# Detection of boundary curves on the piecewise smooth boundary surface of three dimensional solids

Robert Houska

*Department of Mathematics, University of Houston, Houston, Texas 77204, USA*

Demetrio Labate*

*Department of Mathematics, University of Houston, Houston, Texas 77204, USA*

**Abstract**

Suppose that $\Omega$ is a three dimensional solid with boundary surface $S = S_1 \cup \cdots \cup S_q$, where each $S_r$ is a smooth surface with boundary curve $\Gamma_r$. Multiscale directional representation systems (e.g., shearlets) are able to capture the essential geometry of $\Omega$ by precisely identifying the boundary set

$$\mathcal{N} = \{(p, n_r(p)) : p \in S_r, r = 1, \ldots, q\},$$

where $n_r(p)$ denotes the normal vector to the surface $S_r$ at $p$. This property has resulted in the successful application of multiscale directional methods in a variety of image processing problems, since edges and boundary sets are usually the most informative features in many types of multidimensional data. However, existing methods are ill-suited to capture those edge-type singularities in the three-dimensional setting resulting from the intersection of piecewise smooth boundary surfaces. In this paper, we introduce a new multiscale directional system based on a modification of the shearlet framework and prove that the associated continuous transform has the ability to precisely identify both the location and orientation of the boundary curves $\Gamma_r$ from the solid $\Omega$. This paper extends a number of results appeared in the literature in recent years to the challenging problem of extracting curvilinear singularities in 3-dimensional objects and is motivated by image analysis problems arising from areas including biomedical and seismic imaging and astronomy.

*Keywords:* analysis of singularities, continuous wavelet transform, edge detection, shearlets, sparse representations, wavelets

*Corresponding author

*Email addresses:* `rthouska@math.uh.edu` (Robert Houska), `dlabate@math.uh.edu` (Demetrio Labate)

## 1. Introduction

Objects with discontinuities along curvilinear edges and surface boundaries appear in a variety of imaging applications. For example, in biomedical imaging, the objects of interest are cells, tissues and other organs; in this case, changes in molecular structures identifying each object are represented as edges and surfaces. In seismic imaging, the objects of interest are the material properties of the Earth's subsurface as a function of depth and these properties change discontinuously across a system of layer boundaries. In astronomical images, the objects of interest include intricate patterns with filaments, clusters, and sheet-like arrangements of galaxies encompassing large nearly empty regions. Notice that, in all such applications, the discontinuities occurring along edges and surfaces are the most informative features and, in many cases, the only structures one is really interested in recovering from data.

Over the past decade, a number of "directional multiscale systems" were introduced to provide improved framework for the representation of multivariate functions containing edge-type discontinuities. The *ridgelets* [2] and *beamlets* [6], for example, were introduced to represent more efficiently lines crossing an image. Other prominent constructions are the *curvelets* [3] and *shearlets* [24, 14] that provide (near) optimally sparse approximations for images with curvilinear edges by combining multiscale analysis and high directional sensitivity. Due to their ability to sparsely represent curvilinear edges, methods based on these representations are particularly useful for the study of edge-dominated phenomena and often outperform more traditional multiscale methods in many image processing applications (cf. [8, 9]).

Perhaps the true potential of such directional multiscale systems is best illustrated when the associated continuous transforms are applied to the analysis of singularities. The continuous curvelet transform, in particular, resolves the *wavefront set* of a distribution in two dimensions [4]. The continuous shearlet transform, in addition to satisfying the latter property [22], has the ability to precisely identify the set of discontinuities of a large class of multivariate functions. More precisely, let $f = \chi_\Omega$, where $\Omega$ is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a piece-wise smooth boundary $S = \partial\Omega$. Then the continuous shearlet transform of $f$ identifies both the location and orientation of the boundary set $S$ by its asymptotic decay at fine scales [15, 16, 17, 19]. These theoretical results have lead to a number of successful applications in problems of edge detection and feature extraction [5, 23, 26, 30]. Note however that the ability to detect the set of singularities of functions and distributions is useful beyond these applications. Consider, for example, the problem of "geometric separation" which aims to break up complex data into geometrically distinct components. It was recently shown that the solution to this problem relies on the ability to detect and separate different types of singularities, e.g., pointwise singularities vs. curvilinear ones [7, 21]. These observations are the foundation for several remarkable applications to image inpainting and morphological component analysis [13, 20, 28, 27].

Motivated by the same types of applied problems, in this paper, we examine the more challenging problem of extracting curvilinear singularities in 3-dimensional objects, which is not covered by existing results. To be more precise about our setting, suppose that $\Omega$ is a three dimensional solid with boundary surface $S = S_1 \cup \cdots \cup S_q$, where each $S_r$ is a smooth surface with boundary curve $\Gamma_r$. Let $n_r(p)$ denote the normal vector to $S_r$ at $p$ and $t_r(p)$ denote the tangent vector to $\Gamma_r$ at $p$ and write

$$\mathcal{N} = \{(p, n_r(p)) : p \in S\}$$

and

$$\mathcal{T} = \{(p, t_r(p)) : p \in \Gamma_r, r = 1, \ldots, q\}.$$

The goal of this paper is to extract the collection $\mathcal{T}$ from the solid region $\Omega$.

On the surface, our setting is similar to reference [29] that deals with the application of directional multiscale transforms to astronomical data restoration. In this reference, the authors heuristically introduce a variant of the curvelet transform for handling singularities forming one-dimensional structures in $\mathbb{R}^3$ and apply this system to problems of denoising and inpainting of astronomical data. Note that, while several numerical illustrations are presented in [29], their approach is purely heuristic. By contrast, in this paper we develop a rigorous theoretical framework for the detection of singularities forming one-dimensional structures in $\mathbb{R}^3$ which are subsets of singularities forming 2-dimensional structures in $\mathbb{R}^3$.

It turns out that, while existing directional multiscale methods do an excellent job of detecting the boundary set $\mathcal{N}$ from $\Omega$, they cannot detect $\mathcal{T}$ since they are designed to deal with different types of geometric structures (cf. §1.1). It was therefore necessary that we develop a new construction intrinsic to the problem at hand. Similar to the classical shearlet approach, our new system is generated by applying anisotropic dilations, shear operations, and translations to a finite set of generating functions, but with some important changes in the choice of shear matrices. Using this approach, we obtain a variant of the continuous shearlet transform which, by its decay at fine scales, can precisely identify the set $\mathcal{T}$. We remark that, while our overall setup bears a superficial resemblance to that of [17], most of our technical results and individual arguments are significantly different.

This paper is organized as follows: For the remainder of the introduction, we motivate our choice of shear and dilation matrices and set down some notation (§1.1 and §1.2); next we state our main theorem about the detection of curvilinear singularities in 3D (§1.3); we also give an example of a "nice" generating function satisfying a reproducing property and examine the properties of the corresponding system (§1.4 and §1.5). In §2, we develop several technical results to prove our main theorem. In §3, we consider generalizations of our main results to other dilation matrices (§3.1) and extensions to higher dimensions (§3.2).

*1.1. Motivation for our choice of dilation and shear matrices*

As mentioned in the previous section, our new analyzing system is generated by applying anisotropic dilations, shear operations, and translations to a finite set of generating functions. In this section, we give some further motivation for the dilation and shear matrices we adopt.

We recall that the analyzing functions associated with the curvelet and shearlet systems are highly anisotropic. In dimension $n = 3$, in particular, dilation matrices are used that make the generating functions "plate-like" at fine scales (essentially supported on parallelepipeds of size $a^2 \times a \times a$, for $0 < a < 1$) and shear and translation operators are then used to move these plates to all locations and "plate" orientations (cf. [1, 17, 18]). This is a natural choice as they are designed to capture surface boundaries. On the other hand, if we now focus on singularities along curve-like structures, a natural approach (and the approach the authors first attempted) is to choose dilation matrices which make our generating functions "stick-like" at fine scales (essentially supported on parallelepipeds of size $a^2 \times a^2 \times a$, for $0 < a < 1$) and then make use of appropriate shear and translation operators to move these sticks to all locations and "stick" orientations. One such dilation/shear matrices combination is

$$a(\alpha) = \begin{pmatrix} \alpha^{\beta_1} & 0 & 0 \\ 0 & \alpha^{\beta_2} & 0 \\ 0 & 0 & \alpha^{\beta_3} \end{pmatrix} \quad \text{and} \quad b(s_1, s_2) = \begin{pmatrix} 1 & 0 & s_1 \\ 0 & 1 & s_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $\beta_1 = \beta_2 > \beta_3$. However, to make much headway into our arguments, we quickly need to assume either $\beta_1 > \beta_2$ or $\beta_1 < \beta_2$. The resulting two new sets of dilation matrices make our generating functions "plank-like" at fine scales. Coupling each of them with sets of shear matrices that orient the respective planks in all "plank" directions, (1.1) is replaced with

$$a^1(\alpha) = \begin{pmatrix} \alpha^{\beta_1} & 0 & 0 \\ 0 & \alpha^{\beta_2} & 0 \\ 0 & 0 & \alpha^{\beta_3} \end{pmatrix} \quad \text{and} \quad b^1(s) = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$a^2(\alpha) = \begin{pmatrix} \alpha^{\beta_2} & 0 & 0 \\ 0 & \alpha^{\beta_1} & 0 \\ 0 & 0 & \alpha^{\beta_3} \end{pmatrix} \quad \text{and} \quad b^2(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix},$$

where $\beta_1 > \beta_2$. The above two dilation/shear matrices combinations (along with, of course, translations) form the basis for the approach we adopted. Note that this approach has the added benefit of significantly reducing redundancy of the shear parameter over that of the "stick-like" shearlets approach.

*1.2. Definitions and notation*

$\Omega$ will always denote a bounded and Lebesgue measurable subset of $\mathbb{R}^3$ and $S_\Omega$ (note the minor change in notation from the introduction) will denote its measure theoretic boundary (see §2.1). For notational convenience, we represent elements of $\mathbb{R}^n$ by both $n \times 1$ column vectors and $1 \times n$ row vectors. If pertinent, the convention adopted in any particular instance is clear from context. For $y \in \mathbb{R}^n$ and $c \in GL_n(\mathbb{R})$, we define the operators $T_y, D_c : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T_y(f)(x) = f(x - y) \quad \text{and} \quad D_c(f)(x) = |\det c|^{-1/2} f(c^{-1}x),$$

We use the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} dx\, f(x) e^{-2\pi \imath \xi \cdot x};$$

$f^\vee$ will denote the inverse Fourier transform of $f \in L^2(\mathbb{R}^n)$.

Fix $\beta_1 > \beta_3 > \beta_2 > 0$ and write $\beta_0 = (\beta_1 - \beta_2 - \beta_3)/2$. For $\alpha > 0$ and $s \in \mathbb{R}$, we define the following matrices:

$$b^{21}(s) = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b^{12}(s) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b^{13}(s) = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$b^{31}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & 0 & 1 \end{pmatrix} \quad b^{32}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{pmatrix} \quad b^{23}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$$

$$a^{21}(\alpha) = \begin{pmatrix} \alpha^{\beta_3} & 0 & 0 \\ 0 & \alpha^{\beta_1} & 0 \\ 0 & 0 & \alpha^{\beta_2} \end{pmatrix} a^{12}(\alpha) = \begin{pmatrix} \alpha^{\beta_1} & 0 & 0 \\ 0 & \alpha^{\beta_3} & 0 \\ 0 & 0 & \alpha^{\beta_2} \end{pmatrix} a^{13}(\alpha) = \begin{pmatrix} \alpha^{\beta_1} & 0 & 0 \\ 0 & \alpha^{\beta_2} & 0 \\ 0 & 0 & \alpha^{\beta_3} \end{pmatrix}$$

$$a^{31}(\alpha) = \begin{pmatrix} \alpha^{\beta_3} & 0 & 0 \\ 0 & \alpha^{\beta_2} & 0 \\ 0 & 0 & \alpha^{\beta_1} \end{pmatrix} a^{32}(\alpha) = \begin{pmatrix} \alpha^{\beta_2} & 0 & 0 \\ 0 & \alpha^{\beta_3} & 0 \\ 0 & 0 & \alpha^{\beta_1} \end{pmatrix} a^{23}(\alpha) = \begin{pmatrix} \alpha^{\beta_2} & 0 & 0 \\ 0 & \alpha^{\beta_1} & 0 \\ 0 & 0 & \alpha^{\beta_3} \end{pmatrix}$$

$$\sigma^{21} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \sigma^{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \sigma^{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma^{31} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \sigma^{32} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \sigma^{23} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\psi \in L^2(\mathbb{R}^3)$, and, for $p \in \mathbb{R}^3$, define

$$\psi^{ij}_{\alpha s p} = T_p D_{b^{ij}(s)} D_{a^{ij}(\alpha)} D_{\sigma^{ij}} \psi. \tag{1.2}$$

Suppose, for the remainder of this paper, unless otherwise specified, that $n$ (the ambient dimension) is 3. The collection $\{\psi^{ij}_{\alpha s p}\}$ induces the 6 continuous transforms $\{\mathcal{S}^{ij}\}$, where

$$\mathcal{S}^{ij}(\Omega)(\alpha, s, p) = \langle \chi_\Omega, \psi^{ij}_{\alpha s p} \rangle, \tag{1.3}$$

and $\chi_\Omega$ denotes the characteristic function of $\Omega$.

