Advances in Computational Mathematics manuscript No.
(will be inserted by the editor)

Smooth projections and the construction of smooth Parseval frames of shearlets

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Received: date / Accepted: date

Abstract Smooth orthogonal projections with good localization properties were originally studied in the wavelet literature as a way to both understand and generalize the construction of smooth wavelet bases on $L^2(\mathbb{R})$. Smoothness plays a critical role in the construction of wavelet bases and their generalizations as it is instrumental to achieve excellent approximation properties. In this paper, we extend the construction of smooth orthogonal projections to higher dimensions, a challenging problem in general for which relatively few results are found in the literature. Our investigation is motivated by the study of multidimensional nonseparable multiscale systems such as shearlets. Using our new class of smooth orthogonal projections, we construct new smooth Parseval frames of shearlets in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$.

Keywords Smooth projections · Parseval frames · Shearlets · Frames · Wavelets · Sparse Representations

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1 Introduction

Smooth orthogonal projections on the real line have played an important role in the wavelet literature due to their connection with the construction of well localized wavelet bases. Wavelets became very popular in the 90's because of their remarkable ability to approximate piecewise regular functions on the real line [7,8]. To achieve such ability, the localization properties of the basis are critical, that is, the basis elements are required to have compact support and rapid decay in the Fourier domain, or rapid decay and compact support in the Fourier domain, or rapid decay in both domains [7]. When one of the three situations happen, the basis is called 'well-localized'.

A very important example of such bases are the local Fourier bases of Coifman and Meyer [6]. The main idea of their construction is to start from a complex exponential basis on a finite interval and to generate a well localized basis of $L^2(\mathbb{R})$ by piecing together dilated copies of the same basis. The challenge here is about piecing together the different sub-systems without losing the regularity of the overall system. Clearly, one cannot simply put together bases after multiplication by the characteristic function of an appropriate interval since this would introduce discontinuities. Instead one has to define smooth localization windows. One solution to this problem is to construct orthogonal projections P_k, $k \in \mathbb{Z}$ with the property that (i) $L^2(\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} P_k L^2(\mathbb{R})$ and (ii) the orthogonal projections P_k are smooth in the sense that each $P_k f$ preserves the regularity of f.

A detailed study of the construction of smooth orthogonal projections on the real line was carried out by Auscher, Weiss and Wickerhauser [1] (see also [15]) by generalizing results by Coifman and Meyer in [6]. The extension of this approach to higher dimensions though is highly nontrivial and only a few results are available in the literature. We recall the extension of smooth projections to manifolds in [3,4] including an interesting result about localized Parseval frames on the sphere.

The main goal of this paper is to extend the construction of smooth orthogonal projections to higher dimensions. For this, we introduce a novel approach based on cyclic permutation operators on $\mathbb{R}^N$ to define smooth orthogonal projections associate with certain partitions of $\mathbb{R}^N$. The construction presented here shows that the complex periodization with quarter-rotations used in the case of $\mathbb{R}^2$ [2] can be replaced by an explicitly real symmetrization procedure. In addition, this paper provides insights into the treatment of $\mathbb{R}^N$ with $N \geq 3$.

Our study is motivated in part by applications in the theory of shearlets [19,22], a multidimensional extension of wavelets that has received a significant interest in harmonic analysis during the last decade. Similar to curvelets [5] and other systems of parabolic molecules [17], shearlets owe their success to their ability to provide nearly optimally sparse approximations for a large class of multivariate functions, outperforming conventional multiscale methods [10, 16,21,27]. They form a collection of well-localized waveforms defined not only over a range of scales and locations, like wavelets, but also over multiple orientations so that they are more apt at representing edges and other anisotropic features. While curvelets use rotations, the directionality of shearlets is controlled by the action of a shear matrix. One advantage of using shearing is that – unlike rotations – this operation is integer lattice invariant and this invariance property can be very beneficial in discrete application. However, while the group of rotations is compact, this property is not true for the shear group. As a result, a shearlet system on $L^2(\mathbb{R}^N)$ is typically constructed by combining N cone-based shearlet systems, each one defined by truncating the shear parameter on a compact interval. To ensure that the system obtained by the union of the cone-based shearlet systems is a Parseval frame of well-localized waveforms, special handling is required. For instance, the widely used approach originally proposed in [13] requires to construct a set of 'boundary shearlets' that are specially defined to ensure that the combination of cone-based shearlets is still smooth.

Using the new class of smooth orthogonal projections introduced in this paper, we define a novel approach to construct smooth Parseval frames of cone-bases shearlets on $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$. This approach, that consists in taking the union of different shearlet systems that have been smoothly projected into appropriate subspaces, is not only simpler and more straightforward than the classical construction involving boundary shearlets but also more flexible.

The rest of the paper is organized as follows. Below we set the notation and state basic definitions that will be used in the paper. In Section 2 we present an approach to define smooth orthogonal projections on $L^2(\mathbb{R}^2)$ and a general approach to construct smooth orthogonal projections on $L^2(\mathbb{R}^N)$. In Section 3 we apply our method for a new construction of smooth Parseval frames of shearlets of $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^3)$.

1.1 Notation and background

We recall some definitions and set some notation that will be used in the rest of the paper.

We adopt the convention that $x \in \mathbb{R}^N$ is a column vector, i.e., $x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$, and that $\xi \in \widehat{\mathbb{R}}^N$ is a row vector, i.e., $\xi = (\xi_1, \ldots, \xi_N)$. A vector x multiplying a matrix $M \in GL_N(\mathbb{R})$ on the right is understood to be a column vector, while a vector ξ multiplying M on the left is a row vector. Thus, $Mx \in \mathbb{R}^N$ and $\xi M \in \widehat{\mathbb{R}}^N$.

Let $f \in L^2(\mathbb{R}^N)$. For $y \in \mathbb{R}^N$, the *translation operator* T_y is defined by $T_y f(x) = f(x - y)$; for $M \in GL_N(\mathbb{R})$, the *dilation operator* D_M is defined by $D_M f(x) = |\det M|^{-1/2} f(M^{-1}x)$; for $\nu \in \widehat{\mathbb{R}}^N$.

1.1.1 Fourier transform

The Fourier transform is a map $f \mapsto \hat{f}$ that, for $f \in L^1(\mathbb{R}^N)$ and any $\xi \in \widehat{\mathbb{R}}^N$, is defines as

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)\, e^{-2\pi i \xi x}\, dx.$$

It is well-known that this operator has a unique extension to $L^2(\mathbb{R}^N)$ that is a unitary operator. The inverse Fourier transform is the map $f \mapsto \check{f}$ defined as

$$\check{f}(x) = \int_{\widehat{\mathbb{R}}^N} f(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

By the Plancherel theorem, the Fourier transform is an isometry on $L^2(\mathbb{R}^N)$, that is,

$$\int_{\mathbb{R}^N} |f(x)|^2\, dx = \int_{\widehat{\mathbb{R}}^N} |\hat{f}(\xi)|^2 d\,\xi.$$

We recall the following useful properties:

$$(T_y\, f)^\wedge(\xi) = (M_y\, \hat{f})(\xi)$$
$$(D_M f)^\wedge(\xi) = (\widehat{D}_M \hat{f})(\xi) \equiv |\det M|^{1/2}\, \hat{f}(\xi M).$$

We will consider subspaces of $L^2(\mathbb{R}^N)$ associated with functions whose essential Fourier support is contained in a region $S \subset \mathbb{R}^N$ and use the notation:

$$L^2(S)^\vee = \{f \in L^2(\mathbb{R}^N) :\, \operatorname{ess\,supp} \hat{f} \subset S \subset \mathbb{R}^N\}.$$

1.1.2 Frames

Let $\mathcal{H}$ be a Hilbert with inner product $\langle\, ,\, \rangle$. A frame is a generalization of a basis where the frame elements are not necessarily linearly independent yet an approximate reproducing property still holds. A set $\{\gamma_k\}_{k\in I}$ in $\mathcal{H}$, where I is a countable indexing set, is a frame for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{k\in I} |\langle f, \gamma_k\rangle|^2 \le B\|f\|^2.$$

A is called the lower frame bound and B the upper frame bound of the frame. If $A = B$ then the frame is called a tight frame; if, in addition, $A = B = 1$ then it is called 1-tight frame or Parseval frame. If γ_k is a Parseval frame then, for any $f \in \mathcal{H}$, we have

$$\|f\|^2 = \sum_{k\in I} |\langle f, \gamma_k\rangle|^2.$$

One property of frames that will be critical in this paper is that the frame condition is preserved under the action of an orthogonal projection. Recall that an operator P on a Hilbert space $\mathcal{H}$ is a projection if $P = P^2$ and it is an orthogonal projection if, in addition, it is self-adjoint, i.e., $P = P^*$. We have the following known result whose proof is reported for completion.

ro:proj_parseval⟩ **Proposition 1** *Let $\{\gamma_k : k \in I\}$ be a frame of $\mathcal{H}$ with frame bounds $0 < A \leq B < \infty$ and P be an orthogonal projection mapping $\mathcal{H}$ onto the subspace $V \subset \mathcal{H}$, then V is closed and $\{P\gamma_k :\in I\}$ is a frame for $V \subset \mathcal{H}$ with the same frame bounds A, B.*

Proof. Any $g \in V$ can be written as $g = Pf$ where $f \in \mathcal{H}$. Hence, using the frame property of $\{\gamma_k : k \in I\}$, we have that for any $g \in V$

$$A\,\|g\|^2 = A\,\|Pf\|^2 \leq \sum_{k\in I} |\langle Pf, \gamma_k\rangle|^2 \leq B\,\|Pf\|^2 = B\,\|g\|^2$$

Since P is an orthogonal projections, for any $g \in V$, we have that $\langle g, P\gamma_k\rangle = \langle Pf, P\gamma_k\rangle = \langle Pf, \gamma_k\rangle$. Hence, for any $g \in V$,

$$A\,\|g\|^2 \leq \sum_{k\in I} |\langle g, P\gamma_k\rangle|^2 \leq B\,\|g\|^2.$$

The closedness of V follows since it is the image of an orthogonal projection. □

2 Smooth projections on $L^2(\mathbb{R}^N)$

⟨sec:1-2⟩ As indicated above, a method for constructing smooth projections on $L^2(\mathbb{R})$ was originally discussed in [15] with the goal to build localized wavelet bases. We start by recalling the basic ideas in the uni-variate setting and, specifically, the following result from [15, Section 1.3].

