CONSTRUCTION OF REGULAR AND IRREGULAR SHEARLET FRAMES

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Abstract. In this paper, we study the construction of irregular shearlet systems, i.e., systems of the form $\mathcal{SH}(\psi,\Lambda) = \{a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(x-t)) : (a,s,t) \in \Lambda\}$, where $\psi \in L^2(\mathbb{R}^2)$, Λ is an arbitrary sequence in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, A_a is a parabolic scaling matrix and S_s a shear matrix. These systems are obtained by appropriately sampling the Continuous Shearlet Transform. We derive sufficient conditions for such a discrete system to form a frame for $L^2(\mathbb{R}^2)$, and provide explicit estimates for the frame bounds. Among the examples of such discrete systems, one is the Parseval frame of shearlets previously introduced by the authors, which is optimal in approximating 2-D smooth functions with discontinuities along C^2-curves. This study provides the framework for the construction of a variety of discrete directional multiscale systems with the ability to detect orientations inherited from the Continuous Shearlet Transform.

1. Introduction

In recent years, there has been a flurry of activities in the study of variants of the affine systems which contain basis elements with many more locations, scales and directions than traditional wavelets. In fact, due to their limited directional sensitivity, traditional wavelets are not very efficient in dealing with distributed discontinuities such as the edges occurring in natural images or the boundaries of solid bodies. In many image processing and partial differential equation applications, it is of fundamental importance to overcome these limitations if one wants to design faster and more efficient computational algorithms.

Several variations of the wavelet scheme have been recently proposed to address these issues, including the *directional wavelets* [1], the *complex wavelets* [12], the *bandelets* [15], and the *contourlets* [8]. One of the major breakthroughs in this direction was made by Candès and Donoho, who introduced the *ridgelets* [3] and then the *curvelets* [4]. The curvelets, in particular, provide a truly directional multiscale representation of multidimensional data, and are able to achieve an essentially optimal approximation property for 2-D smooth functions with discontinuities along C^2-curves.

The *shearlets*, recently introduced by the authors and their collaborators, provide an alternative approach to the curvelets, and exhibit some very distinctive features. In fact, similarly to the curvelets, the shearlets are a multiscale directional system and are also optimal in approximating 2-D smooth functions with discontinuities along C^2-curves [11]. However, unlike the curvelets, the shearlets form an *affine system*. That is, they are generated by dilating and translating one single generating function, where the dilation matrix is the product of a parabolic scaling matrix and a shear matrix. In particular, the shearlets can

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be regarded as coherent states arising from a unitary representation of a particular locally compact group, called the *shearlet group*. This allows one to employ the theory of uncertainty principles to study the accuracy of the shearlet parameters [7]. Another consequence of the group structure of the shearlets is that they are associated with a generalized Multiresolution Analysis, and this is particularly useful in both their theoretical and numerical applications [14, 9, 16].

Similarly to the theory of affine systems, we distinguish between *continuous* and *discrete* shearlet systems: the continuous shearlet systems are associated with the whole range of scaling, shear, and translation indices $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, whereas the discrete shearlet systems are associated with a sequence $\Lambda \subset \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ of "discrete" scaling, shear, and translation indices.

In [13], the authors introduced the *Continuous Shearlet Transform* for functions and distributions on $\mathbb{R}^2$, which is defined by $\mathcal{SH}_\psi f(a, s, t) = \langle f, \psi_{ast} \rangle$, where the analyzing elements ψ_{ast} are dilated, sheared and translated copies of a single generating function ψ, depending continuously on the scale parameter a, the shear parameter s and the location t. This transform is derived within the framework of the multidimensional wavelet transform and has the ability to identify not only the location of the singular points of a distribution f, but also the orientation of their distributed singularities. The goal of this paper is to show that, by appropriately sampling the Continuous Shearlet Transform, one obtains a variety of discrete representations exhibiting exactly those desirable properties of multiscale analysis, directional sensitivity, and localization described above. As a special case of this discretization, we obtain the regular discrete shearlet systems previously introduced in [10, 11]. However, many more similar systems can be obtained, including systems defined by an irregular sequence of indices. In particular, we derive sufficient conditions for a discrete shearlet system to form a frame for $L^2(\mathbb{R}^2)$ with explicit estimates of the associated frame bounds.