We are now in a position to define the main transform of this paper, $\mathcal{S}_{(3,1)}$, which we call the $(3,1)$-*continuous shearlet transform*. The "$(3,1)$" indicates that the transform is designed to capture singularities along 1-dimensional structures in the 3-dimensional ambient space (following this terminology, the continuous transform from [17] would be the $(3,2)$-*continuous shearlet transform*). Let $\mathbb{V}$ denote $(\mathbb{R}^3 \setminus \{0\})/\sim$, where $v \sim w$ if $v = cw$ for some $c \in \mathbb{R} \setminus \{0\}$. Write

$$\mathcal{K} = \{(2,1),(3,1),(1,2),(3,2),(1,3),(2,3)\}$$

and, for $(i,j) \in \mathcal{K}$, define

$$\mathcal{P}_{ij} = \begin{cases} [-1,1], & \text{if } (i,j) = (1,3),(2,3),(1,2) \\ (-1,1), & \text{otherwise} \end{cases}.$$

If $v \in \mathbb{V}$, there exists a unique $j = j(v) \in \{1,2,3\}$ such that $v_j \neq 0$ and $v_i/v_j \in \mathcal{P}_{ij}$, for all $i$, with the quantities $j$ and $v_i/v_j$ well-defined with respect to $\sim$. If $\alpha > 0$, $v \in \mathbb{V}$, and $p \in \mathbb{R}^3$, we define

$$\mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) = \prod_{i \in \{1,2,3\} \setminus \{j\}} S^{ij}(\Omega)(\alpha, v_i/v_j, p).$$

*1.3. Main results*

To formulate and prove our main results, we need the following two definitions to state precisely the notions of *piecewise regular surface*:

**Definition 1.1.** *Let $p \in S_\Omega$ and $K \in \mathbb{Z}^+ \cup \{\infty\}$. We say that $S_\Omega$ is $C^K$ at $p$ if there exists an open set $U \subset \mathbb{R}^3$ with $p \in U$ and $F \in C^K(U,\mathbb{R})$ with $\nabla F(p) \neq 0$ such that*

$$\Omega \cap U = \{x \in U : F(x) < 0\}$$

*(in the a.e. sense; we use a.e. as an abbreviation for almost every[where]). In this case, we call $\mathcal{O}_\Omega(p) = \nabla F(p)$ the orientation of $S_\Omega$ at $p$. Note that $\mathcal{O}_\Omega(p)$ is well-defined (up to nonzero scalar multiplication).*

**Definition 1.2.** *Let $K \in \mathbb{Z}^+ \cup \{\infty\}$ and $p \in S_\Omega$. We say that $S_\Omega$ is piecewise $C^K$ at $p$ if there exists an open set $U \subset \mathbb{R}^3$ with $p \in U$ and $F, G \in C^K(U,\mathbb{R})$ with $F(p) = G(p) = 0$ and $\{\nabla F(p), \nabla G(p)\}$ linearly independent such that*

$$\Omega \cap U = \{x \in U : F(x) < 0\} \Box \{x \in U : G(x) < 0\} \tag{1.4}$$

*(in the a.e. sense), where the symbol $\Box$ can be either $\cap$ or $\cup$. In this case, we call*

$$\mathcal{O}_\Omega(p) = \nabla F(p) \times \nabla G(p)$$

*(where $\times$ is the vector cross product) the orientation of $S_\Omega$ at $p$. Note that $\mathcal{O}_\Omega(p)$ is well-defined (up to nonzero scalar multiplication) and equals the tangent vector at $p$ to the curve defined by $\{x : F(x) = G(x) = 0\}$ near $p$.*

Throughout §2, the assumptions we require the generating function $\psi$ to satisfy vary significantly–from very mild assumptions in Theorem 2.2 to relatively strong assumptions in Theorem 2.9. To handle this, we define three different "admissibility conditions" on $\psi$ in §2. The third such admissibility condition is defined below.

**Definition 1.3.** *Let $K_1, K_2 \in \mathbb{Z}^+$ with $K_1 \geq 2$. We say that $\psi$ is $(K_1, K_2, 3)$-admissible (the "3" indicates that this is the third admissibility condition) if there exists $\psi_q \in L^2(\mathbb{R})$ ($q = 1, 2, 3$) and $r \in \{0, 1\}$ with*

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2)\hat{\psi}_3(\xi_3/\xi_1^r),$$

*for a.e. $\xi$, such that*

*(i)* *$\hat{\psi}_1$ belongs to $C^{K_1}(\mathbb{R})$ and vanishes on an open set containing the origin, $\hat{\psi}_1^{(k)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ ($0 \leq k \leq K_1$), and*

$$\int_{\mathbb{R}} d\xi \, \frac{\hat{\psi}_1(\xi)}{\xi}, \int_{\mathbb{R}} d\xi \, \frac{\hat{\psi}_1(\xi)}{\xi^2} \neq 0.$$

*(ii)* *$\hat{\psi}_2$ belongs to $C^{K_1}(\mathbb{R})$ and is compactly supported, $\hat{\psi}_2^{(q)}(0) = 0$ ($q = 0, 1, 2$), and*

$$\psi_2(0), \int_{\mathbb{R}} d\xi \, \frac{\hat{\psi}_2(\xi)}{\xi} \neq 0.$$

*(iii)* *$\hat{\psi}_3 \in C^{K_1}(\mathbb{R})$, $\hat{\psi}_3^{(k)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ ($0 \leq k \leq K_1$), $\xi^{K_2-1}\hat{\psi}_3 \in L^1(\mathbb{R})$, and $\hat{\psi}_3(0) \neq 0$.*

*If $r = 1$, we also require that*

*(iv)* *$\xi\hat{\psi}_1^{(k)} \in L^1(\mathbb{R})$ ($0 \leq k \leq K_1$).*

*(v)* *$\xi^{K_1}\hat{\psi}_3^{(k)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ ($0 \leq k \leq K_1$).*

*We say that $\psi$ is $(\infty, K_2, 3)$-admissible if $\psi$ is $(K, K_2, 3)$ admissible, for all $K \in \{2, 3, 4, \dots\}$; $(K_1, \infty, 3)$ and $(\infty, \infty, 3)$ admissibility are defined similarly.*

We can now state our main result, which shows that $\mathcal{S}_{(3,1)}$ precisely identifies both $p$ and $\mathcal{O}_\Omega(p)$ when $S_\Omega$ is piecewise $C^\infty$ at $p$.

**Theorem 1.4.** *Let $p \in \mathbb{R}^3$. Suppose that $\beta_1 < 2\beta_2$ and that $\psi$ is $(\infty, \infty, 3)$-admissible. We have the following:*

- *If $p \notin \overline{S_\Omega}$, where $\overline{S_\Omega}$ denotes the closure of $S_\Omega$, or if $S_\Omega$ is $C^\infty$ at $p$, then*

$$\lim_{\alpha \to 0^+} \alpha^{-K} \mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) = 0,$$

*for all $K > 0$ and all $v \in \mathbb{V}$.*

- *Let $v \in \mathbb{V}$ and assume $S_\Omega$ is piecewise $C^\infty$ at $p$. If $v \sim \mathcal{O}_\Omega(p)$, then*

$$\lim_{\alpha \to 0^+} \alpha^{-2(\beta_1+\beta_3+\beta_0)} \mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) \in \mathbb{C} \cup \{\infty\} \setminus \{0\};$$

  *otherwise,*

$$\lim_{\alpha \to 0^+} \alpha^{-K} \mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) = 0,$$

  *for all $K > 0$.*

The arguments and technical tools needed to derive these results are discussed in §2. We make the following remarks regarding the above theorem:

- For simplicity, we state Theorem 1.4 for $(\infty, \infty, 3)$-admissible functions and (piecewise) $C^\infty$ surfaces. However, a version of Theorem 1.4 also holds for $(K_1, K_2, 3)$-admissible functions and (piecewise) $C^{K_2}$ surfaces (see §2).
- Theorem 1.4 does not, in particular, apply to the case $p \in \overline{S_\Omega} \setminus S_\Omega$. If $S_\Omega$ were the topological boundary of $\Omega$, then this case would be vacuous, since then $S_\Omega$ would be closed. However, we make use of a measure theoretic version of the Divergence Theorem in which $S_\Omega$ is the measure theoretic boundary of $\Omega$ (see §1.2). In this more general setup, $S_\Omega$ need not be closed. For example, if $\Omega$ is a solid hyperbolic 3D cone, $S_\Omega$ does not contain the tip of the cone.

The statement of Theorem 1.4 is rather compact and some of its notation rather involved. Additionally, earlier in this section, we remarked that existing (e.g., plate-like) 3D systems, while performing excellently at detecting the piecewise smooth boundary of a 3D solid, are insufficient to characterize its boundary curves. For both of these reasons, we now examine the result of Theorem 1.4 in the context of a simple example and compare this result to what is achievable with existing state-of-the-art 3D plate-like systems. Suppose, then, that $\beta_1 < 2\beta_2$, that $\psi$ is $(\infty, \infty, 3)$-admissible (we construct an example in the next section), and that

$$\Omega \cap U = \{x \in U : x_1 < 0\} \cap \{x \in U : x_2 < 0\},$$

where $U = (-1, 1)^3$. Write $\Gamma = \{x \in U : x_1 = x_2 = 0\}$,

$$S_1 = \{x \in U : x_1 = 0, x_2 < 0\}, \quad \text{and} \quad S_2 = \{x \in U : x_1 < 0, x_2 = 0\}.$$

Then, it follows that

- $\overline{S_\Omega} \cap U = S_1 \cup \Gamma \cup S_2$
- $S_\Omega$ is $C^\infty$ at $p$, for all $p \in S_1 \cup S_2$
- $S_\Omega$ is piecewise $C^\infty$ at $p$, with $\mathcal{O}_\Omega(p) = (0, 0, 1)$ and $j(\mathcal{O}_\Omega(p)) = 3$, for all $p \in \Gamma$.

Thus, Theorem 1.4 implies that

- If $p \in U \setminus \Gamma$, then

$$\lim_{\alpha \to 0^+} \alpha^{-K} \mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) = 0,$$

  for all $K > 0$ and all $v \in \mathbb{V}$.

- Assume $p \in \Gamma$. If $v \sim (0,0,1)$,

$$\lim_{\alpha \to 0^+} \alpha^{-2(\beta_1+\beta_3+\beta_0)} \mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) \in \mathbb{C} \cup \{\infty\} \setminus \{0\};$$

  otherwise,

$$\lim_{\alpha \to 0^+} \alpha^{-K} \mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p) = 0, \tag{1.5}$$

  for all $K > 0$.

In other words, the (3,1)-continuous shearlet transform characterizes both the location and orientation of the singularity curve $\Gamma$ through its asymptotic decay at fine scales. This is illustrated in Figure 1 showing that the slow asymptotic decay of the transform characterizes the location and orientation of $\Gamma$.

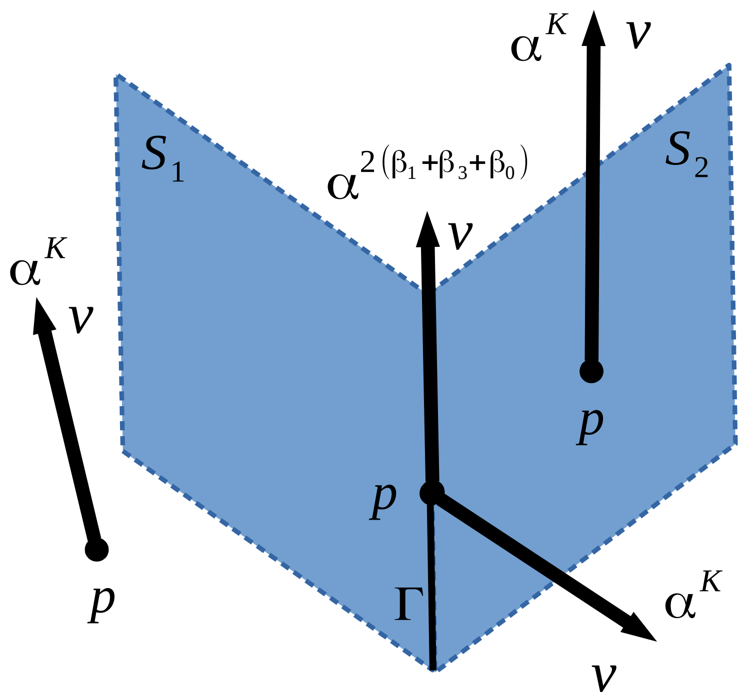


Figure 1: Asymptotic decay rates of $\mathcal{S}_{(3,1)}(\Omega)(\alpha, v, p)$ for various values of $v$ and $p$.

We now apply Theorem 3.1 of [17] to the example of the previous paragraph.

Theorem 3.1 of [17], which regards the detection of the piecewise smooth boundary of a 3D solid by plate-like shearlet systems, is one of the most precise results of its kind available. The shearlet transform of of [17] applied to $\Omega$ will be denoted by $\mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p)$, where $\alpha > 0$ indexes scale, $v \in \mathbb{V}$ orientation, and $p \in \mathbb{R}^3$ location. We have the following results:

- If $p \in U \setminus (\Gamma \cup S_1 \cup S_2)$, then
$$\lim_{\alpha\to 0^+} \alpha^{-K} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) = 0, \tag{1.6}$$
for all $v \in \mathbb{V}$ and all $K > 0$.