⟨pro.HW⟩ **Proposition 2** *Let $s \in C^\infty(\mathbb{R})$ be a function satisfying*

$$|s(x)|^2 + |s(-x)|^2 = 1$$

Then the operator $P_s^\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$P_s^\pm f(x) = \overline{s(x)}\,(s(x)f(x) \pm s(-x)f(-x))$$

is an orthogonal projection, that is, $P_s^\pm$ is idempotent and self-adjoint.

An example of a real valued function satisfying Proposition 2 is $s(x) = \sin(\theta(x))$ where $\theta(x) = \int_{-\infty}^{x} \phi(t)\,dt$ and $\phi \in C^\infty(\mathbb{R})$ is an even function with $\mathrm{supp}\,(\phi) \subset [-\epsilon, \epsilon]$ and $\int_{-\epsilon}^{\epsilon} \phi(t)\,dt = \frac{\pi}{2}$. Another example is $c(x) = \cos(\theta(x))$. These functions are illustrated in Fig. 1 where the plot shows that s is supported on $[-\epsilon, \infty]$ and c is supported on $[-\infty, \epsilon]$.

For $s(x) = \sin(\theta(x))$ the operator P_s^+ is an orthogonal projection onto the subspace $H^+ = P_s^+ L^2(\mathbb{R})$ and for $c(x) = \cos(\theta(x))$ the operator P_c^- is an orthogonal projection onto the subspace $H^- = P_c^- L^2(\mathbb{R})$. Note that if $f \in H^+$ then $f = \overline{s}g$ where $g \in L^2(\mathbb{R})$. Similarly, if $f \in H^-$ then $f = \overline{c}g$ where $g \in L^2(\mathbb{R})$. In particular, if $f \in H^+$ then $\mathrm{supp}\,(f) \subset [-\epsilon, \infty)$ and if $f \in H^-$ then $\mathrm{supp}\,(f) \subset (-\infty, \epsilon]$. Since ϵ can be chosen to be arbitrarily small, we say

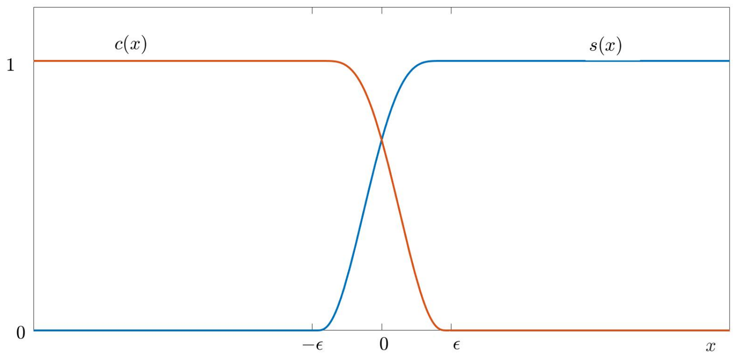

Fig. 1 Graphs of smooth window functions s and c.
⟨2ds1⟩

that the operator P_s^+ is an orthogonal projection *associated with the positive reals* and P_c^- is an orthogonal projection *associated with the negative reals.*

Since $\theta(x) + \theta(-x) = \pi/2$, then it is easy to verify that $s(x) = c(-x)$. It follows from this property that, for any $f \in L^2(\mathbb{R})$, we have

$$P_s^+ f + P_c^- f = f.$$

Note that, due to the definition of s and c, the projections P_s^+ and P_c^- preserve the regularity of f. That is, if $f \in C^m(\mathbb{R})$ for some $m \in \mathbb{N}$, then $P_s f$, $P_c f$ are also in $C^m(\mathbb{R})$. Therefore, P_s and P_c are smooth orthogonal projections associated with the positive and negative reals, respectively.

This idea is further elaborated in [15] to build smooth projections onto more general intervals of $\mathbb{R}$.

2.1 Smooth projection in $L^2(\mathbb{R}^2)$

⟨sec:1⟩ In this section, we show how to adapt the one-dimensional construction above to build smooth orthogonal projections of $L^2(\mathbb{R}^2)$ *associated with* half-planes and cone-shaped domains in $\mathbb{R}^2$ (we use the wording ‘associated with’ in the sense defined above).

Similar to Proposition 2, for $f \in L^2(\mathbb{R}^2)$ we define

$$P_{s,x}^{\pm} f(x,y) = \overline{s(x)}(s(x)f(x,y) \pm s(-x)f(-x,y)), \tag{1}$$ def.psx2d

where $s \in C^\infty(\mathbb{R})$ is given as in Proposition 2. We start with the following observation.

⟨lem.lsx⟩ **Lemma 1** *Let s be as in Proposition 2. Then $P_{s,x}^{\pm}$, given by* (1)*, is an orthogonal projection on $L^2(\mathbb{R}^2)$.*

Proof. The proof is similar to Proposition 2 which is given in [15]. □

Similar to the observations made in the one-dimensional case, if $s(x) = \sin(\theta(x))$ the operator $P^+_{s,x}$ is an orthogonal projection onto the subspace $H^+ = P^+_{s,x}L^2(\mathbb{R}^2)$ and for $c(x) = \cos(\theta(x))$ the operator $P^-_{c,x}$ is an orthogonal projection onto the subspace $H^- = P^-_{c,x}L^2(\mathbb{R}^2)$. We have that $f \in H^+$ implies $f = \bar{s}g$ where $g \in L^2(\mathbb{R}^2)$ and $f \in H^-$ implies $f = \bar{c}g$ where $g \in L^2(\mathbb{R}^2)$; also, if $f \in H^+$ then $\operatorname{supp}(f) \subset [-\epsilon, \infty) \times \mathbb{R}$ and if $f \in H^-$ then $\operatorname{supp}(f) \subset (-\infty, \epsilon] \times \mathbb{R}$. Hence, similar to the one-dimensional case, we say that the operator $P^+_{s,x}$ is an orthogonal projection *associated with the half-plane* $x > 0$ and $P^-_{c,x}$ is an orthogonal projection *associated with the half-plane* $x < 0$. In addition, the operators $P^+_{s,x}$ and $P^-_{c,x}$ preserve the regularity of the function on which are they are acting in the sense described above. That is, they are smooth projection operators.

The next Lemma shows that $P^+_{s,x}$ and $P^-_{c,x}$ are orthogonal and complementary.

⟨comp.proj⟩ **Lemma 2** *Let $s(x) = \sin(\theta(x))$ and $c(x) = s(-x)$ where θ is defined as in Proposition 2. Then $P^+_{s,x}$ and $P^-_{c,x}$ are mutually orthogonal and complementary on $L^2(\mathbb{R}^2)$ in the sense that, for any $f \in L^2(\mathbb{R}^2)$, we have $P^+_{s,x}f + P^-_{c,x}f = f$.*

Proof. Since $P^+_{s,x}$ is an orthogonal projection, so is $I - P^+_{s,x}$, and $P^+_{s,x}(I - P^+_{s,x}) = 0$.

We claim that $P^-_{c,x} = I - P^+_{s,x}$. In fact, since $s^2(x) + s^2(-x) = 1$, we have:

$$\begin{aligned}
f(x,y) - P^+_{s,x}f(x,y) &= f(x,y) - s(x)[s(x)f(x,y) + s(-x)f(-x,y)] \\
&= f(x,y)[1 - s^2(x)] - s(x)s(-x)f(-x,y) \\
&= f(x,y)[s^2(-x)] - c(-x)c(x)f(-x,y) \\
&= f(x,y)[c^2(x)] - c(-x)c(x)f(-x,y) \\
&= c(x)[c(x)f(x,y) - c(-x)f(-x,y)] \\
&= P^-_{c,x}f(x,y). \quad \square
\end{aligned}$$

The following result is a direct consequence of Lemmata 1 and 2.

⟨cor.comp.proj⟩ **Corollary 1** *Let $s(x) = \sin(\theta(x))$ and $c(x) = s(-x)$ where θ is defined as in Proposition 2. Then $P^+_{s,y}$ and $P^-_{c,y}$, defined as*

$$P^+_{s,y}f(x,y) = \overline{s(y)}[s(y)f(x,y) + s(-y)f(x,-y)]$$

and

$$P^-_{c,y}f(x,y) = \overline{c(y)}[c(y)f(x,y) - c(-y)f(x,-y)],$$

are mutually orthogonal. In addition, for any $f \in L^2(\mathbb{R}^2)$,

$$P^+_{s,y}f + P^-_{c,y}f = f.$$

Similar to the projection operators $P^+_{s,x}$ and $P^-_{c,x}$, also the operators $P^+_{s,y}$ and $P^-_{c,y}$ are smooth projections associated with the half-planes $y > 0$ and $y < 0$, respectively.

We next show how to define smooth projection operators associated cone-spaced domains. We define the regions

$$
\begin{aligned}
Q_1 &= \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } |x| \geq |y|\} \\
Q_2 &= \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } |x| \geq |y|\} \\
Q_3 &= \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |x| \leq |y|\} \\
Q_4 &= \{(x, y) \in \mathbb{R}^2 : y \leq 0 \text{ and } |x| \leq |y|\} \\
Q_h &= Q_1 \cup Q_2 \text{ and } Q_v = Q_3 \cup Q_4.
\end{aligned}
$$

The motivation for choosing such regions comes from the theory of shearlets where, as discussed in Sec. 3 below, it is useful to partition $\mathbb{R}^2$ into the 'horizontal' and 'vertical' cones Q_h and Q_v respectively.