The paper is organized as follows. In Section 2 we recall the definitions of continuous and discrete shearlet systems, and the basic definitions from frame theory. The main result concerning sufficient conditions for irregular shearlet systems to form a frame is presented in Section 3. Finally, in Section 4 we discuss the application of this result to the construction of regular shearlet systems.

2. Shearlet Systems

Let us recall the basic notation and definitions related with shearlet systems (cf. [7, 13]).

2.1. **Continuous Shearlet Systems.**

We start with the continuous setting. For each $a > 0$ and $s \in \mathbb{R}$, let A_a denote the *parabolic scaling matrix*

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix},$$

and S_s denotes the *shear matrix*

$$S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Also, let $T_t f(x) = f(x-t)$, $t \in \mathbb{R}^2$ and $D_M f(x) = |\det M|^{-\frac{1}{2}} f(M^{-1}x)$, $M \in GL(2,\mathbb{R})$ denote the *translation* and *dilation operator* on $L^2(\mathbb{R}^2)$, respectively. Then the *(continuous) shearlet system* generated by $\psi \in L^2(\mathbb{R}^2)$ is defined by

$$\{\psi_{ast} = T_t D_{S_s A_a} \psi = a^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1}(\cdot\ - t)) : a \in \mathbb{R}^+,\ s \in \mathbb{R},\ t \in \mathbb{R}^2\}, \tag{1}$$

and the associated *Continuous Shearlet Transform* of some $f \in L^2(\mathbb{R}^2)$ is the mapping

$$\mathcal{SH}_\psi f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$$

given by

$$\mathcal{SH}_\psi f(a,s,t) = \langle f, \psi_{ast} \rangle. \tag{2}$$

In other word, the Continuous Shearlet Transform projects the function f onto the functions ψ_{ast}, at scale a, orientation s and location t. This definition is similar to the classical definition of the continuous wavelet transform, where the function f is projected onto the elements ψ_{at}, indexed by scale a and location t.

Similarly to continuous wavelets, a function $\psi \in L^2(\mathbb{R}^2)$ is called a *continuous shearlet* if it satisfies the *admissibility condition*

$$\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\nu_x, \nu_y)|^2}{\nu_x^2}\, d\nu_y\, d\nu_x < \infty. \tag{3}$$

In this case, one can show that each function $f \in L^2(\mathbb{R}^2)$ can be reconstructed from its shearlet coefficients $(\langle f, \psi_{ast}\rangle)_{ast}$. We refer to [7] and [13] for additional details about these properties. In [13] one can also find a construction of functions ψ satisfying the admissibility condition (3).

Another perspective into the study of shearlet systems is provided by the analysis of their group–theoretical properties. The associated locally compact group – the so-called *Shearlet group* $\mathbb{S}$ – is defined to be the set $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ endowed with multiplication given by

$$(a,s,t)\cdot(a',s',t') = (aa', s + s'\sqrt{a}, t + S_s A_a t').$$

Let $\sigma : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^2))$ be the unitary representation of this group on $L^2(\mathbb{R}^2)$, given by

$$\sigma(a,s,t)\,\psi(x) = a^{-\frac{3}{4}}\,\psi(A_a^{-1} S_s^{-1}(x-t)).$$

Then the elements of the shearlet system are obtained as:

$$\psi_{ast} = \sigma(a,s,t)\,\psi.$$

2.2. **Discrete Shearlet Systems.** In [10, 11], the *discrete shearlets* were introduced as the (discrete) systems of the form:

$$\{\psi_{jkm}(x) = |\det A_4|^{-j/2}\,\psi(S_1^{-k}\, A_4^{-j} x - m) :\ j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \tag{4}$$

where $\psi \in L^2(\mathbb{R}^2)$. It is shown that, provided ψ is appropriately chosen, the system (4) is a Parseval frame of $L^2(\mathbb{R}^2)$. As we will show in this paper, this is one example of a more general class of *regular shearlet systems*, whose precise definition will be given later on.