- Assume $p \in S_1$. If $v = (1, 0, 0)$ (i.e., the normal vector of $S_1$), then
$$\lim_{\alpha\to 0^+} \alpha^{-1} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) \neq 0;$$
otherwise,
$$\lim_{\alpha\to 0^+} \alpha^{-K} \mathcal{S}_{(3,2)(\Omega)}(\alpha, v, p) = 0,$$
for all $K > 0$.

- Assume $p \in S_2$. If $v = (0, 1, 0)$ (i.e., the normal vector of $S_2$), then
$$\lim_{\alpha\to 0^+} \alpha^{-1} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) \neq 0;$$
otherwise,
$$\lim_{\alpha\to 0^+} \alpha^{-K} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) = 0,$$
for all $K > 0$.

- Assume $p \in \Gamma$. If $v \in \{(1, 0, 0), (0, 1, 0)\}$, then
$$\lim_{\alpha\to 0^+} \alpha^{-1} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) \neq 0;$$
otherwise,
$$\limsup_{\alpha\to 0^+} \alpha^{-3/2} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) < \infty. \tag{1.7}$$

We thus see that $\mathcal{S}_{(3,2)}$ is able to detect the location of $\Gamma$ as all $p$ such that the condition
$$\lim_{\alpha\to 0^+} \alpha^{-K} \mathcal{S}_{(3,2)}(\Omega)(\alpha, v, p) = 0, \text{ for all } K > 0$$
fails for two at least two $v$. In this case, $\mathcal{S}_{(3,2)}$ can then detect the orientation, $(0, 0, 1)$, of $\Gamma$ as the vector cross product of the two unique $v$, $(1, 0, 0)$ and $(0, 1, 0)$, for which (1.7) fails. Comparing these results to the those in the previous paragraph (particularly, (1.5) to (1.7)), we see that while $\mathcal{S}_{(3,2)}$ can detect the location of $\Gamma$ just as precisely as $\mathcal{S}_{(3,1)}$, the latter is much better able to precisely identify the orientation of $\Gamma$.

*1.4. Example of an admissible function satisfying a reproducing property*

Let $0 < \alpha_0, s_0 \le \infty$. In this section, we formulate a reproducing condition on the collection

$$\{\psi^{ij}_{\alpha sp} : (i,j) \in \mathcal{K}, 0 < \alpha < \alpha_0, |s| < s_0, p \in \mathbb{R}^3\} \tag{1.8}$$

and construct a $(\infty, \infty, 3)$-admissible function for which (1.8) satisfies this reproducing property.

It is often desirable that the collection (1.8) forms a so-called continuous reproducing system; i.e., that one can recover the function $f \in L^2(\mathbb{R}^3)$ from the inner products

$$\{\langle f, \psi^{ij}_{\alpha sp}\rangle : (i,j) \in \mathcal{K}, 0 < \alpha < \alpha_0, |s| < s_0, p \in \mathbb{R}^3\}.$$

For our particular setup, it is convenient to formulate the continuous reproducing property as follows: Let $\mu$ be a measure on $\mathbb{R}^3 \times (-s_0, s_0) \times (0, \alpha_0)$ and let $E$ be a Lebesgue measurable subset of $\mathbb{R}^3$. We say (1.8) forms a continuous reproducing system for $L^2(E)^\vee = \{f : \int_E |\hat{f}|^2 < \infty\}$ (with respect to $\mu$) if there exists a collection $\{E_{ij} : (i,j) \in \mathcal{K}\}$ of Lebesgue measurable subsets of $\mathbb{R}^3$ with $E = \bigcup_{(i,j)\in\mathcal{K}} E_{ij}$ (in the a.e. sense) such that

$$f(x) = \int_{\mathbb{R}^3 \times (-s_0,s_0)\times(0,\alpha_0)} d\mu(p,s,\alpha) \langle f, \psi^{ij}_{\alpha sp}\rangle \psi^{ij}_{\alpha sp}(x), \tag{1.9}$$

in the sense of weak convergence in $L^2(E_{ij})^\vee$, for all $f \in L^2(E_{ij})^\vee$ and all $(i,j) \in \mathcal{K}$. The relationship

$$T_p D_{b^{ij}(s)} D_{a^{ij}(\alpha)} T_{\overline{p}} D_{b^{ij}(\overline{s})} D_{a^{ij}(\overline{\alpha})} = T_{p + b^{ij}(s)a^{ij}(\alpha)\overline{p}} D_{b^{ij}(s+\alpha^{\beta_1-\beta_3}\overline{s})} D_{a^{ij}(\alpha\overline{\alpha})}$$

induces a group structure on $G_{ij} = \mathbb{R}^3 \times \mathbb{R} \times (0,\infty)$ with multiplication

$$(p,s,\alpha)(\overline{p},\overline{s},\overline{\alpha}) = (p + b^{ij}(s)a^{ij}(\alpha)\overline{p}, s + \alpha^{\beta_1-\beta_3}\overline{s}, \alpha\overline{\alpha}).$$

A computation shows that $G_{ij}$ has left Haar measure $\lambda$, where $d\lambda(p,s,\alpha) = dp\, ds\, d\alpha/\alpha^{2\beta_1+\beta_2+1}$ (in particular, $\lambda$ does not depend on $i$ and $j$). Often, one chooses $\mu = \lambda$ in (1.9).

We first note that if $\psi$ is $(K_1, K_2, 3)$-admissible, then (1.8) cannot form a continuous reproducing system for $L^2(\{\xi \in \mathbb{R}^3 : |\xi| > R\})^\vee$, for any $R > 0$. Indeed, if there exists $0 < \epsilon, M < \infty$ such that

$$\operatorname{supp}(\hat{\psi}) \subset ((-\infty, -\epsilon] \cup [\epsilon, \infty)) \times [-M, M] \times (-\infty, \infty),$$

then, using (1.16), it follows that

$$\operatorname{supp}(\hat{\psi}^{ij}_{\alpha sp}) \subset \left\{ \xi \in \mathbb{R}^3 : |\xi_k| \le M \left(\frac{\epsilon}{|\xi_i|}\right)^{-\beta_2/\beta_1} \right\},$$

where $\{i,j,k\} = \{1,2,3\}$. Write

$$N = \left\{ \xi \in \mathbb{R}^3 : |\xi_k| > M \left( \frac{\epsilon}{|\xi_i|} \right)^{-\beta_2/\beta_1} \text{ for all } i,k \in \{1,2,3\} \right\}.$$

Then, $N$ is open and unbounded (since $\beta_1 > \beta_2$). Moreover, by the above containment and equality, we have $N \cap \operatorname{supp}(\hat{\psi}^{ij}_{\alpha sp}) = \emptyset$, for all $i,j,\alpha,s,p$. The above assertion now follows.

Despite the negative result of the previous paragraph, it is possible for (1.8) to form a continuous reproducing system for a certain subspace of $L^2(\mathbb{R}^3)$ when $\psi$ is $(\infty,\infty,3)$-admissible. To see this, let $0 < \epsilon < M_1 < \infty$ and $0 < M_2 < M_3 < \infty$ be such that

$$\frac{M_3}{M_2} > \left( \frac{M_1}{\epsilon} \right)^{\beta_2/\beta_1}. \tag{1.10}$$

Choose $\theta_1, \theta_2 \in C^\infty(\mathbb{R}, [0,\infty))$ such that

$$\operatorname{supp}(\theta_1) \subset [\epsilon, M_1], \qquad \int_0^\infty \frac{d\alpha}{\alpha} \theta(\alpha)^2 = \beta_1/2, \tag{1.11}$$

$\theta_2$ is compactly supported in $(0,\infty)$, and

$$|\theta_2(\xi)| = 1, \quad \text{for all } M_2 \le \xi \le M_3. \tag{1.12}$$

For $q = 1,2$, define

$$\theta_q^{\text{even}}(\xi) = \begin{cases} \theta_q(\xi), & \text{if } \xi \ge 0 \\ \theta_q(-\xi), & \text{if } \xi < 0 \end{cases},$$

$$\theta_q^{\text{odd}}(\xi) = \begin{cases} \theta_q(\xi), & \text{if } \xi \ge 0 \\ -\theta_q(-\xi), & \text{if } \xi < 0 \end{cases},$$

and $\psi_q \in L^2(\mathbb{R})$ by $\hat{\psi}_q = \theta_q^{\text{even}} + i\theta_q^{\text{odd}}$. Let $0 < M_4 < \infty$ and choose $\psi_3 \in L^2(\mathbb{R})$ such that $\hat{\psi}_3$ is even, belongs to $C^\infty(\mathbb{R},\mathbb{R})$, and satisfies $\hat{\psi}_3(0) \neq 0$,

$$\operatorname{supp}(\hat{\psi}_3) \subset [-M_4, M_4], \quad \text{and} \quad \|\psi_3\| = 1. \tag{1.13}$$

Define $\psi \in L^2(\mathbb{R}^3)$ by $\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2)\hat{\psi}_3(\xi_3/\xi_1)$. Then, it follows that $\psi$ is $(\infty,\infty,3)$-admissible and real-valued and that $\hat{\psi}$ belongs to $C^\infty(\mathbb{R}^3)$ and is compactly supported. Fix $\alpha_0 < \infty$ and

$$s_0 > \alpha_0^{\beta_1-\beta_3} M_4, \tag{1.14}$$

and define $E_{ij}$ as the set

$$\{\xi \in \mathbb{R}^3 \colon |\xi_i| \ge \tfrac{M_1}{\alpha_0^{\beta_1}},\ M_2(\frac{|\xi_i|}{\epsilon})^{\frac{\beta_2}{\beta_1}} \le |\xi_k| \le M_3(\frac{|\xi_i|}{M_1})^{\frac{\beta_2}{\beta_1}},\ |\xi_j| \le (s_0 - \alpha_0^{\beta_1-\beta_3} M_4)|\xi_i|\}, \tag{1.15}$$

for all $(i,j) \in \mathcal{K}$, where $\{i,j,k\} = \{1,2,3\}$. Write $E = \cup_{(i,j)\in\mathcal{K}} E_{ij}$.

We claim that (1.8) forms a continuous reproducing system for $L^2(E)^\vee$ with respect to $\lambda$ (where $\lambda$ is as defined above). To verify this, it suffices to show that

$$I(\xi) = \int_{(-s_0,s_0)\times(0,\alpha_0)} \frac{ds\,d\alpha}{\alpha^{\beta_1-\beta_3+1}} |\hat{\psi}_1(\alpha^{\beta_1}\xi_1)|^2 |\hat{\psi}_2(\alpha^{\beta_2}\xi_2)|^2 |\hat{\psi}_3(\alpha^{\beta_3-\beta_1}(\tfrac{\xi_3}{\xi_1}+s))|^2 = 1,$$

for a.e. $\xi \in E_{13}$ (see, for instance, [22]). If $\xi \in E_{13}$, we have

$$\begin{aligned} I(\xi) &= \int_0^{\alpha_0} \frac{d\alpha}{\alpha^{\beta_1-\beta_3+1}} |\hat{\psi}_1(\alpha^{\beta_1}\xi_1)|^2 |\hat{\psi}_2(\alpha^{\beta_2}\xi_2)|^2 \int_{-s_0}^{s_0} ds\, |\hat{\psi}_3(\alpha^{\beta_3-\beta_1}(\tfrac{\xi_3}{\xi_1}+s))|^2 \\ &= \int_0^{\alpha_0} \frac{d\alpha}{\alpha} |\hat{\psi}_1(\alpha^{\beta_1}\xi_1)|^2 |\hat{\psi}_2(\alpha^{\beta_2}\xi_2)|^2 \int_{\alpha^{\beta_3-\beta_1}(\xi_3/\xi_1-s_0)}^{\alpha^{\beta_3-\beta_1}(\xi_3/\xi_1+s_0)} dt\, |\hat{\psi}_3(t)|^2 \\ &= \int_0^{\alpha_0} \frac{d\alpha}{\alpha} |\hat{\psi}_1(\alpha^{\beta_1}\xi_1)|^2 |\hat{\psi}_2(\alpha^{\beta_2}\xi_2)|^2 \\ &= \frac{1}{\beta_1} \int_0^{\alpha_0^{\beta_1}|\xi_1|} \frac{d\gamma}{\gamma} |\hat{\psi}_1(\operatorname{sgn}(\xi_1)\gamma)|^2 \left| \hat{\psi}_2\left(\frac{\gamma^{\beta_2/\beta_1}\xi_2}{|\xi_1|^{\beta_2/\beta_1}}\right)\right|^2 = 1, \end{aligned}$$

where the second equality follows from the change of variable $t = \alpha^{\beta_3-\beta_1}(\xi_3/\xi_1 + s)$, the third equality from (1.14) and (1.13), the fourth equality from the change of variable $\gamma = \alpha^{\beta_1}|\xi_1|$, and the fifth from (1.10), (1.11), and (1.12). This verifies the above assertion.