We start by composing out projections operators to generate smooth projections associated with the orthogonal quadrants.

Proposition 3 *Let $s, P^+_{s,x}, P^-_{c,x}$ be as in Lemma 2 and $P^+_{s,y}$ and $P^-_{s,y}$ as in corollary 1. Then $P^+_{s,x}P^+_{s,y}$, $P^+_{s,x}P^-_{c,y}$, $P^-_{c,x}P^+_{s,y}$,and $P^-_{c,x}P^-_{s,y}$ are projections and*

$$P^+_{s,x}P^+_{s,y}f + P^+_{s,x}P^-_{s,y}f + P^-_{c,x}P^+_{s,y}f + P^-_{c,x}P^-_{s,y}f = f$$

Proof. It follows directly from the definition that $P^+_{s,x}$, $P^+_{s,y}$, $P^-_{c,x}$ and $P^-_{s,y}$ commute with each other. Since $P^+_{s,x}$ and $P^+_{s,y}$ are projections,

$$
\begin{aligned}
(P^+_{s,x}P^+_{s,y})^2 &= (P^+_{s,x}P^+_{s,y})(P^+_{s,x}P^+_{s,y}) = P^+_{s,x}P^+_{s,x}P^+_{s,y}P^+_{s,y} \\
&= (P^+_{s,x})^2(P^+_{s,y})^2 = P^+_{s,x}P^+_{s,y}
\end{aligned}
$$

and

$$(P^+_{s,x}P^+_{s,y})^* = (P^+_{s,y})^*(P^+_{s,x})^* = P^+_{s,x}P^+_{s,x} = P^+_{s,x}P^+_{s,y}$$

Hence, $P^+_{s,x}P^+_{s,y}$ is an orthogonal projection. An identical argument shows that also the operators $P^+_{s,x}P^-_{s,y}$, $P^-_{c,x}P^+_{s,y}$ and $P^-_{c,x}P^-_{s,y}$ are orthogonal projections. Finally using Lemma 2 and Corollary 1 we conclude that

$$
\begin{aligned}
&P^+_{s,x}P^+_{s,y}f + P^+_{s,x}P^-_{s,y}f + P^-_{c,x}P^+_{s,y}f + P^-_{c,x}P^-_{s,y}f \\
&= P^+_{s,x}(P^+_{s,y}f + P^-_{s,y}f) + P^-_{c,x}(P^+_{s,y}f + P^-_{s,y}f) \\
&= P^+_{s,x}f + P^-_{c,x}f \\
&= f. \quad \square
\end{aligned}
$$

With appropriate rotations, we can now define smooth orthogonal projections *associated with* the cones $Q_i, i = 1, \ldots, 4$. Let $R_{\frac{\pi}{4}}$ be the rotation by the angle of $\frac{\pi}{4}$, where the rotation operator in two dimensions is given by $R_\theta f(x_1, x_2) = f(x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$. Then, we obtain:

1. $R_{-\frac{\pi}{4}}P^+_{s,x}P^+_{s,y}R_{\frac{\pi}{4}}$, associated with the cone Q_1;
2. $R_{-\frac{\pi}{4}}P^-_{c,x}P^-_{c,y}R_{\frac{\pi}{4}}$, associated with the cone Q_2;
3. $R_{-\frac{\pi}{4}}P^-_{c,x}P^+_{s,y}R_{\frac{\pi}{4}}$, associated with the cone Q_3;

4. $R_{-\frac{\pi}{4}} P^{+}_{s,x} P^{-}_{c,y} R_{\frac{\pi}{4}}$, associated with the cone Q_4.

Using the above four projections, we obtain a smooth projection P_1 *associated with the horizontal cone* Q_h, as

$$P_1 = R_{-\frac{\pi}{4}} (P^{+}_{s,x} P^{+}_{s,y} + P^{-}_{c,x} P^{-}_{c,y}) R_{\frac{\pi}{4}} f, \qquad (2)$$ eq:P1_1

and a smooth projection P_2 *associated with the horizontal cone* Q_v, as

$$P_2 = R_{-\frac{\pi}{4}} (P^{-}_{c,x} P^{+}_{s,y} + P^{+}_{s,x} P^{-}_{c,y}) R_{\frac{\pi}{4}} f. \qquad (3)$$ eq:P2_1

By construction, we have $P_2 f + P_1 f = f$.

We remark that the smooth orthogonal projections P_1 and P_2 are *associated with the cones* Q_h and Q_v in the sense defined above but they are not projections onto subspaces of L^2 with support on Q_h and Q_v. Namely, by taking ϕ in Proposition 2 to have arbitrarily small ϵ support, the smooth function s approximates the characteristic function of $\mathbb{R}^+$ within ϵ. Hence, for any function f with support in $Q^{\epsilon}_h = \{(x,y) \in \mathbb{R}^2 : |y| < |x| - \epsilon\}$ we have that $P_1 f = f$; similarly for any for any function f with support in $Q^{\epsilon}_v = \{(x,y) \in \mathbb{R}^2 : |x| < |y| - \epsilon\}$ we have that $P_2 f = f$. Also $P_1 f(x,y) = 0$ if $||y| > |x| + \epsilon$ and $P_2 f(x,y) = 0$ if $||x| > |y| + \epsilon$.

2.2 Projections on $L^2(\mathbb{R}^N)$

⟨sec:2⟩ We present a new general procedure to generate orthogonal projections into subspaces of $L^2(\mathbb{R}^N)$ associated with partitions of the domain $\mathbb{R}^N$. We will use this procedure to build projections in $L^2(\mathbb{R}^N)$ and an alternative method to build smooth projections associated with cone-shaped domains on $\mathbb{R}^2$.

Let $N \in \mathbb{N}$ and T be the cyclic coordinate permutation operator on $\mathbb{R}^N$. Let $x \in \mathbb{R}^N$, $s_1 \in C^\infty(\mathbb{R}^N)$ and $s_j = s_1(T^{j-1}(\cdot))$ for $j = 1, 2, ...N$ such that

$$\sum_{j=1}^{N} |s_j|^2 = 1.$$

For $f \in L^2(\mathbb{R}^N)$, define V_j as

$$V_j f = \frac{1}{\sqrt{N}} \overline{s_j} \sum_{\tau=0}^{N-1} w^{\tau j} f(T^\tau(\cdot)),$$

where $w = e^{\frac{2\pi i}{N}}$. We then have the following result.

⟨thm1⟩ **Theorem 1** *For V_j, T, and s_j defined as above, the operator P_j given by*

$$P_j f(x) = V_j V_j^* f(x) = \overline{s_j} \sum_{\tau=0}^{N-1} \overline{w^{\tau j}} s_j(T^\tau(x)) f(T^\tau(x))$$

is an orthogonal projection on $L^2(\mathbb{R}^N)$*. If for each* $\tau \in \{0,1,2,\dots,N-1\}$*,*

$$\sum_{j=1}^{N} w^{\tau j}\overline{s_j}(x)s_{j+\tau}(x) = \delta_{\tau,0}\,, \tag{4}$$ eq.sjsi

we also have that, for any $f \in L^2(\mathbb{R}^N)$*,* $f = \sum_{j=1}^N P_j f$*.*

Remark 1 There are simple solutions to (4). Clearly, the constant functions $s_j = w\frac{1}{\sqrt{N}}$, for every j, and $s_j = \frac{1}{\sqrt{N}}$, for every j, satisfy equation (4). A non-smooth solution can be obtained by using indicator functions. If U_j are disjoint subsets of $\mathbb{R}^N$ such that $\chi_{U_{j+1}} = \chi_{U_j}(T(\cdot))$ with $\bigcup_{\tau=}^N U_\tau = R^N$, then $s_j = \chi_{U_j}$ is a solution of (4).

We also remark that there is a relationship between the solutions of (4) and the solution of the filter equations associated with M-band wavelets [24]. The exploration of this connection is beyond the scope of this paper and will be investigated in a separate work.

Before proving Theorem 1, we calculate V_j^*. Applying the change of variable $T^\tau x = y$ on the third line below and by the change of index $k = N-\tau$ we get:

$$\begin{aligned}
\langle V_j^* f, g\rangle &= \langle f, V_j g\rangle \\
&= \int_{\mathbb{R}^N} f(x)\,\overline{\frac{1}{\sqrt{N}}\overline{s_j(x)}\sum_{\tau=0}^{N-1} w^{\tau j}\, g(T^\tau(x))}\,dx \\
&= \frac{1}{\sqrt{N}}\sum_{\tau=0}^{N-1}\int_{\mathbb{R}^N} \overline{w^{\tau j}}\, s_j(x)\,\overline{g(T^\tau(x))}\, f(x)\,dx \\
&= \frac{1}{\sqrt{N}}\sum_{\tau=0}^{N-1}\int_{\mathbb{R}^N} w^{(N-\tau)j} |\det J_\tau|\, s_j(T^{N-\tau}(y))\, f(T^{N-\tau}(y))\,\overline{g(y)}\,dy \\
&= \int_{\mathbb{R}^N} \big(\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} w^{kj}\, s_j(T^k(y))\, f(T^k(y))\big)\,\overline{g(y)}\,dy.
\end{aligned}$$

Thus $V_j^* f = \frac{1}{\sqrt{N}}\sum_{\tau=0}^{N-1} w^{\tau j}\, s_j(T^\tau(\cdot))f(T^\tau(\cdot))$.

