\phi,\theta)\cdot(G_1(u),u)})\,\frac{\Theta(\phi,\theta)\cdot(-1,\nabla G_1(u))}{\Theta(\phi,\theta)\cdot(G_{1u_2}(u),0,1)}du_2du_1
\end{aligned}$$

Integrating by parts twice for the inner integral and neglecting the endpoint terms evaluated at $u_2 = g(u_1)+\epsilon$ (by Lemma 4.1) we have the estimate

$$\begin{aligned}
&-\int_{g(u_1)}^{g(u_1)+\epsilon}\frac{\partial}{\partial u_2}(e^{-2\pi i\frac{\rho}{a}\Theta(\phi,\theta)\cdot(G_1(u),u)})\,\frac{\Theta(\phi,\theta)\cdot(-1,\nabla G_1(u))}{\Theta(\phi,\theta)\cdot(G_{1u_2}(u),0,1)}du_2\\
&=e^{-2\pi i\frac{\rho}{a}\Theta(\phi,\theta)\cdot(G_1(u_1,g(u_1)),u_1,g(u_1))}\frac{\Theta(\phi,\theta)\cdot(-1,\nabla G_1(u_1,g(u_1)))}{\Theta(\phi,\theta)\cdot(G_{1u_2}(u_1,g(u_1)),0,1)}+O(a).
\end{aligned}$$

Introducing the notation

$$\Gamma(\phi,\theta,u_1) = \frac{\Theta(\phi,\theta)\cdot(-1,\nabla G_1(u_1,g(u_1)))}{\Theta(\phi,\theta)\cdot(G_{1u_2}(u_1,g(u_1)),0,1)},$$

$$H(\phi,\theta,u_1) = \Theta(\phi,\theta)\cdot(G_1(u_1,g(u_1)),u_1,g(u_1)),$$

and splitting the integral in θ over $[-\frac{\pi}{2},\frac{\pi}{2}]$ and $[\frac{\pi}{2},\frac{3\pi}{2}]$, we can write:

$$I_{11}(a,0,0,0) = \frac{1}{(2\pi)^2}\,I_{111}(a,0,0,0)+\frac{1}{(2\pi)^2}\,I_{112}(a,0,0,0),$$

where (using again that $\hat{\psi}_1$ is odd and $\hat{\psi}_2$ is even)

$$I_{111}(a,0,0,0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\infty} \hat{\psi}_1(\rho\sin\phi\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}\tan\theta)$$

$$\times\,\hat{\psi}_2(a^{-\frac{1}{2}}\cot\phi\sec\theta)\int_\alpha^\beta e^{-2\pi i\frac{\rho}{a}H(\phi,\theta,u_1)}\Gamma(\phi,\theta,u_1)\,du_1\sin\phi\,d\rho\,d\phi\,d\theta$$

$$I_{112}(a,0,0,0) = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\infty} \hat{\psi}_1(\rho\sin\phi\cos\theta)\,\hat{\psi}_2(a^{-\frac{1}{2}}\tan\theta)$$

$$\times\hat{\psi}_2(a^{-\frac{1}{2}}\cot\phi\sec\theta)\int_\alpha^\beta e^{-2\pi i\frac{\rho}{a}H(\phi,\theta+\pi,u_1)}\Gamma(\phi,\theta+\pi,u_1)\,du_1\sin\phi\,d\rho\,d\phi\,d\theta.$$

It is sufficient to examine the term $I_{111}(a,0,0,0)$, since the behavior of $I_{112}(a,0,0,0)$ is similar.

Notice that $g(0)=0$, $G_1(0,g(0))=0$ and hence $H(\phi,\theta,0)=0$ for all a,ρ,ϕ, and θ. Let $\Phi(u_1)=G_1(u_1,g(u_1))$. If we assume $\Phi'(0)\neq 0$, then we have $\frac{\partial}{\partial u_1}(H(\phi_0,\theta_0,0))\neq 0$ and hence we may assume that $\frac{\partial}{\partial u_1}(H(\phi,\theta,u_1)\neq 0$ for all ϕ,θ and u_1. Since $u_1=0$ is an interior point of the interval (α,β), the same argument used in the proof of (ii) in Theorem 3.1 yields that $I_{111}(a,0,0,0)$ has rapid decay as $a\to 0$. It remains to consider the two cases $\Phi'(0)=0,\ \Phi''(0)\neq 0$ and $\Phi'(0)=0,\ \Phi''(0)=0$.