*1.5. Frequency support*

Fix $\beta_1 = 3$, $\beta_2 = 2$, $\beta_3 = 5/2$. Following the notation and approach in §1.4, one can construct a $(\infty,\infty,3)$-admissible function $\psi$ with

$$\epsilon = 1/2 \quad M_1 = 3/2 \quad M_2 = 0.6 \quad M_3 = 1.4 \quad s_0 = 2 \quad \alpha_0 = 1 \quad M_4 = 1$$

and

$$\operatorname{supp}(\hat{\psi}) = ([-3/2,-1/2]\cup[1/2,3/2])^2 \times [-1,1]$$

such that (1.8) forms a continuous reproducing system for $L^2(E)^\vee$ with respect to $\lambda$. Note that $\beta_1,\beta_2,\beta_3$ and $\psi$ satisfy the hypotheses of Theorem 1.4. In this section, we examine the support of the functions $\hat{\psi}^{ij}_{\alpha s p}$ and the structure of the sets $E_{ij}$.

Note that

$$\hat{\psi}^{ij}_{\alpha s p}(\xi) = \alpha^{(\beta_1+\beta_2+\beta_3)/2} e^{-2\pi\imath\xi\cdot p} \hat{\psi}(\alpha^{\beta_1}\xi_i, \alpha^{\beta_2}\xi_k, \alpha^{\beta_3}(\xi_j + s\xi_i)), \tag{1.16}$$

for a.e. $\xi$, where $\{i,j,k\} = \{1,2,3\}$ (unless indicated otherwise, whenever $\xi, \xi_1, \xi_2, \ldots$ appear in the same context, it is assumed that $\xi = (\xi_1, \xi_2, \ldots)$). It follows that $\operatorname{supp}(\hat{\psi}^{ij}_{\alpha s p})$ equals

$$\{\xi \in \mathbb{R}^3 : |\xi_i| \in [\alpha^{-\beta_1}/2, 3\alpha^{-\beta_1}/2], |\xi_k| \in [\alpha^{-\beta_2}/2, 3\alpha^{-\beta_2}/2], |\xi_j/\xi_i + s| \le \alpha^{\beta_1-\beta_3}\}.$$

Informally, $\operatorname{supp}(\hat{\psi}^{ij}_{\alpha sp})$ consists of a plank centered at each of the four points $(\xi_i, \xi_k, \xi_j) = (\alpha^{-\beta_1}, \pm\alpha^{-\beta_2}, -s\alpha^{-\beta_1})$ and $(\xi_i, \xi_k, \xi_j) = (-\alpha^{-\beta_1}, \pm\alpha^{-\beta_2}, s\alpha^{-\beta_1})$. Each plank has long axis in direction $(\xi_i, \xi_k, \xi_j) = (1, 0, -s)$ and short axis in direction $(\xi_i, \xi_k, \xi_j) = (0, 1, 0)$. These planks become more elongated as $\alpha \to 0^+$ (see Figure 2). Plugging our particular parameter values into (1.15), we have

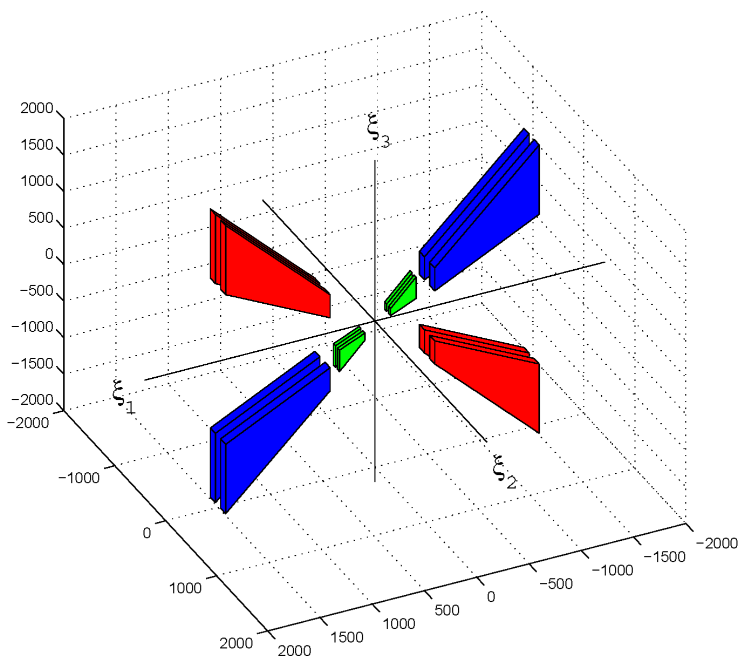


Figure 2: Supports of the functions $\hat{\psi}^{13}_{\alpha sp}$ for $(\alpha, s) = (0.1, 1)$ (blue), $(\alpha, s) = (0.1, -1)$ (red), $(\alpha, s) = (0.16, 1)$ (green).

$$E_{ij} = \{\xi \in \mathbb{R}^3 : |\xi_i| \geq \frac{3}{2},\, 0.6(\frac{1}{2})^{\frac{2}{3}}|\xi_i|^{\frac{2}{3}} \leq |\xi_k| \leq 1.4(\frac{2}{3})^{\frac{2}{3}}|\xi_i|^{\frac{2}{3}},\, |\xi_j| \leq |\xi_i|\}$$

(see Figure 3).

## 2. The transforms $\mathcal{S}^{ij}$

In this section, we examine the asymptotic decay of the transforms $\mathcal{S}^{ij}$. We will prove all our results for the case $(i, j) = (1, 3)$ only. The general case follows from Lemma 2.12. To ease notation, write $b = b^{13}$, $a = a^{13}$, and $\mathcal{S} = \mathcal{S}^{13}$. Our analysis of the asymptotic decay of $\mathcal{S}^{ij}(\Omega)(\alpha, s, p)$ will be split into several cases: $p \notin \overline{S_\Omega}$ (§2.2), $S_\Omega$ $C^K$ at $p$ (§2.3), and $S_\Omega$ piecewise $C^K$ at $p$ (§2.4). In section §2.1, we make use of a clever application (from [17]) of the divergence theorem that allows us to rewrite $\hat{\chi}_\Omega$ as an integral over $S_\Omega$.

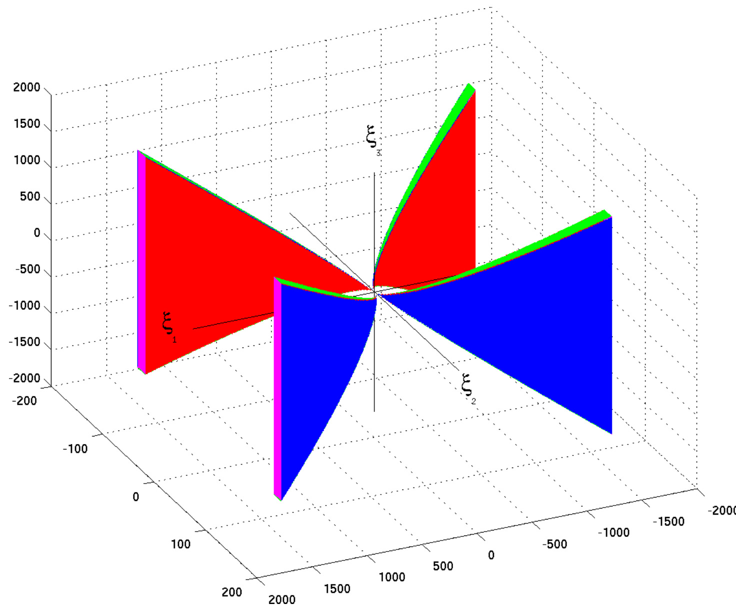


Figure 3: The set $E_{13} \cap \{\xi \in \mathbb{R}^3 : |\xi_1| \leq 1500\}$. The colors only serve to distinguish between the various branches and faces. For visualization purposes, the axes are not to scale.

*2.1. The Divergence Theorem*

Let $h$ denote 2-dimensional Hausdorff measure on $\mathbb{R}^3$ (§2.1 of [10]). $S_\Omega$ is defined by $x \in S_\Omega$ if

$$\limsup_{r\to 0} \frac{|B_r(x) \cap \Omega|}{r^3} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{|B_r(x) \setminus \Omega|}{r^3} > 0, \tag{2.1}$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$ and $B_r(x) = \{y \in \mathbb{R}^3 : \|y - x\| < r\}$ (norms without subscripts are assumed to be $L^2$-norms). Then, $S_\Omega$ is a Borel measurable subset of $\mathbb{R}^3$ (§5.11 of [10]) and hence $h$-measurable. Assume $h(S_\Omega) < \infty$. Using Theorem 1 of §5.11 of [10], the divergence theorem (the Gauss-Green Theorem of [10]), and that $\Omega$ is bounded, it follows that

$$\begin{aligned} \hat{\chi}_\Omega(\xi) &= \int_{\mathbb{R}^3} dx\, \chi_\Omega(x) e^{-2\pi\imath\xi\cdot x} \\ &= \int_\Omega dx \,\operatorname{div}\left[\frac{e^{-2\pi\imath\xi\cdot x}}{-2\pi\imath\|\xi\|^2}\xi\right] \\ &= -\frac{1}{2\pi\imath\|\xi\|^2}\int_{S_\Omega} dh(x) e^{-2\pi\imath\xi\cdot x}\xi\cdot n_\Omega(x), \end{aligned} \tag{2.2}$$

for all $\xi \neq 0$, where $n_\Omega \in L^\infty((S_\Omega, h), \mathbb{R}^3)$ is the measure theoretic unit outward normal to $\Omega$. That is, for $h$-a.e. $x \in S_\Omega$, $n_\Omega(x)$ is uniquely determined by:

$\|n_\Omega(x)\| = 1$,

$$\lim_{r\to 0} \frac{|B_r(x)\cap\Omega\cap H(x,n_\Omega(x))|}{r^3} = 0, \text{ and } \lim_{r\to 0} \frac{|(B_r(x)\setminus\Omega)\cap H(x,-n_\Omega(x))|}{r^3} = 0, \tag{2.3}$$

where, for $p,\nu \in \mathbb{R}^3$, $H(p,\nu) = \{y\in\mathbb{R}^3 : \nu\cdot(y-p)\geq 0\}$ (cf. §5.1, §5.7, §5.8 of [10]).

Using (1.16) and (2.2), it follows that (recall that $\mathcal{S} = \mathcal{S}^{13}$)

$$\begin{aligned}\mathcal{S}(\Omega)(\alpha,s,p) &= \langle \hat\chi_\Omega, \hat\psi^{13}_{\alpha sp}\rangle \\ &= \int_{\mathbb{R}^3} d\xi\, \hat\chi_\Omega(\xi)\overline{\hat\psi^{13}_{\alpha sp}(\xi)} \\ &= -\frac{\alpha^{(\beta_1+\beta_2+\beta_3)/2}}{2\pi\imath}\int_{\mathbb{R}^3} d\xi\, \frac{\overline{\hat\psi(\alpha^{\beta_1}\xi_1,\alpha^{\beta_2}\xi_2,\alpha^{\beta_3}(\xi_3+s\xi_1))}}{\xi_1^2+\xi_2^2+\xi_3^2} \\ &\qquad\times\int_{S_\Omega} dh(x)e^{-2\pi\imath\xi\cdot(x-p)}\xi\cdot n_\Omega(x).\end{aligned} \tag{2.4}$$

*2.2. Fast decay away from S*

In this section, we prove that $\mathcal{S}^{ij}(\Omega)(\alpha,s,p)$ decays "fast" when $p\notin\overline{S_\Omega}$ (Theorem 2.2). Theorem 2.2 follows from Lemma 2.3, which is our version of the so-called *localization lemma* from [17]. Lemma 2.3 also enables us, in the main results of §2.3 and §2.4, to localize the inner intergral of (2.4) near $p$. Below is our first admissibility condition.

**Definition 2.1.** *Let $K\in\{1,2,3,\dots\}$. We say that $\psi$ is $(K,1)$-admissible if*

- *(i) $\hat\psi\in C^K(\mathbb{R}^3)$*
- *(ii) $\partial^\omega\hat\psi\in L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, for all $0\leq|\omega|\leq K-1$ (we are using multi-index notation as in §8.1 of [12]), and that*
- *(iii) $\partial^\omega\hat\psi/\xi_1^{K+1-|\omega|}\in L^1(\mathbb{R}^3)$, for all $0\leq|\omega|\leq K$.*

Below is the main result of this subsection.

**Theorem 2.2.** *Let $K\in\mathbb{Z}^+$ and suppose that $\psi$ is $(K,1)$-admissible. If $p\notin\overline{S_\Omega}$, then*

$$\limsup_{\alpha\to 0^+}\alpha^{-K\beta_2-\beta_0}|\mathcal{S}^{ij}(\Omega)(\alpha,s,p)|<\infty,$$

*for all $i$, $j$, and $s$.*

Theorem 2.2 follows directly from Lemma 2.12, allowing us to reduce the proof to the case of $\mathcal{S}^{13}$, equality (2.4) and the estimate of Lemma 2.3 below.