To prove Theorem 1, we use the fact that $P_j = V_j V_j^*$ is a projection if $(V_j^* V_j)V_j^* = V_j^*$. In fact, if $(V_j^* V_j)V_j^* = V_j^*$, then

$$P_j^2 = V_j V_j^* V_j V_j^* = V_j (V_j^* V_j) V_j^* = V_j V_j^*$$

and the self adjointness of $P_j = V_j V_j^*$ is immediate from the definition.

Proof of Theorem 1.

Clearly $P_j^* = P_j$. To prove that P_j is a projection, we will show that $V_j^* V_j V_j^* = V_j^*$. In the calculation below we make an explicit use of the fact that w and the operator T are cyclic of order N. Using the expression of $V_j^* f$ from above and letting $m = k + \tau$ on the third line below , we have

$$
\begin{aligned}
V_j V_j^* f &= V_j\Big(\frac{1}{\sqrt{N}} \sum_{\tau=0}^{N-1} w^{\tau j}\, s_j(T^\tau(\cdot))\, f(T^\tau(\cdot)) \Big) \\
&= \frac{1}{N} \overline{s_j} \sum_{k=0}^{N-1} \Big(w^{kj} \sum_{\tau=0}^{N-1} w^{\tau j}\, s_j(T^k T^\tau(\cdot))\, f(T^k T^\tau(\cdot)) \Big) \\
&= \frac{1}{N} \overline{s_j} \sum_{k=0}^{N-1} \sum_{\tau=0}^{N-1} \Big(w^{(k+\tau)j} s_j(T^{k+\tau}(\cdot)) f(T^{k+\tau}(\cdot)) \Big) \\
&= \frac{1}{N} \overline{s_j} \sum_{k=0}^{N-1} \sum_{m=k}^{N+k-1} \Big(w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big) \\
&= \frac{1}{N} \overline{s_j} \Bigg(\sum_{k=0}^{N-1} \sum_{m=k}^{N-1} \Big(w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big) \\
&+ \sum_{k=0}^{N-1} \sum_{m=N}^{N+k-1} \Big(w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big) \Bigg) \\
&= \frac{1}{N} \overline{s_j} \Bigg(\sum_{k=0}^{N-1} \sum_{m=k}^{N-1} \Big(w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big) \\
&+ \sum_{k=0}^{N-1} \sum_{m=0}^{k-1} \Big(w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big) \Bigg) \\
&= \frac{1}{N} \overline{s_j} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \Big(w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big) \\
&= \frac{1}{N} \overline{s_j}\, N \sum_{m=0}^{N-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \\
&= \overline{s_j} \sum_{m=0}^{N-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& V_j^*(V_j V_j^* f) \\
&= \frac{1}{\sqrt{N}} \sum_{\tau=0}^{N-1} \Big(w^{\tau j} s_j(T^\tau(x)) \Big(\overline{s_j}(T^\tau(\cdot)) \sum_{k=0}^{N-1} w^{kj} s_j(T^\tau T^k(\cdot)) f(T^\tau T^k(\cdot)) \Big) \Big) \\
&= \frac{1}{\sqrt{N}} \sum_{\tau=0}^{N-1} \Big(|s_j(T^\tau(\cdot))|^2 \sum_{k=0}^{N-1} w^{(k+\tau)j} s_j(T^{k+\tau}(\cdot)) f(T^{k+\tau}(\cdot)) \Big) \\
&= \frac{1}{\sqrt{N}} \sum_{\tau=0}^{N-1} \Big(|s_j(T^\tau(\cdot))|^2 \sum_{m=i}^{N+\tau-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Big)
\end{aligned}
$$

$$= \frac{1}{\sqrt{N}} \sum_{\tau=0}^{N-1} \Bigg(|s_j(T^\tau(\cdot))|^2 \Bigg(\sum_{m=\tau}^{N-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot))$$

$$+ \sum_{m=N}^{N+\tau-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Bigg) \Bigg)$$

$$= \frac{1}{\sqrt{N}} \sum_{\tau=0}^{N-1} \Bigg(|s_j(T^\tau(\cdot))|^2 \Bigg(\sum_{m=\tau}^{N-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot))$$

$$+ \sum_{m=0}^{\tau-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Bigg) \Bigg)$$

$$= \frac{1}{\sqrt{N}} \Bigg(\sum_{\tau=0}^{N-1} |s_j(T^\tau(\cdot))|^2 \Bigg) \Bigg(\sum_{m=0}^{N-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) \Bigg)$$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} w^{mj} s_j(T^m(\cdot)) f(T^m(\cdot)) = V_j^* f.$$

Hence P_j is an orthogonal projection.

Finally, when for each $\tau \in \{0, 1, 2, \ldots, N-1\}$,

$$\sum_{j=1}^{N} w^{\tau j} \overline{s_j}(x) s_{j+\tau}(x) = \delta_{\tau,0}$$

we have

$$\sum_{j=1}^{N} P_j f(x) = \sum_{j=1}^{N} \overline{s_j}(x) \Bigg(\sum_{\tau=0}^{N-1} w^{\tau j} s_j(T^\tau(x)) f(T^\tau(x)) \Bigg)$$

$$= \sum_{\tau=0}^{N-1} \sum_{j=1}^{N} w^{\tau j} \overline{s_j}(x) s_{j+\tau}(x) f(T^\tau(x))$$

$$= f(x). \quad \square$$

2.3 Smooth projections on $L^2(\mathbb{R}^2)$. Alternative construction.

⟨ss_smooth⟩ Let $s_1 \in C^\infty(\mathbb{R}^2)$ and T be the cyclic permutation operator on $\mathbb{R}^2$ such that $T^2 = T$ and $s_1(T(\cdot)) = s_2$ with $|s_1|^2 + |s_2|^2 = 1$.

For $f \in L^2(\mathbb{R}^2)$, following the result in Theorem 1, the operators P_1 and P_2 given $P_1 f = \overline{s_1}(s_1 f + s_2 f(T(\cdot)))$ and $P_2 f = \overline{s_2}(s_1 f - s_2 f(T(\cdot)))$ are orthogonal projections. Notice that

$$\begin{aligned} P_1 f + P_2 f &= \overline{s_1}(s_1 f - s_2 f(T(\cdot))) + \overline{s_2}(s_2 f + s_1 f(T(\cdot))) \\ &= (|s_1|^2 + |s_2|^2) f + (\overline{s_2} s_1 - \overline{s_1} s_2) f(T(\cdot)) \\ &= f + (\overline{s_2} s_1 - \overline{s_1} s_2) f(T(\cdot)). \end{aligned} \quad (5) \text{ eq.s1s2}$$

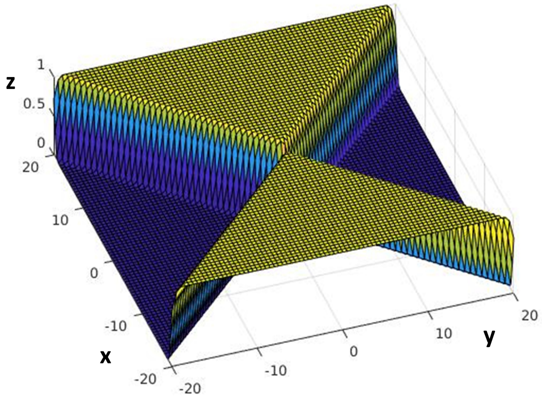

Fig. 2 Graph of the smooth window function s_1.
⟨fig.2d⟩

Assuming s is a real valued function, then we have that $P_1 f + P_2 f = f$.

By constructing appropriate smooth real functions s_1 and s_2 satisfying $s_1^2 + s_2^2 = 1$, we can generate smooth projections associated with the cone shaped domains Q_i, $i = 1, \ldots 4$. For that purpose, we will adapt the construction of a smooth function from Section 2.1 to the 2-dimensional setting. Similar to Section 2.1, let $\phi \in C^\infty(\mathbb{R})$ be an even function with $\text{supp}(\phi) \subset [-\epsilon, \epsilon]$ and $\int_{-\epsilon}^{\epsilon} \phi(t)dt = \sqrt{\frac{\pi}{2}}$. Define

$$\theta(x,y) = \int_{-\infty}^{x-y} \phi(t)dt \int_{-\infty}^{x+y} \phi(t)dt + \int_{-\infty}^{-x-y} \phi(t)dt \int_{-\infty}^{-x+y} \phi(t)dt.$$

Since $\phi \in C^\infty(\mathbb{R})$ is supported on $[-\epsilon, \epsilon]$, it follows that θ is supported in the region $\{(x,y) \in \mathbb{R}^2 : |y| < |x| + \epsilon\}$. In addition, the properties of ϕ imply that

$$\theta(x,y) = \frac{\pi}{2} \text{ if } |y| < |x| - \epsilon.$$

Moreover we claim that $\theta(x,y) + \theta(y,x) = \frac{\pi}{2}$. Indeed,

$$\begin{aligned}
&\theta(x,y) + \theta(y,x) \\
&= \int_{-\infty}^{x-y} \phi(t)dt \int_{-\infty}^{x+y} \phi(t)dt + \int_{-\infty}^{-x-y} \phi(t)dt \int_{-\infty}^{-x+y} \phi(t)dt \\
&+ \int_{-\infty}^{y-x} \phi(t)dt \int_{-\infty}^{x+y} \phi(t)dt + \int_{-\infty}^{-x-y} \phi(t)dt \int_{-\infty}^{x-y} \phi(t)dt. \\
&= \int_{-\infty}^{x+y} \phi(t)dt \left(\int_{-\infty}^{x-y} \phi(t)dt + \int_{-\infty}^{y-x} \phi(t)dt \right) \\
&+ \int_{-\infty}^{-x-y} \phi(t)dt \left(\int_{-\infty}^{y-x} \phi(t)dt + \int_{-\infty}^{x-y} \phi(t)dt \right) \\
&= \int_{-\infty}^{x+y} \phi(t)dt \left(\int_{-\infty}^{x-y} \phi(t)dt + \int_{x-y}^{\infty} \phi(t)dt \right)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{-x-y}\phi(t)dt\left(\int_{-\infty}^{y-x}\phi(t)dt+\int_{y-x}^{\infty}\phi(t)dt\right)\\
&=\sqrt{\frac{\pi}{2}}\left(\int_{-\infty}^{x+y}\phi(t)dt+\int_{-\infty}^{-x-y}\phi(t)dt\right)\\
&=\sqrt{\frac{\pi}{2}}\left(\int_{-\infty}^{x+y}\phi(t)dt+\int_{x+y}^{\infty}\phi(t)dt\right)\\
&=\sqrt{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}}=\frac{\pi}{2}.
\end{aligned}
$$