It is a simple calculation to show that the system (4) can be derived from (1) by sampling the continuous parameters a, s, t on a discrete set. To do this, we replace $a > 0$ by the sequence $a_j = 4^j$, $j \in \mathbb{Z}$. Next, for each $j \in \mathbb{Z}$, we replace $s \in \mathbb{R}$ by the discrete sequence $s_{jk} = k\,2^j$, $k \in \mathbb{Z}$. Notice that the shear parameter is allowed to change with the scale

a_j. Finally, for each $j,k \in \mathbb{Z}$, we sample the location parameter $t \in \mathbb{R}^2$ at the points $t_{jkm} = S_{k\,2^j} A_{4^j} m$, $m \in \mathbb{Z}^2$.

Observe that, for each $M \in GL(2,\mathbb{R})$ and $t \in \mathbb{R}^2$, $T_{Mt}\, D_M = D_M\, T_t$. Also notice that $S_{k2^j} A_{4^j} = A_{4^j} S_k = A_4^j S_1^k$. Using these observations, by evaluating the element ψ_{ast}, given by (1), on the discrete set

$$\{(a_j, s_{jk}, t_{jkm}) = (4^j, k2^j, S_{k\,2^j} A_{4^j} m) : \, j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\},$$

we have

$$\begin{aligned} \psi_{a_j s_{jk} t_{jkm}}(x) &= T_{S_{k2^j} A_{4^j} m}\, D_{S_{k2^j} A_{4^j}}\, \psi(x) \\ &= D_{S_{k2^j} A_{4^j}}\, T_m\, \psi(x) \\ &= D_{A_{4^j} S_k}\, T_m\, \psi(x). \end{aligned}$$

The last expression is exactly the element ψ_{jkm} in (4).

It is clear that the same sampling procedure can be described in more generality. For a function $\psi \in L^2(\mathbb{R}^2)$, set $a > 1$ and $b, c > 0$. Then the shearlet system derived by sampling the Continuous Shearlet Transform at the points

$$\Gamma = \{(a^j, bka^{\frac{j}{2}}, S_{bka^{j/2}} A_{a^j} cm) : j,k \in \mathbb{Z},\, m \in \mathbb{Z}^2\}$$

is called *regular (discrete) shearlet system* and is given by

$$\{T_{S_{bka^{j/2}} A_{a^j} cm} D_{S_{bka^{j/2}} A_{a^j}} \psi = D_{S_{bka^{j/2}} A_{a^j}} T_{cm} \psi : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

As shown above, the system (4) is an example for these systems. More generally, given an arbitrary discrete sequence $\Lambda \subseteq \mathbb{S}$, the associated *irregular shearlet system* is defined by

$$\mathcal{SH}(\psi, \Lambda) = \{\psi_{ast} : (a,s,t) \in \Lambda\}.$$

Clearly, this system coincides with the regular shearlet system when $\Lambda = \Gamma$.

Further, we wish to remark that, in the following, we will use the notation $\Lambda \subseteq \mathbb{S}$ to denote a countable sequence of points in $\mathbb{S}$ and not merely a subset. That is, we allow repetitions of points in Λ.

2.3. **Frames.** We briefly recall the definition and basic properties of frames in Hilbert spaces.

A sequence $\{f_i\}_{i \in I}$ of elements in a separable Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\, \|f\|^2 \le \sum_{i \in I} |\, \langle f, f_i \rangle \,|^2 \le B\, \|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. A frame is called *tight frame*, if A and B can be chosen as $A = B$; in the case $A = B = 1$, it is a *Parseval frame*.

Given a frame $\{f_i\}_{i\in I}$, the *frame operator* $Sf = \sum_{i \in I} \langle f, f_i \rangle\, f_i$ is a bounded, positive, and invertible mapping of $\mathcal{H}$ onto itself. The *canonical dual frame* is $\{\tilde{f}_i\}_{i \in I}$, where $\tilde{f}_i = S^{-1} f_i$. Then, for each $f \in \mathcal{H}$ we have the *frame expansions*

$$f = \sum_{i \in I} \langle f, f_i \rangle\, \tilde{f}_i = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i.$$

This equation shows that a frame provides a basis-like representation. In general, however, the elements of a frame need not be independent, and a frame or even a Parseval frame need not be a basis. We refer to [5] for additional information about frames.