**Lemma 2.3.** *Let $K\in\mathbb{Z}^+$, let $\theta\in L^\infty(S_\Omega,h)$, and let $U\subset\mathbb{R}^3$ be open with $0\in U$. Suppose that $\psi$ is $(K,1)$-admissible. For $\alpha>0$, define*

$$I(\alpha) = \frac{-\alpha^{(\beta_1+\beta_2+\beta_3)/2}}{2\pi\imath}\int_{\mathbb{R}^3} d\xi\frac{\overline{\hat\psi(\alpha^{\beta_1}\xi_1,\alpha^{\beta_2}\xi_2,\alpha^{\beta_3}\xi_3)}}{\xi_1^2+\xi_2^2+\xi_3^2}\int_{S_\Omega\setminus U} dh(x)\theta(x)e^{-2\pi\imath\xi\cdot x}\xi\cdot n_\Omega(x).$$

*Then*

$$\limsup_{\alpha\to 0^+} \alpha^{-K\beta_2-\beta_0}|I(\alpha)| < \infty.$$

We need the following "repeated integration by parts" lemma (whose proof follows easily from induction and the standard integration by parts result) in the proof of Lemma 2.3.

**Lemma 2.4.** *Let $J \in \mathbb{Z}^+$ and let $f, g \in C^J(\mathbb{R})$ be such that $f^{(j)}g^{(J-1-j)}$ vanishes at $\infty$, for all $j = 0, \dots, J-1$, and $f^{(j)}g^{(J-j)} \in L^1(\mathbb{R})$, for all $j = 0, \dots, J$. Then,*

$$\int_{\mathbb{R}} dx\, f(x)g^{(J)}(x) = (-1)^J \int_{\mathbb{R}} dx\, f^{(J)}(x)g(x).$$

We now prove Lemma 2.3.

*Proof of Lemma 2.3.* Fix $0 < \alpha \le 1$. Using the change of variable $\eta_q = \alpha^{\beta_q}\xi_q$ ($q = 1, 2, 3$), it follows that

$$\begin{aligned}
I(\alpha) &= -\frac{\alpha^{(-\beta_1-\beta_2-\beta_3)/2}}{2\pi\imath} \int_{\mathbb{R}^3} d\eta\, \frac{\overline{\hat{\psi}(\eta)}}{\alpha^{-2\beta_1}\eta_1^2 + \alpha^{-2\beta_2}\eta_2^2 + \alpha^{-2\beta_3}\eta_3^2} \\
&\times \int_{S_\Omega\setminus U} dh(x)\, \theta(x)(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot n_\Omega(x) e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot x} \\
&= -\frac{\alpha^{\beta_0}}{2\pi\imath} \int_{\mathbb{R}^3} d\eta\, \frac{\overline{\hat{\psi}(\eta)}}{\eta_1^2 + \alpha^{2(\beta_1-\beta_2)}\eta_2^2 + \alpha^{2(\beta_1-\beta_3)}\eta_3^2} \qquad (2.5) \\
&\times \int_{S_\Omega\setminus U} dh(x)\, \theta(x)(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)\cdot n_\Omega(x) e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot x}.
\end{aligned}$$

Note also that

$$\begin{aligned}
&\int_{\mathbb{R}^3} d\eta \left| \frac{\overline{\hat{\psi}(\eta)}}{\eta_1^2 + \alpha^{2(\beta_1-\beta_2)}\eta_2^2 + \alpha^{2(\beta_1-\beta_3)}\eta_3^2} \right| \\
&\quad \times \int_{S_\Omega\setminus U} dh(x) \left| \theta(x)(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)\cdot n_\Omega(x) e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot x} \right| \\
&\quad \le \|\theta\|_\infty h(S_\Omega) \int_{\mathbb{R}^3} d\eta\, \frac{|\hat{\psi}(\eta)|}{(\eta_1^2 + \alpha^{2(\beta_1-\beta_2)}\eta_2^2 + \alpha^{2(\beta_1-\beta_3)}\eta_3^2)^{1/2}} \qquad (2.6) \\
&\quad \le \|\theta\|_\infty h(S_\Omega) \int_{\mathbb{R}^3} d\eta\, \frac{|\hat{\psi}(\eta)|}{|\eta_1|} < \infty,
\end{aligned}$$

where, in the last inequality, we have used properties (ii) and (iii) in Definition 2.1. Choose $\epsilon > 0$ and pairwise disjoint Borel measurable subsets $S_q \subset \mathbb{R}^3$, for $q = 1, 2, 3$, satisfying

$$S_q \subset \{x \in \mathbb{R}^3 : |x_q| \ge \epsilon\}, \qquad (2.7)$$

for all $q$, where $S_\Omega \setminus U = S_1 \cup S_2 \cup S_3$. Then, using (2.5), (2.6), and the

Fubini-Tonelli theorem it follows that

$$I(\alpha) = -\frac{\alpha^{\beta_0}}{2\pi\imath} \sum_{q=1,2,3} \int_{S_q} dh(x)\,\theta(x) \int_{\mathbb{R}^3} d\eta\, f_\alpha(x,\eta) e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot x}, \tag{2.8}$$

where $f_\alpha : S \times \mathbb{R}^3 \to \mathbb{C}$ is defined by

$$f_\alpha(x,\eta) = \frac{(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)\cdot n_\Omega(x)}{\eta_1^2 + \alpha^{2(\beta_1-\beta_2)}\eta_2^2 + \alpha^{2(\beta_1-\beta_3)}\eta_3^2} \overline{\hat{\psi}(\eta)},$$

for a.e. $(x,\eta)$. We require the following claim, whose proof is a straightforward application of induction and the quotient rule.

**Claim**. For each $q \in \{1,2,3\}$ and $k \in \{0,\dots,K\}$, there exists $L_k^q \in \mathbb{Z}^+$ and, for each $l = 1,\dots,L_k^q$, there exist $\gamma_l^{qk} \geq 0$, $c_l^{qk} \in L^\infty(S_\Omega, h)$ not depending on $\alpha$ or $\eta$, $m_l^{qk} : \mathbb{R}^3 \to \mathbb{R}$, a monomial not depending on $\alpha$ or $x$, and $\omega_l^{qk}$ a multi-index with $|\omega_l^{qk}| \leq k$ and $|\omega_l^{qk}| = \deg(m_l^{qk}) - 2^{k+1} + k + 1$ such that

$$\frac{\partial^k}{\partial \eta_q^k} f_\alpha(x,\eta) = \sum_{l=1}^{L_k^q} \frac{\alpha^{\gamma_l^{qk}} c_l^{qk}(x) m_l^{qk}(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)}{(\eta_1^2 + \alpha^{2(\beta_1-\beta_2)}\eta_2^2 + \alpha^{2(\beta_1-\beta_3)}\eta_3^2)^{2^k}} \overline{\partial^{\omega_l^{qk}} \hat{\psi}(\eta)},$$

for a.e. $(x,\eta)$. We are using monomial in the strict sense (i.e., $\eta_1\eta_3$ is a monomial but $-\eta_1\eta_3$ and $2\eta_1\eta_3$ are not). $\square$

If $q \in \{1,2,3\}$, choose $r$ and $s$ such that $\{q,r,s\} = \{1,2,3\}$. If $m : \mathbb{R}^3 \to \mathbb{R}$ is a monomial and $\gamma \in \mathbb{R}$, then, by switching to spherical coordinates, it is clear that $|m(\eta)|/\|\eta\|^\gamma \leq 1/\|\eta\|^{\gamma - \deg(m)}$, for all $\eta \neq 0$. Using this and the claim, if $k \in \{0,\dots,K\}$, we have

$$\begin{aligned} \left| \frac{\partial^k}{\partial \eta_q^k} f_\alpha(x,\eta) \right| &\leq \sum_{l=1}^{L_k^q} \|c_l^{qk}\|_\infty \left| \frac{m_l^{qk}(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)}{\eta_1^2 + \alpha^{2(\beta_1-\beta_2)}\eta_2^2 + (\alpha^{2(\beta_1-\beta_3)}\eta_3^2)^{2^k}} \right| |\partial^{\omega_l^{qk}} \hat{\psi}(\eta)| \\ &\leq \sum_{l=1}^{L_k^q} \|c_l^{qk}\|_\infty \frac{|\partial^{\omega_l^{qk}} \hat{\psi}(\eta)|}{\|(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)\|^{k+1-|\omega_l^{qk}|}} \\ &\leq \sum_{l=1}^{L_k^q} \|c_l^{qk}\|_\infty \frac{|\partial^{\omega_l^{qk}} \hat{\psi}(\eta)|}{|\eta_1|^{k+1-|\omega_l^{qk}|}}, \end{aligned} \tag{2.9}$$

for a.e. $(x,\eta)$. The second inequality, together with the claim and property (ii) of Definition 2.1, implies that $\frac{\partial^k}{\partial \eta_q^k} f_\alpha(x,\cdot)$ vanishes at $\infty$, for $k = 0,\dots,K-1$ and $h$-a.e. $x$. The third inequality, together with the claim and properties (ii) and (iii) of Definition 2.1 implies that $\frac{\partial^k}{\partial \eta_q^k} f_\alpha(x,\cdot) \in L^1(\mathbb{R}^3)$, for $k = 0,\dots,K$ and $h$-a.e. $x$. Using these observations, the Fubini-Tonelli theorem, Lemma 2.4, (2.7), (2.9), the claim, and property (i) of Definition 2.1, we obtain

$$\begin{aligned}
&\left| \int_{S_q} dh(x)\, \theta(x) \int_{\mathbb{R}^3} d\eta\, f_\alpha(x,\eta) e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot x} \right| \\
&\le \|\theta\|_\infty \int_{S_q} dh(x) \int_{\mathbb{R}^2} d\eta_r \otimes d\eta_s \left| \int_{\mathbb{R}} d\eta_q\, f_\alpha(x,\eta) e^{-2\pi\imath\alpha^{-\beta_q}\eta_q x_q} \right| \\
&= \|\theta\|_\infty \int_{S_q} dh(x) \int_{\mathbb{R}^2} d\eta_r \otimes d\eta_s \left| \int_{\mathbb{R}} d\eta_q\, f_\alpha(x,\eta) \frac{\partial^K}{\partial \eta_q^K} \left( \frac{e^{-2\pi\imath\alpha^{-\beta_q}\eta_q x_q}}{\{-2\pi\imath\alpha^{-\beta_q} x_q\}^K} \right) \right| \\
&\le \frac{\|\theta\|_\infty \alpha^{K\beta_q}}{(2\pi\epsilon)^K} \int_{S_q} dh(x) \int_{\mathbb{R}^3} d\eta \left| \frac{\partial^K}{\partial \eta_q^K} f_\alpha(x,\eta) \right| \\
&\le \frac{\|\theta\|_\infty h(S_\Omega) \alpha^{K\beta_q}}{(2\pi\epsilon)^K} \sum_{l=1}^{L_K^q} \|c_l^{qK}\|_\infty \|\partial^{\omega_l^{qK}} \hat{\psi}/|\eta_1|^{K+1-|\omega_l^{qK}|}\|_1.
\end{aligned}$$

The lemma follows from the above inequality and (2.8). □

*2.3. Fast Decay at Smooth Boundary Points*

In this section, we prove that $\mathcal{S}^{ij}(\Omega)(\alpha, s, p)$ decays "fast" when $S_\Omega$ is $C^K$ at $p$, for most values of $s$ (Theorem 2.6). Below is our second admissibility condition.

**Definition 2.5.** *Let $K \in \mathbb{Z}^+$. We say that $\psi$ is $(K,2)$-admissible if*

$$\frac{\xi_2^{k_2}\xi_3^{k_3}\hat{\psi}}{\xi_1^K} \in L^1(\mathbb{R}^3),$$

*for all $|(k_2,k_3)| \le K-1$ (we are considering $(k_2,k_3)$ as a multi-index).*

Below is the main result of this section.

**Theorem 2.6.** *Let $p \in S_\Omega$ and $K_1, K_2 \in \mathbb{Z}^+$ with $K_2 \ge 2$. Suppose that $\psi$ is $(K_1,1)$- and $(K_2,2)$-admissible and that $S_\Omega$ is $C^{K_2}$ at $p$. Then,*

$$\limsup_{\alpha\to 0^+} \alpha^{-\beta-\beta_0} |\mathcal{S}^{ij}(\Omega)(\alpha,s,p)| < \infty,$$

*where*

*(i) $\beta = \min\{K_1\beta_2, (K_2-1)(\beta_1-\beta_3)\}$, if $s\mathcal{O}_\Omega(p)_i + \mathcal{O}_\Omega(p)_j \neq 0$.*

*(ii) $\beta = \min\{K_1\beta_2, (K_2-1)(\beta_1-\beta_2)\}$, if $\mathcal{O}_\Omega(p)_k \neq 0$, where $k$ is such that $\{i,j,k\} = \{1,2,3\}$.*

We require the following lemma in the proof of Theorem 2.6.