Now we set $s_1(x,y) = \sin(\theta(x,y))$ and $s_2(x,y) = \sin(\theta(y,x))$. Then,

$$s_2(x,y) = \sin(\theta(y,x)) = \sin(\frac{\pi}{2} - \theta(x,y)) = \cos(\theta(x,y)).$$

Therefore,

$$|s_1(x,y)|^2 + |s_2(y,x)|^2 = \sin^2(\theta(y,x)) + \cos^2(\theta(x,y)) = 1.$$

The projections P_1 and P_2 we have so constructed are smooth orthogonal projections associated with the horizontal cone Q_h and the vertical cone Q_v, respectively, in the sense discussed at the end of Section 2.1. The smooth function s_1 is illustrated in Figure 2 and it shows that - as expected - it approximates the characteristic function of the cone Q_h.

Note that the projections we constructed in this section are very similar but not identical to the projections P_1 and P_2 defined in Section 2.1.

3 Smooth Parseval frames of shearlets

⟨sec:3⟩ We now apply our newly constructed smooth projection operators to construct Parseval frames of shearlets that are localized on appropriate cone-shaped domains of $\mathbb{R}^2$ and $\mathbb{R}^3$. The general case $\mathbb{R}^N$, $N > 3$, is technically challenging since – as shown in Sec. 2.2 – the cyclic coordinate permutation operators T^τ do not commute.

3.1 Construction of shearlets

We recall some basic facts about 2-dimensional shearlets from [10,13]. Given a function $\psi \in L^2(\mathbb{R}^2)$, a system of 2-dimensional shearlets is a collection

$$\{\psi_{j,\ell,k} = |\det A_{(1)}|^{-\frac{j}{2}}\psi(B_{(1)}^{\ell}A_{(1)}^{-j}\cdot -k) : j,\ell \in \mathbb{Z}, k \in \mathbb{Z}^2\} \tag{6}$$ phi.one

where $A_{(1)} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ and $B_{(1)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The matrix $A_{(1)}$ is an *anisotropic dilation matrix*, so-called since it produces a different dilation factor along the horizontal and vertical directions. The non-expanding matrix $B_{(1)}$ is called a *shear matrix* and its integer powers control the directional properties of the shearlet system. The generator function ψ can be chosen to be a well-localized function such that the corresponding system (6) is a Parseval frame of $L^2(\mathbb{R}^2)$ [20]. Due to the action of the integer powers of the anisotropic dilation and shear matrices, the system (6) forms a Parseval frame of waveforms defined at various scales, orientations and locations, controlled by j, ℓ, and k, respectively. The frequency tiling associated with this system is shown in Figure 3 (left). We refer the interested reader to [9,14,18,25] for additional constructions of discrete shearlets and other related systems.

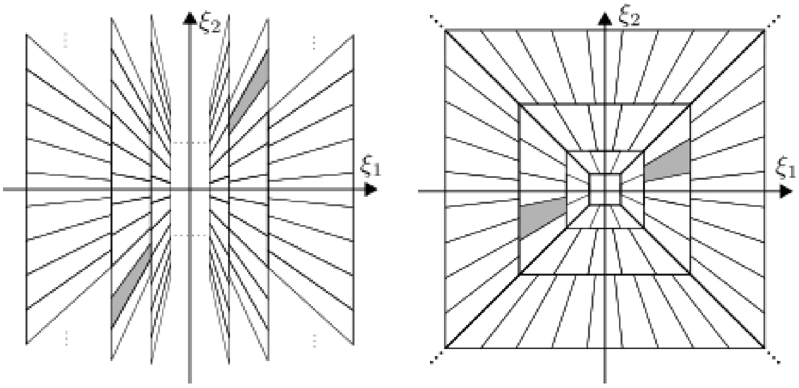

Fig. 3 Frequency tiling corresponding to horizontal shearlets (left) and to the system of cone-based shearlets (right).
⟨fig.shear1⟩

3.2 Cone-based shearlet on $L^2(\mathbb{R}^2)$

?⟨sec.coneb⟩? As suggested by Figure 3 (left), the system of shearlets (6) has a directional bias. Namely, due to the action of the shear matrix $B_{(1)}$, the elements of this system are not very efficient at representing functions whose support is concentrated along the ξ_2 axis in the Fourier domain and to accurately represent such functions one needs $\ell \to \pm\infty$. This behavior is a limitation in applications from sparse approximations [10,11,16] where it is desirable to uniformly represent information associated to any direction. For this reason, a modified system of shearlets, called *cone-based shearlets* was introduced [13,20]. We recall this construction below.

Let ϕ be a C^∞ bivariate function such that

$$0 \le \hat{\phi} \le 1,\ \hat{\phi} = 1 \text{ on } [-\tfrac{1}{16}, \tfrac{1}{16}]^2, \text{ and } \hat{\phi} = 0 \text{ on } \mathbb{R}^2 \setminus [-\tfrac{1}{8}, \tfrac{1}{8}]^2 \qquad (7)$$ eq.phi2

and let

$$w(\xi) = \sqrt{\widehat{\phi}^2(2^{-2}\xi) - \widehat{\phi}^2(\xi)}. \qquad (8)$$ eq.w2

It follows that

$$\widehat{\phi}^2(\xi) + \sum_{j\geq 0} w^2(2^{-2j}\xi_2) = 1 \quad \text{for } \xi \in \mathbb{R}^2. \tag{9}$$
eq.phi2+w

Each function $w^2(2^{-2j}\cdot)$ has support into the Cartesian corona

$$K_j = [-2^{2j-1}, 2^{2j-1}]^2 \setminus [-2^{2j-4}, 2^{2j-4}]^2 \subset \mathbb{R}^2.$$

Also, let $v \in C^\infty(\mathbb{R})$ be chosen so that

$$\operatorname{supp} v \subset [-1,1] \text{ and } \sum_{m\in\mathbb{Z}} |v(u-m)|^2 = 1 \quad \text{for } u \in \mathbb{R}. \tag{10}$$
eq.v123

Using this notation we state the following definition.

Definition 1 For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $V_{(1)}(\xi_1,\xi_2) = v(\frac{\xi_2}{\xi_1})$, a *horizontal shearlet system* for $L^2(\mathbb{R}^2)$ is a collection of functions

$$\Psi_1 = \{\psi^{(1)}_{j,\ell,k} : j \geq 0, \ell \in \mathbb{Z}, k \in \mathbb{Z}^2\}, \tag{11}$$
def.ns

where

$$\hat{\psi}^{(1)}_{j,\ell,k}(\xi) = |\det A_{(1)}|^{-j/2}\, w(2^{-2j}\xi)\, V_{(1)}(\xi A^{-j}_{(1)} B^{-\ell}_{(1)})\, e^{2\pi i \xi A^{-j}_{(1)} B^{-\ell}_{(1)} k} \tag{12}$$
def.nsb

and $A_{(1)}$, $B_{(1)}$ are given after (6). Similarly, a *vertical shearlet system* for $L^2(\mathbb{R}^2)$ is a collection

$$\Psi_2 = \{\psi^{(2)}_{j,\ell,k} : j \geq 0, \ell \in \mathbb{Z}, k \in \mathbb{Z}^2\}, \tag{13}$$
def.ns2

where

$$\hat{\psi}^{(2)}_{j,\ell,k}(\xi) = |\det A_{(2)}|^{-j/2}\, w(2^{-2j}\xi)\, V_{(2)}(\xi A^{-j}_{(2)} B^{-\ell}_{(2)})\, e^{2\pi i \xi A^{-j}_{(2)} B^{-\ell}_{(2)} k}, \tag{14}$$
def.ns2b

$A_{(2)} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, $B_{(2)} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $V_{(2)}(\xi) = v(\frac{\xi_1}{\xi_2})$.

It is shown in [13,20] that a horizontal shearlet system Ψ_1, with the range of ℓ restricted to $[-2^j, 2^j]$, forms a Parseval frame of the subspace $L^2(Q_{ht})^\vee$ corresponding to the truncated horizontal cone $Q_{ht} = Q_h \setminus [-\frac{1}{8}, \frac{1}{8}]^2$. That is, given any $f \in L^2(Q_{ht})^\vee$, we have that

$$\|f\|^2 = \sum_{j\geq 0} \sum_{\ell=-2^j}^{2^j} \sum_{k\in\mathbb{Z}^2} |\langle f, \psi^{(1)}_{j,\ell,k}\rangle|^2.$$

Similarly, a vertical shearlet system Ψ_2, with ℓ-range restricted to $[-2^j, 2^j]$, forms a Parseval frame of the subspace $L^2(Q_{vt})^\vee$ where $Q_{vt} = Q_v \setminus [-\frac{1}{8}, \frac{1}{8}]^2$. One can obtain a Parseval frame of $L^2(\mathbb{R})$ by appropriately projecting and combining these systems with a system of coarse scale functions $\{\phi(2\cdot - k), k \in \mathbb{Z}^2\}$, where ϕ satisfies (7). Namely, the cone-based shearlet system

$$SH = P_{Q_0}\Phi(\phi) \cup P_{Q_{ht}}\Psi_1 \cup P_{Q_{vt}}\Psi_2, \tag{15}$$
shearlet:2dc

where $P_{Q_0} f = (\chi_{Q_0} \hat{f})^\vee$, $P_{Q_{ht}} f = (\chi_{Q_{ht}} \hat{f})^\vee$ and $P_{Q_{vt}} f = (\chi_{Q_{vt}} \hat{f})^\vee$, is a Parseval frame of $L^2(\mathbb{R}^2)$ [20]. The corresponding tiling in the Fourier domain is shown in Fig. 3.