3. Sufficient Conditions for Irregular Shearlet Frames

In the following, we will consider irregular shearlet systems $\mathcal{SH}(\psi,\Lambda)$ with $\psi \in L^2(\mathbb{R}^2)$ and with an associated sequence Λ in $\mathbb{S}$ of the form

$$\Lambda = \{(a_j, s_{jk}, S_{s_{jk}} A_{a_j} cm) : j,k \in \mathbb{Z},\, m \in \mathbb{Z}^2\},$$

where $a_j \in \mathbb{R}^+$ and $s_{jk} \in \mathbb{R}$ for $j,k \in \mathbb{Z}$, and $c > 0$. This choice induces a natural change of the dilation and translation operator leaving a system with still arbitrarily sampled parameters:

$$\mathcal{SH}(\psi,\Lambda) = \{T_{S_{s_{jk}} A_{a_j} cm} D_{S_{s_{jk}} A_{a_j}} \psi = D_{S_{s_{jk}} A_{a_j}} T_{cm} \psi : j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \tag{5}$$

where $\psi \in L^2(\mathbb{R}^2)$. Hence the only restrictive assumption is that the translation parameter is sampled over a lattice $c\mathbb{Z}^2$ of varying size $c > 0$.

Our aim now is to derive sufficient conditions on the sequence $\Lambda \subseteq \mathbb{S}$ and the function $\psi \in L^2(\mathbb{R}^2)$ for $\mathcal{SH}(\psi,\Lambda)$ to be a frame. In doing this, we will also prove estimates for the frame bounds of $\mathcal{SH}(\psi,\Lambda)$. The following result uses several ideas from [6, Prop. 3.3.2], adapted to the action of the shearlet group.

Theorem 3.1. *Let $c > 0$ be fixed and, for each $j,k \in \mathbb{Z}$, let $a_j \in \mathbb{R}^+$ and $s_{jk} \in \mathbb{R}$. Define $\Lambda \subseteq \mathbb{S}$ to be $\Lambda = \{(a_j, s_{jk}, S_{s_{jk}} A_{a_j} c\, m) : j,k \in \mathbb{Z},\, m \in \mathbb{Z}^2\}$. Further, let $\psi \in L^2(\mathbb{R}^2)$ and set*

$$\phi(\omega) = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^2} \sum_{j,k \in \mathbb{Z}} |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)||\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \omega)| \quad \text{for a.e. } \omega \in \mathbb{R}^2. \tag{6}$$

If there exist $0 < \alpha \le \beta < \infty$ such that

$$\alpha \le \sum_{j,k \in \mathbb{Z}} |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)|^2 \le \beta \quad \text{for a.e. } \xi \in \mathbb{R}^2$$

and

$$\sum_{n \in \mathbb{Z}^2, n \neq 0} \sqrt{\phi(\tfrac{1}{c} n)\, \phi(-\tfrac{1}{c} n)} =: \gamma < \alpha, \tag{7}$$

then $\mathcal{SH}(\psi,\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$ with frame bounds A, B satisfying

$$\frac{1}{c^2}(\alpha - \gamma) \le A \le B \le \frac{1}{c^2}(\alpha + \gamma).$$

Proof. Let $f \in L^2(\mathbb{R}^2)$. Using the Plancherel theorem, we obtain

$$\sum_{j,k,m} |\langle f, D_{S_{s_{jk}} A_{a_j}} T_{cm} \psi\rangle|^2 = \sum_{j,k,m} \Big| \int_{\mathbb{R}^2} \hat{f}(\xi) a_j^{\frac{3}{4}} e^{-2\pi i \langle S_{s_{jk}} A_{a_j} cm, \xi\rangle} \overline{\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)}\, d\xi \Big|^2.$$