**Lemma 2.7.** *Let $U$ be an open subset of $\mathbb{R}^3$, let $f \in C^2(\mathbb{R}^2, \mathbb{R})$, let $\triangle \in \{<, >\}$, and write*

$$\sigma = \begin{cases} 1, & \text{if } \triangle = < \\ -1, & \text{if } \triangle = > \end{cases}.$$

*(i) If*

$$\Omega \cap U = \{x \in U : x_3 \triangle f(x_1, x_2)\}, \tag{2.10}$$

*(in the a.e. sense), then*

$$S_\Omega \cap U = \{x \in U : x_3 = f(x_1, x_2)\} \tag{2.11}$$

*and, for h-a.e. $p \in S_\Omega \cap U$, we have*

$$n_\Omega(p) = \frac{\sigma(-\partial_1 f(p_1, p_2), -\partial_2 f(p_1, p_2), 1)}{\sqrt{\partial_1 f(p_1, p_2)^2 + \partial_2 f(p_1, p_2)^2 + 1}}. \tag{2.12}$$

*(ii) If*

$$\Omega \cap U = \{x \in U : x_2 \triangle f(x_1, x_3)\},$$

*(in the a.e. sense), then*

$$S_\Omega \cap U = \{x \in U : x_2 = f(x_1, x_3)\}$$

*and, for h-a.e. $p \in S_\Omega \cap U$, we have*

$$n_\Omega(p) = \frac{\sigma(-\partial_1 f(p_1, p_3), 1, -\partial_2 f(p_1, p_3))}{\sqrt{\partial_1 f(p_1, p_3)^2 + \partial_2 f(p_1, p_3)^2 + 1}}.$$

*(iii) If*

$$\Omega \cap U = \{x \in U : x_1 \triangle f(x_2, x_3)\},$$

*(in the a.e. sense), then*

$$S_\Omega \cap U = \{x \in U : x_1 = f(x_2, x_3)\}$$

*and, for h-a.e. $p \in S_\Omega \cap U$, we have*

$$n_\Omega(p) = \frac{\sigma(1, -\partial_1 f(p_2, p_3), -\partial_2 f(p_2, p_3))}{\sqrt{\partial_1 f(p_2, p_3)^2 + \partial_2 f(p_2, p_3)^2 + 1}}.$$

*Proof of Lemma 2.7.* We assume $\triangle = <$ and only prove part (i); the other cases follow from this special case and Lemma 2.12. Suppose (2.10) holds in the a.e. sense. We first verify (2.11). To show the first containment, assume that $p \notin \{x \in U : x_3 = f(x_1, x_2)\}$. We want to show that $p \notin S_\Omega \cap U$. We may assume that $p \in U$. Then, $p \in U^- \cup U^+$, where

$$U^- = \{x \in U : x_3 < f(x_1, x_2)\} \quad \text{and} \quad U^+ = \{x \in U : x_3 > f(x_1, x_2)\}.$$

Since $f \in C(\mathbb{R}^2, \mathbb{R})$ and $U$ is open, $U^-$ and $U^+$ are both open. Also, it follows from (2.10) that $|U^- \setminus \Omega| = |U^+ \cap \Omega| = 0$. By examining (2.1), it follows that $p \notin S_\Omega$. This shows $S_\Omega \cap U \subset \{x \in U : x_3 = f(x_1, x_2)\}$. To verify the second containment, assume

$$p \in \{x \in U : x_3 = f(x_1, x_2)\}. \tag{2.13}$$

We want to show that $p \in S_\Omega \cap U$, for which it suffices to verify (2.1). We show that

$$\limsup_{r \to 0} \frac{|B_r(p) \cap \Omega|}{r^3} > 0. \tag{2.14}$$

The other inequality of (2.1) is verified in a similar fashion. Let $r > 0$ be small enough such that $B_r(p) \subset U$. Using that $f \in C^1(\mathbb{R}^2, \mathbb{R})$, Taylor's theorem, (2.13), and the equivalence of norms on $\mathbb{R}^2$, it follows that there exists $A \geq 0$ such that $f(x_1, x_2) \geq p_3 - A\sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2}$, for all $(x_1 - p_1)^2 + (x_2 - p_2)^2 < r^2$. Using (2.10), the above inequality, and the translation invariance of Lebesgue measure, we have

$$\begin{aligned}
|B_r(p) \cap \Omega| &= |\{x \in B_r(p) : x_3 < f(x_1, x_2)\}| \\
&\geq |\{x \in B_r(p) : x_3 < p_3 - A\sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2}\}| \\
&= |\{x \in B_r(0) : x_3 < -A\sqrt{x_1^2 + x_2^2}\}| \\
&= \frac{r^3}{3}\left(1 - \frac{A}{\sqrt{1 + A^2}}\right),
\end{aligned}$$

where the last equality is obtained by integrating in cylindrical coordinates. Note that $1 - A/\sqrt{1 + A^2} > 0$ for all $A \geq 0$; (2.14) follows.

To verify (2.12), let $p \in S_\Omega \cap U$ and write

$$\nu = \frac{(-\partial_1 f(p_1, p_2), -\partial_2 f(p_1, p_2), 1)}{\sqrt{\partial_1 f(p_1, p_2)^2 + \partial_2 f(p_1, p_2)^2 + 1}}.$$

We clearly have $\|\nu\| = 1$. We show that

$$\lim_{r \to 0} \frac{|B_r(p) \cap \Omega \cap H(p, \nu)|}{r^3} = 0; \tag{2.15}$$

the other equality of (2.3) is verified similarly. Let $r > 0$ be small enough such that $B_r(p) \subset U$. Using that $f \in C^2(\mathbb{R}^2, \mathbb{R})$, Taylor's theorem, and (2.11), it follows that there exists $A \geq 0$ such that

$$|f(x_1 + p_1, x_2 + p_2) - p_3 - x_1 \partial_1 f(p_1, p_2) - x_2 \partial_2 f(p_1, p_2)| \leq A(x_1^2 + x_2^2), \tag{2.16}$$

for all $(x_1 - p_1)^2 + (x_2 - p_2)^2 < r^2$. Using (2.10) and the translation invariance of Lebesgue measure, we have

$$
\begin{aligned}
&|B_r(p) \cap \Omega \cap H(p,\nu)| \\
&= |\{x \in B_r(p) \colon p_3 + (x_1 - p_1)\partial_1 f(p_1,p_2) + (x_2 - p_2)\partial_2 f(p_1,p_2) \le x_3 < f(x_1,x_2)\}| \\
&= |\{x \in B_r(0) \colon x_1\partial_1 f(p_1,p_2) + x_2\partial_2 f(p_1,p_2) \le x_3 < f(x_1+p_1,x_2+p_2) - p_3\}| \\
&\subset |\{x \in C_r^3(0) \colon x_1\partial_1 f(p_1,p_2) + x_2\partial_2 f(p_1,p_2) \le x_3 < f(x_1+p_1,x_2+p_2) - p_3\}| \\
&\le \frac{A\pi r^4}{2},
\end{aligned}
$$

where $C_r^3(0) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < r^2\}$ and the last inequality follows by integrating in cylindrical coordinates and using (2.16). This verifies (2.15); (2.12) follows. $\square$

We now prove Theorem 2.6.

*Proof of Theorem 2.6.* Choose $F$ and $U$ as in Definition 1.1. It follows from Lemma 2.12 that we may assume $i = 1$, $j = 3$, $s = 0$, and $p = 0$. Moreover, the proofs of (i) and (ii) are similar enough that we only prove (i). Suppose, then, that $\partial_3 F(0) \neq 0$. We require the following claims:

**Claim A**. There exists $g \in C^{K_2}(\mathbb{R}^2,\mathbb{R})$ with $g(0) = 0$ and compactly supported functions $\phi_q \in C^{K_2-1}(\mathbb{R}^2,\mathbb{R})$ $(q = 1,2,3)$ such that the following holds:

$$\mathcal{S}(\Omega)(\alpha,0,0) = \mathcal{S}_0(\alpha) + \mathcal{S}_1(\alpha), \tag{2.17}$$

for all $\alpha > 0$, where $S_j : (0,\infty) \to \mathbb{C}$ $(j = 0,1)$ satisfy

$$\limsup_{\alpha \to 0^+} \alpha^{-K_1\beta_2 - \beta_0} |\mathcal{S}_1(\alpha)| < \infty \tag{2.18}$$

and

$$|\mathcal{S}_0(\alpha)| \le \frac{\alpha^{\beta_0}}{2\pi} \sum_{q=1,2,3} \int_{\mathbb{R}^3} d\xi \frac{|\hat{\psi}(\xi)|}{|\xi_1|} \left| \int_{\mathbb{R}^2} dx\, \phi_q(x) e^{-2\pi\imath(\alpha^{-\beta_1}\xi_1, \alpha^{-\beta_2}\xi_2, \alpha^{-\beta_3}\xi_3)\cdot(x,g(x))} \right|, \tag{2.19}$$

where

$$\partial_q g(0) = \begin{cases} -\partial_1 F(0)/\partial_3 F(0), & \text{if } q = 1 \\ -\partial_2 F(0)/\partial_3 F(0), & \text{if } q = 2 \end{cases}. \tag{2.20}$$

$\square$

**Claim B**. Let $J \in \mathbb{Z}^+$, let $f, g \in C^J(\mathbb{R}^2)$ with $f$ compactly supported, and let $\lambda \in \mathbb{R}$. Then, there exist compactly supported $h_0, \ldots, h_J \in C(\mathbb{R}^2)$ depending on $f$ and $g$ but not on $\lambda$ such that

$$\partial_1^J (f e^{\lambda g}) = e^{\lambda g} \sum_{j=0}^{J} \lambda^j h_j.$$

$\square$

The proof of Claim B is straightforward; we only prove Claim A.

*Proof of Claim A.* Since $S_\Omega$ is $C^{K_2}$ at 0, it follows from ideas in the beginning of the proof of Lemma 2.7 that $F(0) = 0$. By the Implicit Function Theorem (see, for instance, [11]), there exists open sets $V \subset U$ and $W \subset \mathbb{R}^2$ with $0 \in V$ and $0 \in W$ and $f \in C^{K_2}(W, \mathbb{R})$ such that

$$\partial_3 F(x) \neq 0, \tag{2.21}$$

for all $x \in V$, and

$$\{x \in V : F(x) = 0\} = \{(x, f(x)) : x \in W\}. \tag{2.22}$$

Since $V$ and $W$ are open with $0 \in V$ and $0 \in W$, $f$ is continuous, and $f(0) = 0$, we can choose $\delta, \epsilon > 0$ such that with $W_0 = (-\delta, \delta)^2$, $\overline{W_0} = [-\delta, \delta]^2$, $I = (-\epsilon, \epsilon)$, and $\overline{I} = [-\epsilon, \epsilon]$ we have

$$\overline{W_0} \subset W, \quad f(W_0) \subset (-\epsilon/2, \epsilon/2), \quad \text{and} \quad \overline{W_0} \times \overline{I} \subset V. \tag{2.23}$$

Write $V_0 = W_0 \times I$. It follows from (2.22) and (2.23) that

$$\{x \in V_0 : F(x) = 0\} = \{x \in V_0 : x_3 = f(x_1, x_2)\}. \tag{2.24}$$

Write

$$A^- = \{x \in V_0 : F(x) < 0\}, \quad A^+ = \{x \in V_0 : F(x) > 0\},$$

$$B^- = \{x \in V_0 : x_3 < f(x_1, x_2)\}, \quad \text{and} \quad B^+ = \{x \in V_0 : x_3 > f(x_1, x_2)\}.$$

Using that $V_0$ is open, that $F$ is continuous, and (2.24), we have that

$$A^- \text{ and } A^+ \text{ are open; } A^- \cup A^+ = B^- \cup B^+;\ A^- \cap A^+ = \emptyset. \tag{2.25}$$

Define $G \in C^{K_2}(V_0, \mathbb{R}^3)$ by $G(x) = (x_1, x_2, F(x))$. By (2.21), $G'(x)$ is invertible, for all $x \in V_0$. The inverse function theorem (see, for instance, [11]) implies that $G$ is an open mapping. Using also that $0 \in V_0$ and $F(0) = 0$, it follows that

$$A^- \neq \emptyset \neq A^+. \tag{2.26}$$

If $x, y \in B^-$, using (2.23) and that $W_0$ is convex, it follows that $x$ and $y$ can be joined by the continuous piecewise linear path contained in $B^-$ represented by

$$x \to (x_1, x_2, -\epsilon/2) \to (y_1, y_2, -\epsilon/2) \to y.$$

Thus, $B^-$ is path connected and hence connected. A similar argument shows that $B^+$ is connected. These two observations, together with (2.25) and (2.26), imply that either $A^- = B^-$ or $A^- = B^+$. Define

$$\triangle = \begin{cases} <, & \text{if } A^- = B^- \\ >, & \text{if } A^- = B^+ \end{cases}$$

and

$$\sigma = \begin{cases} 1, & \text{if } A^- = B^- \\ -1, & \text{if } A^- = B^+ \end{cases}.$$

Finally, using that $f \in C^{K_2}(W, \mathbb{R})$, that $\overline{W_0}$ is a closed subset of $W$, and standard smooth extension techniques (see, for instance, Lemma 2.27 of [25]), it follows that there exists compactly supported $g \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$ such that $g|_{\overline{W_0}} = f|_{\overline{W_0}}$. Using also Definition 1.1 and that $V_0 \subset U$, we have that

$$\Omega \cap V_0 = \{x \in V_0 : x_3 \triangle g(x_1, x_2)\} \tag{2.27}$$

in the a.e. sense. Note also that $g(0) = 0$ and that (2.20) holds (by (2.24) and the chain rule).