- *Case 1:* $\Phi'(0)=0,\ \Phi''(0)\neq 0$.

In this case, we have that $\frac{\partial^2}{\partial u_1^2}H(\phi_0,\theta_0,0)\neq 0$ and hence we may assume that $\frac{\partial^2}{\partial u_1^2}H(\phi,\theta,u_1)\neq 0$ for all ϕ,θ and u_1. Choose $\eta(u_1)\in C_0^\infty(\alpha,\beta)$ such that $\eta(u_1)=1$ for u_1 near 0 and let

$$\Gamma(\phi,\theta,u_1)=\Gamma_1(\phi,\theta,u_1)+\Gamma_2(\phi,\theta,u_1),$$

where $\Gamma_1(\phi,\theta,u_1)=\Gamma(\phi,\theta,u_1)(1-\eta(u_1))$ and $\Gamma_2(\phi,\theta,u_1)=\Gamma(\phi,\theta,u_1)\eta(u_1)$. For $j=1,2$, let I_{111j} be defined by replacing Γ in I_{111} by Γ_j. For I_{1111}, one can follow the proof of Lemma 4.1 to show that I_{1111} has rapid decay as $a\to 0$. For I_{1112}, one can apply Lemma 4.2 for the integral on u_1, and $t_1=a^{-\frac{1}{2}}\tan\theta$ and $t_2=a^{-\frac{1}{2}}\cot\phi\sec\theta$ for the integral on ϕ and θ to show that $I_{1112}=O(a^{\frac{3}{2}})$ as $a\to 0$.

- *Case 2:* $\Phi'(0)=0,\ \Phi''(0)=0$.

In this case, we have that $\Phi(u_1)=O(u_1^3)$ as $u_1\to 0$. Write $g(u_1)=g'(0)u_1+O(u_1^2)$. Again, letting $t_1=a^{-\frac{1}{2}}\tan\theta$, $t_2=a^{-\frac{1}{2}}\cot\phi\sec\theta$ and $v=a^{-\frac{1}{2}}u_1$, one can verify that

$$\lim_{a\to 0}\frac{\rho}{a}H(\phi,\theta,u_1)=\rho\,(t_1v+t_2g'(0)\,v).$$

It follows that

$$\lim_{a\to 0}a^{-\frac{3}{2}}I_{1112}(a,0,0,0) = -\frac{1}{(2\pi)^2G_{u_2}(0,0)}\int_0^\infty\int_{-1}^1\int_{-1}^1\hat{\psi}_1(\rho)\hat{\psi}_2(t_1)\,\hat{\psi}_2(t_2)\,\times$$

$$\times \int_{-\infty}^{\infty} e^{-2\pi\rho i(t_1 v + t_2 g'(0)v)} dv\, dt_1\, dt_2\, d\rho$$
$$= -\frac{1}{(2\pi)^2 G_{u_2}(0,0)} \int_0^\infty \hat{\psi}_1(\rho) \int_{-\infty}^{\infty} \psi_2(\rho v)\psi_2(\rho g'(0)v)\, dv\, d\rho.$$

Combining Case 1 and Case 2, it follows that

$$I_{111}(a,0,0,0) = O(a^{\frac{3}{2}}).$$

As mentioned above, the behavior of $I_{112}(a,0,0,0)$ is similar, so that we have that

$$I_{11}(a,0,0,0) = O(a^{\frac{3}{2}})$$

This completes the proof of (iv).