For fixed $j,k \in \mathbb{Z}$, let F_{jk} denote a fundamental domain for the lattice $\frac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} \mathbb{Z}^2$ in $\mathbb{R}^2$, i.e., $\mathbb{R}^2 = \bigcup_{n \in \mathbb{Z}^2} (F_{jk} + \frac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n)$, where the union is disjoint. Since $\det(\frac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1}) = \frac{1}{c^2} a_j^{-3/2}$, it follows that $|F_{jk}| = \frac{1}{c^2} a_j^{-3/2}$. Using the change of variables $\xi \mapsto \xi + \frac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n$ and applying Plancherel's theorem once more yields

$$\begin{aligned}
&\sum_{j,k,m} |\langle f, D_{S_{s_{jk}} A_{a_j}} T_{cm} \psi \rangle|^2 \\
&= \sum_{j,k,m} a_j^{\frac{3}{2}} \Big| \sum_{n \in \mathbb{Z}^2} \int_{F_{jk}} \hat{f}(\xi + \tfrac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n)\, \overline{\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)} e^{-2\pi i \langle m, c A_{a_j} S_{s_{jk}}^T \xi \rangle}\, d\xi \Big|^2 \\
&= \frac{1}{c^2} \sum_{j,k} \int_{F_{jk}} \Big| \sum_{n \in \mathbb{Z}^2} \hat{f}(\xi + \tfrac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n)\, \overline{\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)} \Big|^2 d\xi \\
&= \frac{1}{c^2} \sum_{j,k,n} \int_{\mathbb{R}^2} \hat{f}(\xi)\, \overline{\hat{f}(\xi + \tfrac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n)}\, \overline{\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)}\, \hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)\, d\xi \\
&= \frac{1}{c^2} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j,k} |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)|^2 \\
&\quad + \frac{1}{c^2} \sum_{\substack{j,k,n \\ n \neq 0}} \int_{\mathbb{R}^2} \hat{f}(\xi)\, \overline{\hat{f}(\xi + \tfrac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n)}\, \overline{\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)}\, \hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)\, d\xi. \qquad (8)
\end{aligned}$$

We now focus on the second term in the last equality, which we denote by $R(f)$. To obtain an estimate for $R(f)$, we employ the Cauchy–Schwarz inequality twice, which gives

$$\begin{aligned}
|R(f)| &\leq \frac{1}{c^2} \sum_{\substack{j,k,n \\ n \neq 0}} \left[\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)| |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)|\, d\xi \right]^{\frac{1}{2}} \\
&\qquad \cdot \left[\int_{\mathbb{R}^2} |\hat{f}(\xi + \tfrac{1}{c} S_{s_{jk}}^{-T} A_{a_j}^{-1} n)|^2 |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)| |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)|\, d\xi \right]^{\frac{1}{2}} \\
&= \frac{1}{c^2} \sum_{\substack{j,k,n \\ n \neq 0}} \left[\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)| |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)|\, d\xi \right]^{\frac{1}{2}} \\
&\qquad \cdot \left[\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi - \tfrac{1}{c} n)| |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)|\, d\xi \right]^{\frac{1}{2}} \\
&\leq \frac{1}{c^2} \sum_{n \neq 0} \left[\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j,k} |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)| |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi + \tfrac{1}{c} n)|\, d\xi \right]^{\frac{1}{2}} \\
&\qquad \cdot \left[\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j,k} |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi)| |\hat{\psi}(A_{a_j} S_{s_{jk}}^T \xi - \tfrac{1}{c} n)|\, d\xi \right]^{\frac{1}{2}}.
\end{aligned}$$

Using this estimate in (8), we obtain:

$$\inf_{f\in L^2(\mathbb{R}^2), f\neq 0} \|f\|^{-2} \sum_{j,k,m} |\langle f, D_{S_{s_{jk}}A_{a_j}} T_{cm}\psi\rangle|^2$$
$$\geq \frac{1}{c^2}\left(\text{ess inf}_{\xi} \sum_{j,k} |\hat{\psi}(A_{a_j}S^T_{s_{jk}}\xi)|^2 - \sum_{n\neq 0}\big(\phi(\tfrac{1}{c}n)\phi(-\tfrac{1}{c}n)\big)^{\frac{1}{2}}\right)$$

and

$$\sup_{f\in L^2(\mathbb{R}^2), f\neq 0} \|f\|^{-2} \sum_{j,k,m} |\langle f, D_{S_{s_{jk}}A_{a_j}} T_{cm}\psi\rangle|^2$$
$$\leq \frac{1}{c^2}\left(\text{ess sup}_{\xi} \sum_{j,k} |\hat{\psi}(A_{a_j}S^T_{s_{jk}}\xi)|^2 + \sum_{n\neq 0}\big(\phi(\tfrac{1}{c}n)\phi(-\tfrac{1}{c}n)\big)^{\frac{1}{2}}\right).$$