Choose open $N_0 \subset W_0 \times I$ with $0 \in N_0$ and $\theta_0, \theta_1 \in C^\infty(\mathbb{R}^3, [0, 1])$ satisfying

$$\theta_0(x) = 1, \text{ for all } x \in N_0, \ \mathrm{supp}(\theta_0) \subset W_0 \times I, \text{ and} \tag{2.28}$$

$$\theta_0(x) + \theta_1(x) = 1, \text{ for all } x \in \mathbb{R}^3.$$

For $q = 1, 2, 3$, define $\phi_q \in C^{K_2-1}(\mathbb{R}^2, \mathbb{R})$ by

$$\phi_q(x) = \begin{cases} -\sigma \partial_1 g(x) \theta_0(x, g(x)), & \text{if } q = 1 \\ -\sigma \partial_2 g(x) \theta_0(x, g(x)), & \text{if } q = 2 \\ \sigma \theta_0(x, g(x)), & \text{if } q = 3 \end{cases} \tag{2.29}$$

and note that each $\phi_q$ is compactly supported.

For $j = 0, 1$ and $\alpha > 0$, define

$$\begin{aligned} \mathcal{S}_j(\alpha) = -\frac{\alpha^{(\beta_1+\beta_2+\beta_3)/2}}{2\pi\imath} \int_{\mathbb{R}^3} d\xi \, &\frac{\overline{\hat{\psi}(\alpha^{\beta_1}\xi_1, \alpha^{\beta_2}\xi_2, \alpha^{\beta_3}\xi_3)}}{\xi_1^2 + \xi_2^2 + \xi_3^2} \\ &\times \int_{S_\Omega} dh(x) \theta_j(x) e^{-2\pi\imath \xi \cdot x} \xi \cdot n_\Omega(x). \end{aligned} \tag{2.30}$$

Then, (2.4) and (2.28) imply (2.17) and (2.28) and Lemma 2.3 imply (2.18). Using (2.27) and Lemma 2.7, we have

$$S_\Omega \cap V_0 = \{x \in V_0 : x_3 = g(x_1, x_2)\}$$

and, for $h$-a.e. $x \in S_\Omega \cap V_0$, we have

$$n_\Omega(x) = \frac{\sigma(-\partial_1 g(x_1, x_2), -\partial_2 g(x_1, x_2), 1)}{\sqrt{\partial_1 g(x_1, x_2)^2 + \partial_2 g(x_1, x_2)^2 + 1}}.$$

Using that $g \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$ is compactly supported, (2.28), the above two equalities, and the Hausdorff change of variables formula (see §3.2 and §3.3 of [10]), it follows that

$$\int_{S_\Omega} dh(x)\theta_0(x)e^{-2\pi\imath\xi\cdot x}\xi\cdot n_\Omega(x) = \int_{\mathbb{R}^2} dx\, \sigma\theta_0(x, g(x))e^{-2\pi\imath\xi\cdot(x,g(x))}\xi\cdot(-\partial_1 g(x), -\partial_2 g(x), 1)$$

Using (2.29), (2.30), the above equality, the change of variable $\eta_1 = \alpha^{\beta_1}\xi_1$, $\eta_2 = \alpha^{\beta_2}\xi_2$, $\eta_3 = \alpha^{\beta_3}\xi_3$, and arguments similar to those used to derive (2.9), it follows that

$$\begin{aligned}
&|\mathcal{S}_0(\alpha)| \\
&= \frac{\alpha^{(\beta_1+\beta_2+\beta_3)/2}}{2\pi}\Big|\sum_{q=1,2,3}\int_{\mathbb{R}^3} d\xi \frac{\xi_q\overline{\hat{\psi}(\alpha^{\beta_1}\xi_1, \alpha^{\beta_2}\xi_2, \alpha^{\beta_3}\xi_3)}}{\xi_1^2+\xi_2^2+\xi_3^2}\int_{\mathbb{R}^2} dx\, \phi_q(x)e^{-2\pi\imath\xi\cdot(x,g(x))}\Big| \\
&= \frac{\alpha^{\beta_0}}{2\pi}\Big|\sum_{q=1,2,3}\int_{\mathbb{R}^3} d\eta\, \frac{m_q(\eta_1, \alpha^{\beta_1-\beta_2}\eta_2, \alpha^{\beta_1-\beta_3}\eta_3)\overline{\hat{\psi}(\eta)}}{\eta_1^2+\alpha^{2(\beta_1-\beta_2)}\eta_2^2+\alpha^{2(\beta_1-\beta_3)}\eta_3^2} \\
&\quad\times\int_{\mathbb{R}^2} dx\, \phi_q(x)e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot(x,g(x))}\Big| \\
&\le \frac{\alpha^{\beta_0}}{2\pi}\sum_{q=1,2,3}\int_{\mathbb{R}^3} d\eta\, \frac{|\hat{\psi}(\eta)|}{|\eta_1|}\Big|\int_{\mathbb{R}^2} dx\, \phi_q(x)e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1, \alpha^{-\beta_2}\eta_2, \alpha^{-\beta_3}\eta_3)\cdot(x,g(x))}\Big|
\end{aligned}$$

where $m_q : \mathbb{R}^3 \to \mathbb{R}$ is defined by $m_q(\eta) = \eta_q$ $(q = 1, 2, 3)$. This verifies (2.19) and proves the claim. □

Let $g$ and $\phi_q$ $(q = 1, 2, 3)$ be as in Claim A. If $q \in \{1, 2, 3\}$, using Lemma 2.4, that $g \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$, that $\phi_q \in C^{K_2-1}(\mathbb{R}^2, \mathbb{R})$ is compactly supported, and Claim B, it follows that there exist compactly supported $h_0^q, \dots, h_{K_2-1}^q \in C(\mathbb{R}^2)$

such that

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} dx_1\, \phi_q(x) e^{-2\pi\imath\alpha^{-\beta_1}\xi_1 x_1} e^{-2\pi\imath\alpha^{-\beta_3}\xi_3 g(x)} \right| \\
&= \left| \int_{\mathbb{R}} dx_1\, \frac{\partial^{K_2-1}}{\partial x_1^{K_2-1}} \left( \frac{e^{-2\pi\imath\alpha^{-\beta_1}\xi_1 x_1}}{(-2\pi\imath\alpha^{-\beta_1}\xi_1)^{K_2-1}} \right) \phi_q(x) e^{-2\pi\imath\alpha^{-\beta_3}\xi_3 g(x)} \right| \\
&= \left| \frac{\alpha^{(K_2-1)\beta_1}}{(2\pi\xi_1)^{K_2-1}} \int_{\mathbb{R}} dx_1\, e^{-2\pi\imath\alpha^{-\beta_1}\xi_1 x_1} \frac{\partial^{K_2-1}}{\partial x_1^{K_2-1}} \left( \phi_q(x) e^{-2\pi\imath\alpha^{-\beta_3}\xi_3 g(x)} \right) \right| \\
&= \left| \frac{\alpha^{(K_2-1)\beta_1}}{(2\pi\xi_1)^{K_2-1}} \int_{\mathbb{R}} dx_1\, e^{-2\pi\imath\alpha^{-\beta_1}\xi_1 x_1} \left( e^{-2\pi\imath\alpha^{-\beta_3}\xi_3 g(x)} \sum_{k=0}^{K_2-1} (-2\pi\alpha^{-\beta_3}\xi_3)^k h_k^q(x) \right) \right| \\
&\le \sum_{k=0}^{K_2-1} \frac{\alpha^{(K_2-1)\beta_1 - k\beta_3} |\xi_3|^k}{(2\pi)^{K_2-1-k} |\xi_1|^{K_2-1}} \int_{\mathbb{R}} dx_1\, |h_k^q(x)|.
\end{aligned}
$$

If $0 < \alpha \le 1$, using (2.19), the Fubini-Tonelli Theorem, and the above inequality, we obtain

$$
\begin{aligned}
|\mathcal{S}_0(\alpha)| &\le \frac{\alpha^{\beta_0}}{2\pi} \sum_{q=1,2,3} \int_{\mathbb{R}^3} d\xi\, \frac{|\hat{\psi}(\xi)|}{|\xi_1|} \int_{\mathbb{R}} dx_2 \left| \int_{\mathbb{R}} dx_1\, \phi_q(x) e^{-2\pi\imath\alpha^{-\beta_1}\xi_1 x_1} e^{-2\pi\imath\alpha^{-\beta_3}\xi_3 g(x)} \right| \\
&\le \frac{\alpha^{\beta_0}}{2\pi} \sum_{q=1,2,3,k=0,\dots,K_2-1} \frac{\|h_k^q\|_1 \alpha^{(K_2-1)\beta_1 - k\beta_3}}{(2\pi)^{K_2-1-k}} \int_{\mathbb{R}^3} d\xi\, \frac{|\xi_3|^k |\hat{\psi}(\xi)|}{|\xi_1|^{K_2}} \\
&\le \alpha^{\beta_0 + (K_2-1)(\beta_1-\beta_3)} \sum_{q=1,2,3,k=0,\dots,K_2-1} \frac{\|h_k^q\|_1}{(2\pi)^{K_2-k}} \left\| \frac{\xi_3^k \hat{\psi}}{\xi_1^{K_2}} \right\|_1 .
\end{aligned}
$$

Part (i) now follows from (2.17), (2.18), and the above two inequalities. □

*2.4. Decay at piecewise smooth boundary points*

In this section, we prove that $\mathcal{S}^{ij}(\Omega)(\alpha, s, p)$ decays "fast" when $S_\Omega$ is piecewise $C^K$ at $p$, unless the orientation $\mathcal{O}_\Omega(p)$ coincides with the "orientation" of $i$, $j$, and $s$, Below are the two main results of this section.

**Theorem 2.8.** *Let $p \in S_\Omega$ and $K_1, K_2 \in \mathbb{Z}^+$ with $K_2 \ge 2$. Suppose that $\psi$ is $(K_1, 1)$- and $(K_2, 2)$-admissible and that $S_\Omega$ is piecewise $C^{K_2}$ at $p$. If $\mathcal{O}_\Omega(p)_i \ne s\mathcal{O}_\Omega(p)_j$, then*

$$
\limsup_{\alpha \to 0^+} \alpha^{-\beta-\beta_0} |\mathcal{S}^{ij}(\Omega)(\alpha, s, p)| < \infty,
$$

*where*

$$
\beta = \min\{K_1\beta_2, (K_2-1)(\beta_1-\beta_3)\}.
$$

**Theorem 2.9.** *Let $p \in S_\Omega$. Suppose that $\beta_1 < 2\beta_2$, $\psi$ is $(K_1, 3, 3)$-admissible, and that $S_\Omega$ is piecewise $C^{K_2}$ at $p$, with $K_1$ satisfying*

$$K_1 > \max\left\{\frac{3(\beta_1+\beta_3)}{2\beta_3-\beta_1}, \frac{4(\beta_1+\beta_3)}{2\beta_3-\beta_2}, \frac{2(\beta_2+\beta_3)}{\beta_3-\beta_2}, \frac{4(\beta_2+\beta_3)}{2\beta_2-\beta_1}\right\}, \tag{2.31}$$

*and $K_2 \geq 3$. If $\mathcal{O}_\Omega(p)_j \neq 0$ and $s = \mathcal{O}_\Omega(p)_i/\mathcal{O}_\Omega(p)_j$, then*

$$\lim_{\alpha\to 0^+} \alpha^{-\beta_0-\beta-\beta_3}\mathcal{S}^{ij}(\Omega)(\alpha, s, p) \in \mathbb{C}\setminus\{0\},$$

*where*

$$\beta = \begin{cases} \beta_1, & \text{if } \partial_k F(p) \neq 0 \neq \partial_k G(p) \\ \beta_2, & \text{otherwise} \end{cases},$$

*where $k$ is such that $\{i, j, k\} = \{1, 2, 3\}$.*

We require the following lemmas in the proofs of Theorems 2.8 and 2.9. We forewarn the reader that the statment of Lemma 2.10 is rather long as it must cover a wide range of circumstances.