The cone-based Parseval frame of shearlets avoids the directional bias of the systems Ψ_1 and Ψ_2. However, this construction has the drawback of introducing discontinuities in the Fourier domain for those elements whose Fourier supports overlap the boundaries of the regions Q_h and Q_v. In several applications – most notably sparse approximations [10,11,16] – it is important that the shearlet system avoids directional bias and that all its elements are smooth or have several continuous derivatives. Hence, the shearlet system (15) is not acceptable for such applications. To correct this limitation, a modified cone-based construction was introduced in [13] whose main idea consists in re-defining those shearlet elements whose Fourier support overlaps the boundaries of the cones Q_h and Q_v.

3.3 New cone-based shearlet on $L^2(\mathbb{R}^2)$

⟨sec.conec⟩ We derive a novel construction of smooth cone-based shearlets that relies on the new smooth projections we constructed in Section 2.

⟨thm2⟩ **Theorem 2** *Let P_1 and P_2 be the smooth orthogonal projections on $L^2(\mathbb{R}^2)$ defined in Section 2.1 by* (2) *and* (3) *(or their variants defined in Section 2.3), and let Ψ_1 and Ψ_2 be the systems* (11) *and* (13)*, respectively, where the functions w and v in* (12) *and* (14) *are C_c^∞ functions satisfying* (8) *and* (10)*. Also, let*

$$\Phi(\phi) = \{\phi_k = \phi(\cdot - k) \colon k \in \mathbb{Z}^2\}, \tag{16}$$?eqs.phi?

where ϕ satisfies (7)*. Then the shearlet system*

$$\Phi(\phi) \cup \Pi_1 \Psi_1 \cup \Pi_2 \Psi_2, \tag{17}$$ def.spf2

is a smooth Parseval frame of $L^2(\mathbb{R}^2)$, where, $\Pi_d f = (P_d \hat{f})^\vee$ for $d = 1, 2$. That is, for any $f \in L^2(\mathbb{R}^2)$

$$f = \sum_{k \in \mathbb{Z}^2} \langle f, \phi_k \rangle \phi_k + \sum_{j \geq 0} \sum_{|\ell| \leq 2^j} \sum_{k \in \mathbb{Z}^2} \langle f, \Pi_1 \psi^{(1)}_{j,\ell,k} \rangle \Pi_1 \psi^{(1)}_{j,\ell,k}$$
$$+ \sum_{j \geq 0} \sum_{|\ell| \leq 2^j} \sum_{k \in \mathbb{Z}^2} \langle f, \Pi_2 \psi^{(2)}_{j,\ell,k} \rangle \Pi_2 \psi^{(2)}_{j,\ell,k},$$

with convergence in the L^2-norm. Here the functions $\psi^{(1)}_{j,\ell,k}$ and $\psi^{(2)}_{j,\ell,k}$ are the elements of Ψ_1 and Ψ_2, respectively.

Before proving the theorem, we make the following observation.

⟨lem.3⟩ **Lemma 3** *Under the assumptions and notation of Theorem 2, the system $\Phi(\phi) \cup \Psi_1$ is a Parseval frame of $L^2(\mathbb{R}^2)$. Similarly, the system $\Phi(\phi) \cup \Psi_2$ is a Parseval frame of $L^2(\mathbb{R}^2)$.*

Proof. The proof is adapted from [13].

Since $(\xi_1, \xi_2)\, A_{(1)}^{-j} B_{(1)}^{-\ell} = (2^{-2j}\xi_1, 2^{-j}\xi_2 - \ell 2^{-2j}\xi_1)$, we can write $\hat{\psi}^{(1)}_{j,\ell,k}$ in (11) as

$$\hat{\psi}^{(1)}_{j,\ell,k}(\xi_1, \xi_2) = 2^{-\frac{3}{2}j}\, w(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}. \tag{18}$$

eq:psihat

Using the change of variable $\eta = \xi A_{(1)}^{-j} B_{(1)}^{-\ell}$ and the notation $K = [-\frac{1}{2}, \frac{1}{2}]^2$, for $f \in L^2(\mathbb{R})$ we have:

$$\begin{aligned}
&\sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}^{(1)}_{j,\ell,k} \rangle|^2 \\
&= \sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \left| \int_{\mathbb{R}^2} 2^{-\frac{3}{2}j} \hat{f}(\xi)\, w(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}\, d\xi \right|^2 \\
&= \sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \left| \int_{K} 2^{\frac{3}{2}j} \hat{f}(\eta B_{(1)}^{\ell} A_{(1)}^{j})\, w(\eta_1, 2^{-j}(\eta_2 + \ell\eta_1))\, v\left(\tfrac{\eta_2}{\eta_1}\right)\, e^{2\pi i \eta k}\, d\eta \right|^2 \\
&= \sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} \int_{K} 2^{3j} |\hat{f}(\eta B_{(1)}^{\ell} A_{(1)}^{j})|^2\, w^2(\eta_1, 2^{-j}(\eta_2 + \ell\eta_1))\, |v\left(\tfrac{\eta_2}{\eta_1}\right)|^2\, d\eta \\
&= \sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2\, w^2(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, |v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)|^2\, d\xi \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} w^2(2^{-2j}\xi_1, 2^{-2j}\xi_2)\, |v\Big(2^j \frac{\xi_2}{\xi_1} - \ell\Big)|^2\, d\xi \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} w^2(2^{-2j}\xi_1, 2^{-2j}\xi_2)\; d\xi.
\end{aligned} \tag{19}$$

lem3.1

where, in the last equality, we have used (10). A similar (simpler) computation shows that

$$\sum_{k \in \mathbb{Z}^2} |\langle \hat{f}, \hat{\phi}_k \rangle|^2 = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2\, \hat{\phi}^2(\xi)\; d\xi. \tag{20}$$

lem3.2

Using (9), (19) and (20), we now conclude that

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^2} |\langle \hat{f}, \hat{\phi}_k \rangle|^2 + \sum_{j \geq 0} \sum_{\ell \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} |\langle \hat{f}, \hat{\psi}^{(1)}_{j,\ell,k} \rangle|^2 \\
&= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left(\hat{\phi}^2(\xi) + \sum_{j \geq 0} w^2(2^{-2j}\xi_1, 2^{-2j}\xi_2) \right) d\xi \\
&= \|f\|^2.
\end{aligned}$$

The proof for the system $\Phi(\phi) \cup \Psi_2$ is very similar. □

We can now prove Theorem 2.

Proof of Theorem 2. The proof follows from Proposition 1, Lemma 3 and the observation that, for any $g \in L^2(\mathbb{R}^2)$, $g = P_1 g + P_2 g$. In fact, for any $f \in L^2(\mathbb{R}^2)$, we can write

$$\begin{aligned} f &= \mathit{\Pi}_1 f + \mathit{\Pi}_2 f \\ &= \sum_k \langle \mathit{\Pi}_1 f, \phi_k \rangle \, \mathit{\Pi}_1 \phi_k + \sum_{j,\ell,k} \langle \mathit{\Pi}_1 f, \psi^{(1)}_{j,\ell,k} \rangle \, \mathit{\Pi}_1 \psi^{(1)}_{j,\ell,k} \\ &+ \sum_k \langle \mathit{\Pi}_2 f, \phi_k \rangle \, \mathit{\Pi}_2 \phi_k + \sum_{j,\ell,k} \langle \mathit{\Pi}_2 f, \psi^{(2)}_{j,\ell,k} \rangle \, \mathit{\Pi}_2 \psi^{(2)}_{j,\ell,k} \\ &= \sum_k \langle f, \phi_k \rangle \, \phi_k + \sum_{j,\ell,k} \langle f, \mathit{\Pi}_1 \psi^{(1)}_{j,\ell,k} \rangle \, \mathit{\Pi}_1 \psi^{(1)}_{j,\ell,k} + \sum_{j,\ell,k} \langle f, \mathit{\Pi}_2 \psi^{(2)}_{j,\ell,k} \rangle \, \mathit{\Pi}_2 \psi^{(2)}_{j,\ell,k}, \end{aligned}$$

with convergence in L^2 norm. In addition, since projections P_1 and P_2 are smooth, then the Parseval frame (17) consists of smooth functions. □

Remark 2 The Fourier-domain tiling associated to the Parseval frame of shearlets (17) has the same geometry as the one illustrated in Fig. 3 for the cone-shaped shearlets. In particular, we claim that, by choosing the smooth orthogonal projections P_1 and P_2 with a value of ϵ sufficiently small (see Proposition 2 and the comment after (3)), for each scale parameter j only the elements $\psi^{(d)}_{j,\ell,k}$ with $\ell = \pm 2^j$, $\ell = \pm(2^j - 1)$ and $\ell = \pm(2^j + 1)$ are affected by the action of the projections P_1 and P_2. Below, we prove our claim for the projection P_1; the argument for P_2 is very similar.