This settles the claim. □

Next we will give a simple condition under which hypothesis (7) is satisfied. Namely, we we show that, if ψ is band-limited then the shearlet system $\mathcal{SH}(\psi,\Lambda)$, given by (5), forms a frame provided that the sampling constant c is chosen to be small enough. In the following, $B_\infty(x,r)$ denotes the closed ball centered at $x\in\mathbb{R}^2$ with radius $r>0$.

Corollary 3.2. *Let $c<\frac{1}{2r}$ and, for each $j,k\in\mathbb{Z}$, let $a_j\in\mathbb{R}^+$ and $s_{jk}\in\mathbb{R}$. Define $\Lambda\subseteq\mathbb{S}$ to be $\Lambda=\{(a_j,s_{jk},S_{s_{jk}}A_{a_j}cm) : j,k\in\mathbb{Z},\ m\in\mathbb{Z}^2\}$. Further, suppose that $\psi\in L^2(\mathbb{R}^2)$ satisfies $\operatorname{supp}\hat{\psi}\subset B_\infty(0,r)$ and there exist $0<\alpha\leq\beta<\infty$ such that*

$$\alpha\leq\sum_{j,k\in\mathbb{Z}}|\hat{\psi}(A_{a_j}S^T_{s_{jk}}\xi)|^2\leq\beta \quad \text{for a.e. } \xi\in\mathbb{R}^2.$$

Then the shearlet system $\mathcal{SH}(\psi,\Lambda)$ is a frame for $L^2(\mathbb{R}^2)$ with frame bounds A,B satisfying

$$\frac{1}{c^2}\alpha\leq A\leq B\leq\frac{1}{c^2}\beta.$$

In particular, if $\alpha=\beta$, then $\mathcal{SH}(\psi,\Lambda)$ is a tight frame for $L^2(\mathbb{R}^2)$ with frame bound $A=\frac{\alpha}{c^2}$.

Proof. First notice that $|\hat{\psi}(A_{a_j}S^T_{s_{jk}}\xi)|\neq 0$ if and only if $A_{a_j}S^T_{s_{jk}}\xi\in B_\infty(0,r)$, and that $|\hat{\psi}(A_{a_j}S^T_{s_{jk}}\xi+\omega)|\neq 0$ if and only if $A_{a_j}S^T_{s_{jk}}\xi\in B_\infty(-\omega,r)$. Thus, if $\omega\in\mathbb{R}^2$ satisfies $B_\infty(0,r)\cap B_\infty(-\omega,r)=\emptyset$, then $\phi(\omega)=0$ with ϕ being defined as in (6). In particular, this is satisfied for all ω such that $\|\omega\|_\infty>2r$. Hence,

$$\sum_{n\in\mathbb{Z}^2, n\neq 0}\sqrt{\phi(\tfrac{1}{c}n)\,\phi(-\tfrac{1}{c}n)}=0 \quad \text{for all } c<\frac{1}{2r}.$$

The proof now follows from Theorem 3.1. □

4. Examples

In this section, we will apply our results to the case of regular shearlet systems $\mathcal{SH}(\psi, \Lambda)$. The shearlet ψ is a discrete version of the continuous shearlets employed in [13] and is very similar to the one used in [11]. Recall that the shearlet system used in [11] provides optimally sparse representations for 2-D smooth functions with discontinuities along C^2-curves.

Set $a > 1$, $b, c > 0$, and let $\psi \in L^2(\mathbb{R}^2)$ be

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\tfrac{\xi_2}{\xi_1}), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where ψ_1, $\psi_2 \in L^2(\mathbb{R})$ are chosen such that

(i) ψ_1 satisfies

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}_1(a^j \omega)|^2 = 1 \quad \text{for a.e. } \omega \in \mathbb{R}$$

with $\hat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, and

(ii) ψ_2 satisfies

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}_2(bk + \omega)|^2 = 1 \quad \text{for a.e. } \omega \in \mathbb{R}$$

with $\hat{\psi}_2 \in C^\infty(\mathbb{R})$, and supp $\hat{\psi}_2 \subset [-1, 1]$.