Proof of (ii). This is the situation where $p \in \partial\Omega \setminus \bigcup_{j=1}^m \gamma_j$, and the unit vector $\Theta(\theta_0, \phi_0)$ associated with the shearing variables (s_1, s_2) does not correspond to the normal orientation, that is $\Theta(\theta_0, \phi_0) \neq \pm\vec{n}(p)$

As in the proof of (iii), Situation (I), it is sufficient to examine the integral I_1, given by (7), with $p = (0,0,0), \theta_0 = 0, \phi = \frac{\pi}{2}$, which yields $s_1 = s_2 = 0$. In particular, using polar coordinates, we can express the integral $I_1(a,0,0,0)$ as

$$\begin{aligned}
&I_1(a,0,0,0)\\
&= -\frac{1}{2\pi i a} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \hat{\psi}_1(\rho \sin\phi \cos\theta) \hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta)) \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi \sec\theta))\\
&\quad\times \int_U e^{-2\pi i \frac{\rho}{a} \Theta(\phi,\theta)\cdot(G(u),u)}\, \Theta(\phi,\theta)\cdot(-1, \nabla G(u)) du\, \rho \sin\phi\, d\rho\, d\phi\, d\theta.
\end{aligned}$$

As in the proof of (iii), we can express U as $U = \{(u_1, u_2) : \quad \alpha_1 \leq u_1 \leq \beta_1,\ \alpha_2 \leq u_2 \leq \beta_2\}$ with $\alpha_1, \alpha_2 < 0,\ \beta_1, \beta_2 > 0$. Also, as in the proof of (iv), we may assume that $\Theta(\phi,\theta)\cdot(G_{1u_2}(u), 0, 1) \neq 0$ for all ϕ, θ and all $u \in U$. It follows that

$$\begin{aligned}
&\int_U e^{-2\pi i \frac{\rho}{a} \Theta(\phi,\theta)\cdot(G(u),u)}\, \Theta(\phi,\theta)\cdot(-1, \nabla G(u)) du\\
&= -\frac{a}{2\pi i \rho} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{\partial}{\partial u_2}(e^{-2\pi i \frac{\rho}{a} \Theta(\phi,\theta)\cdot(G(u),u)})\, \frac{\Theta(\phi,\theta)\cdot(-1, \nabla G(u))}{\Theta(\phi,\theta)\cdot(G_{u_2}(u), 0, 1)} du_2 du_1.
\end{aligned}$$

Integrating by parts the inner integral, this can be written as:

$$\int_{\alpha_2}^{\beta_2} \frac{\partial}{\partial u_2} \left(e^{-2\pi i \frac{\rho}{a} \Theta(\phi,\theta)\cdot(G(u),u)}\right) \frac{\Theta(\phi,\theta)\cdot(-1, \nabla G(u))}{\Theta(\phi,\theta)\cdot(G_{u_2}(u), 0, 1)}\, du_2 = J_1(u_1) - J_2(u_1),$$

where

$$J_1(u_1) = \left(e^{-2\pi i \frac{\rho}{a}\Theta(\phi,\theta)\cdot(G(u_1,u_2),u_1,u_2)} \frac{\Theta(\phi,\theta)\cdot(-1,\nabla G(u_1,u_2)}{\Theta(\phi,\theta)\cdot(G_{u_2}(u_1,u_2),0,1)} \right)_{u_2=\alpha_2}^{u_2=\beta_2}$$

$$J_2(u_1) = \int_{\alpha_2}^{\beta_2} e^{-2\pi i \frac{\rho}{a}\Theta(\phi,\theta)\cdot(G(u_1,u_2),u_1.u_2)} \frac{\partial}{\partial u_2}\left(\frac{\Theta(\phi,\theta)\cdot(-1,\nabla G(u_1,u_2))}{\Theta(\phi,\theta)\cdot(G_{u_2}(u_1,u_2),0,1)}\right) du_2.$$

Since $\beta_2 \neq 0$, $\alpha_2 \neq 0$, it follows that $(G(u_1,\beta_2),u_1,\beta_2) \neq (0,0,0)$ and that $(G(u_1,\alpha_2),u_1,\alpha_2) \neq (0,0,0)$ for all $u_1 \in [\alpha_1,\beta_1]$. Using the argument of Lemma 4.1, for u_1 fixed, it follows that the J_1 term yields the desired decay a^N. Integrating by parts the integral J_2, an argument by induction shows that also the J_2 term gives the desired decay a^N. This is completes the proof of (ii).