Recall that, for any function g with support in $Q_h^\epsilon = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |y| < |x| - \epsilon\}$, we have that $P_1 g = g$. Hence to prove our claim it will be sufficient to show that: (i) the lines $\xi_2 = \xi_1 - \epsilon$ and $\xi_2 = -\xi_1 + \epsilon$, for $|\xi_1| > 2^{-4}$ are contained inside the supports of the shearlet elements $\hat{\psi}^{(1)}_{j,\ell,k}$ with $\ell = 2^j$ and $\ell = -2^j$, respectively; (ii) for any fixed j, the shearlet elements $\hat{\psi}^{(1)}_{j,\ell,k}$ only overlap those corresponding to $\ell + 1$ and $\ell - 1$.

For (ii), we observe that in the Fourier domain an element of the set Ψ_1, given by (12), is expressed as (18) Hence, by (10), at any fixed scale j, the functions $\hat{\psi}^{(1)}_{j,\ell,k}$ only overlap the adjacent functions $\hat{\psi}^{(1)}_{j,\ell-1,k}$ and $\hat{\psi}^{(1)}_{j,\ell+1,k}$. For (i), a direct calculation shows that the Fourier support of $\hat{\psi}^{(1)}_{j,\ell,k}$ is given by the trapezoidal region

$$Z_{j,\ell} = \{ |\tfrac{\xi_2}{\xi_1} - \ell 2^{-j}| \le 2^{-j}, \, \xi_1 \in [-2^{2j-1}, 2^{2j-1}] \setminus [-2^{2j-4}, 2^{2j-4}] \};$$

in particular the supports of the shearlet elements corresponding to $\ell = 2^j$ are contained in the trapezoids of vertices $(2^{2j-4}, 2^{2j-4} \pm 2^{j-1})$ and $(2^{2j-1}, 2^{2j-1} \pm 2^{j-4})$. A direct calculation shows that if $\epsilon < \frac{1}{\sqrt{2}} 2^{-4}$, then, for $\xi_1 > 2^{-4}$ the line $\xi_2 = \xi_1 - \epsilon$ is fully contained inside the supports of the trapezoidal regions $Z_{j,2^j}$. Similarly, the line $\xi_2 = -\xi_1 + \epsilon$ is fully contained inside the supports of the trapezoidal regions $Z_{j,-2^j}$.

Finally, we observe that, under the choice of ϵ described above, $P_1\psi^{(1)}_{j,\ell,k}=0$ if $|\ell|>2^j+1$ (and similarly for $P_2\psi^{(2)}_{j,\ell,k}$). Hence, in the definition of the cone-based system of shearlets (17), we could restrict the range of ℓ to $|\ell|\le 2^j+1$ without affecting the Parseval frame property.

3.4 Smooth cylindrical shearlets in $L^2(\mathbb{R}^3)$

In this section, we construct a new smooth Parseval frame of shearlets in $L^2(\mathbb{R}^3)$ that we call *cylindrical shearlets*. This system is motivated by applications from hyperspectral imaging where the first two variables are associated with the space domain and the third one with the spectrum [23]. Because of the geometry of such data, it is convenient to use a representation that is directionally sensitive with respect to the space variables only [26]. For this reason, we consider a modified 3-dimensional shearlet system where the shear variable is one-dimensional unlike the conventional case where it is two-dimensional. We expect that cylindrical shearlets could provide improved sparse approximations for image models consistent with hyperspectral data found in applications.

We define our new construction by modifying the approach in [12, 13]. Unlike the classical 3-dimensional shearlets that require to partition the space into 3 pyramids, we associate cylindrical shearlets to two cylindrical pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{R}^3$ defined below:

$$\mathcal{P}_1=\{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3:|\tfrac{\xi_2}{\xi_1}|\le 1\},\quad \mathcal{P}_2=\{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3:|\tfrac{\xi_1}{\xi_2}|\le 1\}.$$

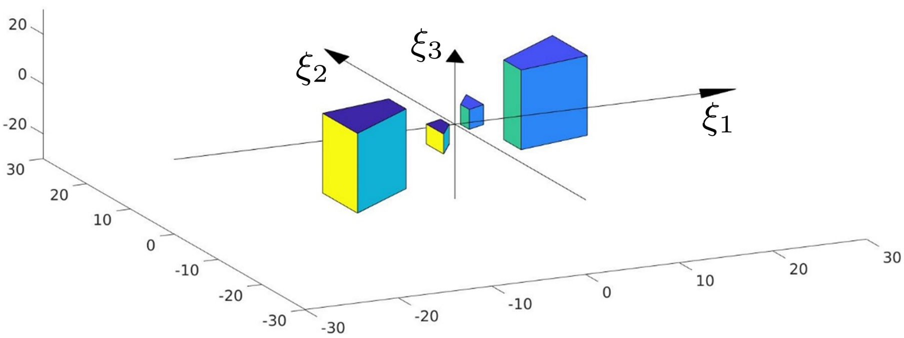

Fig. 4 Representative elements of the cylindrical shearlet system $\{\psi^{(1)}_{j,\ell,k}\}$ for $(j,\ell)=(0,2)$ and $(j,\ell)=(3,2)$, in the Fourier domain.

⟨fig.3d⟩

Similar to the 2-dimensional case, we let $\phi\in L^2(\mathbb{R}^3)$ be such that $\hat\phi\in C_c^\infty$,

$$0\le\hat\phi\le 1,\ \hat\phi=1 \text{ on } [-\frac{1}{16},\frac{1}{16}]^3 \ \text{ and } \ \hat\phi=0 \ \text{ on } \mathbb{R}^3\setminus[-\frac{1}{8},\frac{1}{8}]^3 \qquad (21)$$ eq.phi3

and define w according to (8), so that

$$\widehat{\phi}^2(\xi) + \sum_{j\geq 0} w^2(2^{-2j}\xi) = 1 \ \text{ for } \xi \in \mathbb{R}^3. \tag{22}$$ eq.phi3+w

The functions $w^2(2^{-2j}\cdot)$ have support in the Cartesian coronae

$$K_j = [-2^{2j-1}, 2^{2j-1}]^3 \setminus [-2^{2j-4}, 2^{2j-4}]^3 \subset \mathbb{R}^3$$

and, for $j \geq 0$, produce a smooth tiling of the frequency space.

Definition 2 For $d = 1, 2$, a cylindrical shearlet system is a countable collection of functions of the form

$$\Psi_d^C = \{\psi_{j,\ell,k}^{(d)} : j \geq 0, \ell \in \mathbb{Z}, k \in \mathbb{Z}^3\}, \tag{23}$$ sh.one

where

$$\hat{\psi}_{j,\ell,k}^{(d)}(\xi) = |\det A_{(d)}|^{-\frac{j}{2}} w(2^{-2j}\xi)\, V_{(d)}(\xi A_{(d)}^{-j} B_{(d)}^{-\ell}) e^{2\pi i \xi A_{(d)}^{-j} B_{(d)}^{-\ell} k}, \tag{24}$$ sh.one.f

$V_{(1)}(\xi_1, \xi_2, \xi_3) = v(\frac{\xi_2}{\xi_1})$, $V_{(2)}(\xi_1, \xi_2, \xi_3) = v(\frac{\xi_1}{\xi_2})$, $v \in C_c^\infty(\mathbb{R})$ satisfies (10) and
$A_{(1)} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $B_{(1)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A_{(2)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $B_{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For $d = 1$, observing that $|\det A_{(1)}| = 2^5$ and

$$(\xi_1, \xi_2, \xi_3) A_{(1)}^{-j} B_{(1)}^{-\ell} = (2^{-2j}\xi_1, -2^{-2j}\ell\xi_1 + 2^{-j}\xi_2, 2^{-2j}\xi_3),$$

an element of the cylindrical shearlets system (24) can be written as

$$\hat{\psi}_{j,\ell,k}^{(1)}(\xi) = 2^{-\frac{5j}{2}} w(2^{-2j}\xi)\, v(2^j \frac{\xi_2}{\xi_1} - \ell)\, e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}, \tag{25}$$ explicit:psihat

showing that the Fourier support of $\psi_{j,\ell,k}^{(1)}$ is contained inside the region

$$U_{j,\ell} = \{\xi \in [-2^{2j-1}, 2^{2j-1}]^3 \setminus [-2^{2j-4}, 2^{2j-4}]^3 \subset \mathbb{R}^3 : |\tfrac{\xi_2}{\xi_1} - \ell 2^{-j}| \leq 2^{-j}\}.$$

The Fourier support of a representative element of the cylindrical shearlet system is shown in Fig. 4.

An argument similar to the 2-dimensional case shows that each system Ψ_d^C, $d = 1, 2$, is a Parseval frames of $L^2(\mathcal{P}_d \setminus [-\frac{1}{8}, \frac{1}{8}]^3)^\vee$. To construct a smooth pyramid-based Parseval frame of $L^2(\mathbb{R}^3)$, we will proceed as in Sec. 3.3. We start with the following observation similar to Lemma 3.

⟨pro.4⟩ **Proposition 4** *Let $\Phi(\phi) = \{\phi_k = \phi(\cdot - k) : k \in \mathbb{Z}^3\}$ where ϕ satisfies (21) and Ψ_d^C, $d = 1, 2$ be the systems (23) where w, v satisfy (8) and (10). Then the systems $\Phi(\phi) \cup \Psi_1^C$ and $\Phi(\phi) \cup \Psi_2^C$ are both Parseval frames of $L^2(\mathbb{R}^3)$.*

That is, for any $f \in L^2(\mathbb{R}^3)$,

$$\|f\|^2 = \sum_{k\in\mathbb{Z}^3} \sum_{j\geq 0} \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} |\langle f, \psi_{j.\ell,k}^{(d)}\rangle|^2.$$

Proof. We only prove case $d = 1$ since the arguments for $d = 2$ is similar.