The discrete lattice $\Lambda \subseteq \mathbb{S}$ is defined by

$$\Lambda = \{(a^j, bka^{\frac{j}{2}}, S_{bka^{j/2}} A_{a^j} cm) : j, k \in \mathbb{Z},\, m \in \mathbb{Z}^2\}.$$

We can now verify that the shearlet systems $\mathcal{SH}(\psi, \Lambda)$ satisfy the hypotheses of Corollary 3.2. In fact, ψ is band-limited with supp $\hat{\psi} \subset B_\infty(0, 2)$. In addition, using (i) and (ii), we obtain

$$\begin{aligned} \sum_{j,k \in \mathbb{Z}} |\hat{\psi}(A_{a_j} S^T_{s_{jk}} \xi)|^2 &= \sum_{j,k \in \mathbb{Z}} |\hat{\psi}_1(a^j \xi_1)\hat{\psi}_2(a^{-\frac{j}{2}}(bka^{\frac{j}{2}} + \tfrac{\xi_2}{\xi_1}))|^2 \\ &= \sum_{j \in \mathbb{Z}} |\hat{\psi}_1(a^j \xi_1)|^2 \sum_{k \in \mathbb{Z}} |\hat{\psi}_2(bk + a^{-\frac{j}{2}} \tfrac{\xi_2}{\xi_1})|^2 \\ &= \sum_{j \in \mathbb{Z}} |\hat{\psi}_1(a^j \xi_1)|^2 = 1. \end{aligned}$$

Thus Corollary 3.2 implies the following result.

Corollary 4.1. *For all $c < \frac{1}{4}$, the shearlet system $\mathcal{SH}(\psi, \Lambda)$ defined as above forms a tight frame for $L^2(\mathbb{R}^2)$ with frame bound $A = \frac{1}{c^2}$.*

Hence, for these values of c, the estimates for the frame bounds of the shearlet frame $\mathcal{SH}(\psi, \Lambda)$ are as best as possible.

Next, let us examine the case $c \geq \frac{1}{4}$. We intend to apply Theorem 3.1 to verify that $\mathcal{SH}(\psi, \Lambda)$ forms a frame and will derive estimates for the frame bounds. For this, the exact value of γ needs to be computed. It can easily be seen that we always have $\gamma > 0$. Hence Theorem 3.1 implies that, provided $\gamma < 1$, the shearlet system $\mathcal{SH}(\psi, \Lambda)$ forms a frame for

$L^2(\mathbb{R}^2)$ with the lower frame bound estimate $\frac{1}{c^2}(\alpha-\gamma)$ and upper frame bound estimate $\frac{1}{c^2}(\alpha+\gamma)$.

It is interesting to compare this result with [10, Thm. 2], where it is proven that, for $a=2$, $b=1$, and $c=1$, the system $\mathcal{SH}(\psi,\Lambda)$ is a tight frame for $L^2(\mathbb{R}^2)$. This shows that the estimates for the frame bounds are not sharp in this case. This suggests that there is room for improvements in Theorem 3.1 and Corollary 3.2.

Another interesting issue is whether the hypothesis supp $\hat{\psi} \subset B_\infty(0,r)$ in Corollary 3.2 can be substituted by a certain decay condition on $\hat{\psi}$. The same issue is discussed in [6, Sec. 3.3.2] for wavelet systems.

5. Conclusion

The results and methods developed in this paper set the foundation for the study of a number of questions related to the construction and application of directional multiscale systems, including the following.

- *Study of stability of discrete shearlet systems*: By providing a sufficient condition on the generator and on the set of indices to form a shearlet frame, one can study the robustness of discrete shearlet systems with respect to perturbations of the scale-shear-location indices.
- *Sampling of the Continuous Shearlet Transform*: By sampling the Continuous Shearlet Transform, a variety of discrete shearlet systems is obtained. It is natural to ask how to design these systems so that they inherit the appropriate mathematical properties from the corresponding continuous system.

The study of these issues will be the focus of future investigation.

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