Remark 4.1 In the proof of Theorem 3.1, we made the assumption that $s = (s_1,s_2) = (0,0)$. Let us examine how to modify the argument for the case (iii) - Situation (I) when $s = (s_1,s_2) \neq (0,0)$. For other cases, the modification is either trivial or straight forward.

Since $s = (s_1,s_2) \neq (0,0)$, we have $s_1 = \tan\theta_0$, $s_2 = \cot\theta_0\cot\phi_0$ with $(\theta_0,\phi_0) \neq (0,\frac{\pi}{2})$ (or $(\pi,\frac{\pi}{2})$). In this case, letting $S, G(u), S_0, G_0(u)$ be given as in Lemma 4.5, with $S = (G(u),u)$ and $p = (G(0),0)$ with $G(0) = 0$, and using spherical coordinates, the integral $I_1(a,s_1,s_2,0)$, taken on S_0, can be written as

$$\begin{aligned} I_1(a,s_1,s_2,0) = & -\frac{1}{2\pi i a}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \hat{\psi}_1(\rho\sin\phi\cos\theta)\hat{\psi}_2(a^{-\frac{1}{2}}(\tan\theta-\tan\theta_0)) \\ & \times \hat{\psi}_2(a^{-\frac{1}{2}}(\cot\phi\sec\theta - \sec\theta_0\cot\phi_0)) \int_U e^{-2\pi i\frac{\rho}{a}\Theta(\phi,\theta)\cdot(G_0(u),u)} \\ & \times \Theta(\phi,\theta)\cdot(-1,\nabla G_0(u))\,du\,\rho\sin\phi\,d\rho\,d\phi\,d\theta. \end{aligned}$$

We assume $G_{u_1u_2}(0,0) = 0$ so that

$$G_0(u) = G_{u_1}(0,0)u_1 + G_{u_2}(0,0)u_2 + \frac{1}{2}(Au_1^2 + Bu_2^2),$$

where A, B are given by (10). Since $\Theta(\theta_0,\phi_0) = \pm\vec{n}(p)$, we have that

$$F_1(\phi_0,\theta_0,0) = -\Theta(\theta_0,\phi_0)\cdot(G_{u_1}(0,0),1,0) = 0,$$
$$F_2(\phi_0,\theta_0,0) = -\Theta(\theta_0,\phi_0)\cdot(G_{u_2}(0,0),0,1) = 0.$$

It follows that $G_{u_1}(0,0) = -\tan\theta_0$ and $G_{u_2}(0,0) = -\cot\theta_0\cot\phi_0$. Also we have that

$$F_1(\phi,\theta,u) = -\Theta(\theta,\phi)\cdot(G_{u_1}(0,0) + Au_1,1,0) =$$

$$
\begin{aligned}
&= -\cos\theta\sin\phi\ (G_{u_1}(0,0) + Au_1) - \sin\theta\sin\phi; \\
F_2(\phi,\theta,u) &= -\Theta(\theta,\phi)\cdot(G_{u_2}(0,0) + Bu_2, 0, 1) \\
&= -\cos\theta\sin\phi\ (G_{u_2}(0,0) + Bu_2) - \cos\phi.
\end{aligned}
$$

Solving $u_{1,\theta,\phi}$ from $F_1(\phi,\theta,u) = 0$ and $u_{2,\theta,\phi}$ from $F_2(\phi,\theta,u) = 0$, we obtain that, if $A \neq 0$,

$$
u_{1,\theta,\phi} = -\frac{1}{A}(\tan\theta + G_{u_1}(0,0)) = -\frac{1}{A}(\tan\theta - \tan\theta_0),
$$

and, if $B \neq 0$,

$$
u_{2,\theta,\phi} = -\frac{1}{B}(\sec\theta\cot\phi + G_{u_2}(0,0)) = -\frac{1}{B}(\sec\theta\cot\phi - \sec\theta_0\cot\phi_0).
$$

Now if we let $t_1 = a^{-\frac{1}{2}}(\tan\theta - \tan\theta_0)$, $t_2 = a^{-\frac{1}{2}}(\sec\theta\cot\phi - \sec\theta_0\cot\phi_0)$, then the rest of the argument is the same as for the case $s = (0,0)$.

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