Using the explicit form of the cylindrical shearlet (25), the change of variable $\eta = \xi\, A_{(1)}^{-j} B_{(1)}^{-\ell}$ and the notation $K = [-\frac{1}{2}, \frac{1}{2}]^3$, we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, \hat{\psi}^{(1)}_{j.\ell,k}\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}^3} |\int_{\mathbb{R}^3} 2^{-\frac{5j}{2}} \hat{f}(\xi)\, w(2^{-2j}\xi)\, v(2^j \frac{\xi_2}{\xi_1} - \ell) e^{2\pi i \xi A_{(1)}^{-j} B_{(1)}^{-\ell} k}\, d\xi|^2 \\
&= \sum_{k\in\mathbb{Z}^3} |\int_{\mathbb{R}^3} 2^{\frac{5j}{2}} \hat{f}(\eta A_{(1)}^{j} B_{(1)}^{\ell})\, w(\eta)\, v(\frac{\eta_2}{\eta_1}) e^{2\pi i \eta k}\, d\eta|^2 \\
&= \int_K 2^{5j} |\hat{f}(\eta A_{(1)}^{j} B_{(1)}^{\ell})|^2 |w(\eta)|^2\, |v(\frac{\eta_2}{\eta_1})|^2\, d\eta \\
&= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2\, w^2(2^{-2j}\xi)\, |v(2^j \frac{\xi_2}{\xi_1} - \ell)|^2\, d\xi.
\end{aligned}
$$

In the computation above, we have used the fact that the function $w(\eta_1, \eta_3)\, v(\frac{\eta_2}{\eta_1})$ is supported inside K. Thus, using (10), we have

$$
\begin{aligned}
\sum_{j\ge 0} \sum_{\ell=-2^j}^{2^j} \sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, \hat{\psi}^{(1)}_{j.\ell,k}\rangle|^2 &= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \Big(\sum_{j\ge 0} w^2(2^{-2j}\xi) \sum_{\ell\in\mathbb{Z}} |v(2^j \tfrac{\xi_2}{\xi_1} - \ell)|^2 \Big) d\xi \\
&= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \sum_{j\ge 0} w^2(2^{-2j}\xi)\, d\xi. \qquad (26) \text{ some:eq}
\end{aligned}
$$

As in Lemma 3, a direct computation shows that

$$
\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, \hat{\phi}_k\rangle|^2 = \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2\, \hat{\phi}^2(\xi)\; d\xi. \qquad (27) \text{ lem3.3}
$$

Thus, using (22), (26) and (27), we conclude that, for each $f \in L^2(\mathbb{R}^3)$,

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, \hat{\phi}_k\rangle|^2 + \sum_{j\ge 0} \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^3} |\langle \hat{f}, \hat{\psi}^{(1)}_{j,\ell,k}\rangle|^2 \\
&= \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \left(\hat{\phi}^2(\xi) + \sum_{j\ge 0} w^2(2^{-2j}\xi_1, 2^{-2j}\xi_2) \right) d\xi \\
&= \|f\|^2. \quad \square
\end{aligned}
$$

Similar to the 2-dimensional construction of smooth cone-based shearlets, we can construct a smooth Parseval frame for $L^2(\mathbb{R}^3)$ by appropriately projecting a combination of the cylindrical shearlets (23) and a family of coarse scale functions. To define our orthogonal projections, we slightly modify the result from Section 2.3.

⟨proj:prop⟩ **Proposition 5** *Let $k_1 \in C^\infty(\mathbb{R}^3)$ and T be the commutation operator on $\mathbb{R}^3$ such that $k_1(T(x,y,z)) = k_1(y,x,z) = k_2(x,y,z)$ with*

$$|k_2(x,y,z)|^2 + |k_1(x,y,z)|^2 = 1.$$

For $f \in L^2(\mathbb{R}^3)$, the operators

$$P_1 = k_1(k_1 f + k_2 f(T)) \text{ and } P_2 = k_2(k_1 f - k_2 f(T))$$

are orthogonal projections. Here $f(T) = f(T(x))$ for $x \in \mathbb{R}^3$. In addition, $P_1 f + P_2 f = f$.

Proof. The statement that P_1 and P_2 are orthogonal projections follows directly from Theorem 1. The property that $P_1 f + P_2 f = f$ follows from (5) after observing that $k_1(x,y,z) = s_1(x,y)$. □

As observed in Section 2.3, we can define a smooth function s_1 and hence smooth functions k_1 and k_2 satisfying the assumptions of Proposition 5. The corresponding projection operators P_1 and P_2 are then smooth projections associated with the pyramidal regions $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Hence, similar to Theorem 2, using Propositions 1, 4 and 5, we have the following result.

Theorem 3 *For $d = 1,2$, let Ψ_d^C be as in (23) where w, v satisfy (8), and (10), respectively. Also, let Π_d be defined by $\Pi_d f = (P_d \hat{f})^\vee$, where P_d is given by Proposition 5 and $k_1(x,y,z) = s_1(x,y)$ is the smooth function constructed in Section 2.3. Then*

$$\{\phi(\cdot - k) \colon k \in \mathbb{Z}^3\} \cup \{\Pi_d\, \psi^{(d)}_{j.\ell,k} : j \geq 0, \ell \in \mathbb{Z}, k \in \mathbb{Z}^3, d = 1,2\},$$

where where ϕ satisfies (21), is a Parseval frame of $L^2(\mathbb{R}^3)$.

References

AWW_92 1. P. Auscher, G. Weiss, M. V. Wickerhauser, Local sine and cosine bases of Coifman and Meyer and the construction of smooth wavelets, Wavelets, 237-256, in: Wavelet Anal. Appl., 2, Academic Press, Boston, MA, 1992.
BKZ_15 2. B. G. Bodmann, G. Kutyniok, X. Zhuang, Gabor shearlets, Appl. Comput. Harmon. Anal. 38, 87-114 (2015).
MBKD 3. M. Bownik, K. Dziedziul, Smooth orthogonal projections on sphere, Const. Approx. 41, 2348 (2015).
MBKD2 4. M. Bownik, K. Dziedziul, A. Kamont, Smooth orthogonal projections on Riemannian manifold, arXiv preprint 1803.03634 (2018).
CD04 5. E. J. Candès, D. L. Donoho, New tight frames of curvelets and optimal representations of objects with C^2 singularities, Comm. Pure Appl. Math. 57, 219-266 (2004).
CM_1991 6. R. Coifman, Y. Meyer, Remarques sur l'analyse de Fourier a fenetre. C. R. Acad. Sci. Paris Ser. I Math. 312(3), 259-261 (1991).
Dau92 7. I. Daubechies, Ten Lectures on Wavelets, Society for Industrial and Applied Mathematics, 1992.
Dev92 8. R. A. DeVore, Nonlinear approximation, Acta Numerica, 51-150, 1998.
GK14 9. P. Grohs, G. Kutyniok, Parabolic Molecules, Foundations of Computational Mathematics, 14(2), 299-337 (2014).

GL_SIAM07 10. K. Guo, D. Labate, Optimally Sparse Multidimensional Representation using Shearlets, SIAM J. Math. Anal. 9, 298-318 (2007).
GL_FIO 11. K. Guo, D. Labate, Representation of Fourier Integral Operators using shearlets, J. Fourier Anal. Appl. 14, 327-371 (2008).
GL_3D 12. K. Guo, D. Labate, Optimally sparse representations of 3D data with C2 surface singularities using Parseval frames of shearlets, SIAM J Math. Anal. 44, 851-886 (2012)
GL_MMNP 13. K. Guo, D. Labate. The Construction of Smooth Parseval Frames of Shearlets. Math. Model. Nat. Phenom. 8, 82-105 (2013).
HZ05 14. B. Han, X. Zhuang, Smooth affine shear tight frames with MRA structures, Appl. Comput. Harmon. Anal. 39(2), 300-338 (2015).
HerWeis 15. E. Hernandez, G. Weiss, A first course on wavelets, CRC Press, 2000.
KKZ14 16. E. J. King, G. Kutyniok, X. Zhuang, Analysis of inpainting via clustered sparsity and microlocal analysis, J. Math. Imag. Visi. 48, 205–234 (2014).
KG14 17. P. Grohs, G. Kutyniok, Parabolic Molecules, Found. Comput. Math. 14(2), 299-337 (2014).
KL07 18. G. Kutyniok, D. Labate, Construction of regular and irregular shearlet frames, J. Wavelet Theory Appl. 1, 1–10 (2007).
KL12 19. G. Kutyniok, D. Labate, Shearlets: Multiscale analysis for multivariate data, Birkhäuser, 2012
KL12a 20. G. Kutyniok, D. Labate, Introduction to Shearlets, in Shearlets: Multiscale analysis for multivariate data, Birkhäuser, 2012, p. 1-38.
KL11 21. G. Kutyniok, W. Lim, Compactly supported shearlets are optimally sparse, Journal of Approximation Theory 163(11), 1564–1589 (2011).
LLKW05 22. D. Labate, W. Lim, G. Kutyniok, G. Weiss, Sparse multidimensional representation using shearlets, Proc. SPIE 5914, Wavelets XI (2005).
Lan02 23. D. Landgrebe, Hyperspectral image data analysis, IEEE Signal Processing Magazine 19(1), 17-28 (2002).
LXQH 24. T Lin, S. Xu, Q. Shi, P Haoa, An algebraic construction of orthonormal M-band wavelets with perfect reconstruction, Applied Mathematics and Computation 172(2), 717-730 (2006).
PKL12 25. K. Pisamai, G. Kutyniok, W. Lim, Construction of compactly supported shearlet frames, Const. Approx. 35, 21–72 (2012).
PLC17 26. S. Prasad, D. Labate, M. Cui, Y. Zhang, Morphologically Decoupled Multi-Scale Sparse Representation for Hyperspectral Image Analysis", IEEE Transactions on Geoscience and Remote Sensing 55(8), 4355-4366 (2017).
Ver17 27. D. Vera, Democracy of shearlet frames with applications, Journal of Approximation Theory 213, 23-49 (2017).