**Lemma 2.10.** *Let $K_1, K_2 \in \mathbb{Z}^+$ with $K_2 \geq 2$. Suppose that $\psi$ is $(K_1, 1)$-admissible and that $S_\Omega$ is piecewise $C^{K_2}$ at 0, with $F$, $G$, $U$, and $\Box$ as in Definition 1.2. Suppose $i, j, k$ are such that $i < j$, $\{i, j, k\} = \{1, 2, 3\}$, and*

$$\det\begin{pmatrix} \partial_i F(0) & \partial_j F(0) \\ \partial_i G(0) & \partial_j G(0) \end{pmatrix} \neq 0.$$

*Choose $l, m \in \{i, j\}$ such that $\partial_l F(0), \partial_m G(0) \neq 0$, and $\partial_l G(0) = 0$, if $l \neq m$. Then, there exists bounded open intervals $I_1, I_2, I_3 \subset \mathbb{R}$, with $0 \in I_1 \cap I_2 \cap I_3$, compactly supported $f, g \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$ and $\gamma \in C^{K_2}(\mathbb{R}, \mathbb{R}^3)$, and $\mu(f), \mu(g), \nu(f), \nu(g) \in \{-1, 1\}$ satisfying*

*(i)* $0 \in V = I_1 \times I_2 \times I_3 \subset U$,

*(ii)* $\gamma_k(x) = x$, *for all* $x \in I_k$,

*(iii)* $\overline{\phi(I_{s(\phi)} \times I_{t(\phi)})} \subset I_{r(\phi)}$ $(\phi = f, g)$,

*(iv)* $\gamma(I_k) \subset V$, *and*

*(v)* $\overline{\gamma_{\lambda(\phi)}(I_k)} \subset I_{\lambda(\phi)}$ $(\phi = f, g)$

*such that for any Borel measurable $\theta_0 : \mathbb{R}^3 \to \mathbb{C}$ satisfying*

*(a)* $\theta_0(\mathbb{R}^3) \subset [0, 1]$,

*(b)* $\theta_0(x) = 1$, *for all* $x \in V_0$, *where* $V_0 \subset \mathbb{R}^3$ *is open with* $0 \in V_0$,

*(c)* $\theta_0(x) = 0$, *for all* $x \notin V$,

*we have*

$$\mathcal{S}(\Omega)(\alpha,0,0) = \mathcal{S}_0(\alpha) + \mathcal{S}_1(\alpha), \tag{2.32}$$

*for all* $\alpha > 0$, *where* $S_j : (0,\infty) \to \mathbb{C}$ *(*$j = 0,1$*) satisfy*

$$\limsup_{\alpha\to 0^+} \alpha^{-K_1\beta_2-\beta_0}|\mathcal{S}_1(\alpha)| < \infty \tag{2.33}$$

*and*

$$\begin{aligned}\mathcal{S}_0(\alpha) = \sum_{\phi=f,g} \frac{-\alpha^{\beta_0}}{2\pi\imath}\mu(\phi)\int_{\mathbb{R}^3} d\xi \frac{\overline{\hat{\psi}(\xi)}}{\xi_1^2+\alpha^{2(\beta_1-\beta_2)}\xi_2^2+\alpha^{2(\beta_1-\beta_3)}\xi_3^2}\int_{\mathbb{R}^2} dx_{s(\phi)}\otimes dx_{t(\phi)}\\ \times\, \theta_0(v_\phi(x_{s(\phi)},x_{t(\phi)}))u_\phi(x_k,x_{\lambda(\phi)})e^{-2\pi\imath(\alpha^{-\beta_1}\xi_1,\alpha^{-\beta_2}\xi_2,\alpha^{-\beta_3}\xi_3)\cdot v_\phi(x_{s(\phi)},x_{t(\phi)})}\\ \times\, w_\phi(x_{s(\phi)},x_{t(\phi)})\cdot(\xi_1,\alpha^{\beta_1-\beta_2}\xi_2,\alpha^{\beta_1-\beta_3}\xi_3),\end{aligned} \tag{2.34}$$

*where* $u_\phi, v_\phi \in C^{K_2}(\mathbb{R}^2,\mathbb{R})$ *and* $w_\phi \in C^{K_2-1}(\mathbb{R}^2,\mathbb{R})$ *are defined by*

$$v_\phi(x) = \begin{cases}(\phi(x),x), & \text{if } r(\phi)=1\\ (x_1,\phi(x),x_2), & \text{if } r(\phi)=2\\ (x,\phi(x)), & \text{if } r(\phi)=3\end{cases}, \tag{2.35}$$

$$w_\phi(x) = \begin{cases}(1,-\partial_1\phi(x),-\partial_2\phi(x)), & \text{if } r(\phi)=1\\ (-\partial_1\phi(x),1,-\partial_2\phi(x)), & \text{if } r(\phi)=2\\ (-\partial_1\phi(x),-\partial_2\phi(x),1), & \text{if } r(\phi)=3\end{cases}, \tag{2.36}$$

$$u_\phi(x) = \chi_{(0,\infty)}(\nu(\phi)(x_2-\gamma_{\lambda(\phi)}(x_1))) \tag{2.37}$$

*and* $r$, $s$, $t$, *and* $\lambda$ *are defined by* $r(f) = l$, $r(g) = m$, $\{\lambda(\phi)\} = \{1,2,3\} \setminus \{k, r(\phi)\}$, $\{s(\phi),t(\phi)\} = \{k,\lambda(\phi)\}$, *and* $s(\phi) < t(\phi)$. *Moreover,*

$$f(0) = g(0) = 0 \text{ and } \gamma(0) = 0, \tag{2.38}$$

$$\partial_\omega f(0) = \begin{cases}-\partial_2F(0)/\partial_1F(0), & \text{if } \omega=1 \text{ and } l=1\\ -\partial_3F(0)/\partial_1F(0), & \text{if } \omega=2 \text{ and } l=1\\ -\partial_1F(0)/\partial_2F(0), & \text{if } \omega=1 \text{ and } l=2\\ -\partial_3F(0)/\partial_2F(0), & \text{if } \omega=2 \text{ and } l=2\\ -\partial_1F(0)/\partial_3F(0), & \text{if } \omega=1 \text{ and } l=3\\ -\partial_2F(0)/\partial_3F(0), & \text{if } \omega=2 \text{ and } l=3\end{cases}, \tag{2.39}$$

$$\partial_\omega g(0) = \begin{cases} -\partial_2 G(0)/\partial_1 G(0), & \textit{if } \omega = 1 \textit{ and } m = 1 \\ -\partial_3 G(0)/\partial_1 G(0), & \textit{if } \omega = 2 \textit{ and } m = 1 \\ -\partial_1 G(0)/\partial_2 G(0), & \textit{if } \omega = 1 \textit{ and } m = 2 \\ -\partial_3 G(0)/\partial_2 G(0), & \textit{if } \omega = 2 \textit{ and } m = 2 \\ -\partial_1 G(0)/\partial_3 G(0), & \textit{if } \omega = 1 \textit{ and } m = 3 \\ -\partial_2 G(0)/\partial_3 G(0), & \textit{if } \omega = 2 \textit{ and } m = 3 \end{cases}, \tag{2.40}$$

*and*

$$\gamma'(0) \sim \nabla F(0) \times \nabla G(0). \tag{2.41}$$

**Lemma 2.11.** *Let $q \in \{1, 2\}$, $K \in \mathbb{Z}^+$, $h, h_1, h_2 \in C^K(\mathbb{R}^2)$ with $h$ compactly supported, and $\lambda_1, \lambda_2 \in \mathbb{R}$. Define $H \in C^K(\mathbb{R}^2)$ by $H(x) = h(x)e^{\lambda_1 h_1(x)}e^{\lambda_2 h_2(x)}$. Then, for each $k \in \{0, \ldots, K\}$, there exists a collection $\{\theta_i^k \in C^{K-k}(\mathbb{R}^2) : i \in \mathcal{I}_k\}$ of compactly supported functions not depending on $\lambda_1$ and $\lambda_2$ such that*

$$\partial_q^k H(x) = e^{\lambda_1 h_1(x)}e^{\lambda_2 h_2(x)} \sum_{i=(i_1,i_2)\in\mathcal{I}_k} \lambda_1^{i_1}\lambda_2^{i_2}\theta_i^k(x),$$

*for all $x \in \mathbb{R}^2$.*

Lemmas 2.11 is straightforward; we only prove Lemma 2.10.

*Proof of Lemma 2.10.* The proofs of all cases are completely similar to one of $(i, j, k, l, m) = (2, 3, 1, 3, 3)$ or $(i, j, k, l, m) = (2, 3, 1, 3, 2)$. Moreover, the cases $(i, j, k, l, m) = (2, 3, 1, 3, 3)$ and $(i, j, k, l, m) = (2, 3, 1, 3, 2)$ are similar enough that we only consider $(i, j, k, l, m) = (2, 3, 1, 3, 3)$. Finally, the proofs of $(i, j, k, l, m, \Box) = (2, 3, 1, 3, 3, \cap)$ and $(i, j, k, l, m, \Box) = (2, 3, 1, 3, 3, \cup)$ are similar enough that we only consider $(i, j, k, l, m, \Box) = (2, 3, 1, 3, 3, \cap)$. Suppose, then, that $(i, j, k, l, m, \Box) = (2, 3, 1, 3, 3, \cap)$. Note that, in this case,

$$r(f) = r(g) = 3,\ s(f) = s(g) = 1,\ t(f) = t(g) = 2,\ \lambda(f) = \lambda(g) = 2. \tag{2.42}$$

Under the assumptions of the previous paragraph, the assumptions in the first sentence of the statement of this lemma become

$$\det \begin{pmatrix} \partial_2 F(0) & \partial_3 F(0) \\ \partial_2 G(0) & \partial_3 G(0) \end{pmatrix} \neq 0$$

and $\partial_3 F(0), \partial_3 G(0) \neq 0$. Using also Definition 1.2 and an argument similar to the first paragraph of the proof of Claim A of Theorem 2.6, it follows that there exists compactly supported $f, g \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$ and $\gamma \in C^{K_2}(\mathbb{R}, \mathbb{R}^3)$, bounded open intervals $I_1, I_2, I_3 \subset \mathbb{R}$, and $\triangle_f, \triangle_g \in \{<, >\}$ such that (i) – (v) and (2.38) – (2.41) hold. Moreover, with

$$V_f = \{x \in V : x_3 \triangle_f f(x_1, x_2)\} \text{ and } V_g = \{x \in V : x_3 \triangle_g g(x_1, x_2)\}, \tag{2.43}$$

we have

(vi) $\{x \in V : F(x) = 0\} = \{x \in V : x_3 = f(x_1, x_2)\}$,

(vii) $\{x \in V : F(x) < 0\} = V_f$,

(viii) $\{x \in V : G(x) = 0\} = \{x \in V : x_3 = g(x_1, x_2)\}$,

(ix) $\{x \in V : G(x) < 0\} = V_g$, and

(x) $\{x \in V : F(x) = G(x) = 0\} = \{x \in V : x = \gamma(x_1)\}$.

(1.4), (2.43), (i), (vii), and (ix) imply that

$$\Omega \cap V_f = \{x \in V_f : x_3 \mathrel{\triangle_g} g(x_1, x_2)\}$$

and

$$\Omega \cap V_g = \{x \in V_g : x_3 \mathrel{\triangle_f} f(x_1, x_2)\}$$

(in the a.e. sense). Using that $V_f$ and $V_g$ are open, that $K_2 \geq 2$, the above two equalities, and Lemma 2.7, it follows that there exists $\mu(f), \mu(g) \in \{-1, 1\}$ such that

$$S_\Omega \cap V_f = \{x \in V_f : x_3 = g(x_1, x_2)\} \tag{2.44}$$

and, for $h$-a.e. $x \in S_\Omega \cap V_f$, we have

$$n_\Omega(x) = \frac{\mu(g)(-\partial_1 g(x_1, x_2), -\partial_2 g(x_1, x_2), 1)}{\sqrt{\partial_1 g(x_1, x_2)^2 + \partial_2 g(x_1, x_2)^2 + 1}} \tag{2.45}$$

and

$$S_\Omega \cap V_g = \{x \in V_g : x_3 = f(x_1, x_2)\} \tag{2.46}$$

and, for $h$-a.e. $x \in S_\Omega \cap V_g$, we have

$$n_\Omega(x) = \frac{\mu(f)(-\partial_1 f(x_1, x_2), -\partial_2 f(x_1, x_2), 1)}{\sqrt{\partial_1 f(x_1, x_2)^2 + \partial_2 f(x_1, x_2)^2 + 1}}. \tag{2.47}$$

(2.43) and (2.44) imply that

$$(S_\Omega \cap V_f) \cap (S_\Omega \cap V_g) = \emptyset. \tag{2.48}$$

Assume $x \in (S_\Omega \cap V) \setminus ((S_\Omega \cap V_f) \cup (S_\Omega \cap V_g))$. Since $x \in V \setminus V_f$, (2.43) implies that

$$-x_3 \mathrel{\Delta_f} -f(x_1, x_2) \text{ or } x_3 = f(x_1, x_2).$$

If $-x_3 \mathrel{\triangle_f} -f(x_1, x_2)$, then $W = \{y \in V : -y_3 \mathrel{\triangle_f} -f(y_1, y_2)\}$ is open with $x \in W$ and, by (1.4), (i), and (vii), $|W \cap \Omega| = 0$, implying, by (2.1), that $x \notin S_\Omega$, a contradiction. It follows that $x_3 = f(x_1, x_2)$. A similar argument shows that $x_3 = g(x_1, x_2)$. Using these two equalities, that $x \in V$, (i), (vi), (viii), and (x), it follows that $x = \gamma(x_1) \in \gamma(I_1)$. This shows that

$$(S_\Omega \cap V) \setminus ((S_\Omega \cap V_f) \cup (S_\Omega \cap V_g)) \subset \gamma(I_1). \tag{2.49}$$

Since $\gamma \in C^1(\mathbb{R}, \mathbb{R}^3)$ is compactly supported, it follows from Theorem 1 of §2.4.1 of [10] that $h(\gamma(\overline{I_1})) = 0$ (note that $\gamma(\overline{I_1})$ is compact and hence $h$-measurable). Combining this observation with (2.49), we have

$$h((S_\Omega \cap V) \setminus ((S_\Omega \cap V_f) \cup (S_\Omega \cap V_g))) = 0. \tag{2.50}$$

Using (i), (ii), (iv), (vi), (viii), and (x), it follows that

$$\begin{aligned}\{x \in V : x_3 = f(x_1, x_2) = g(x_1, x_2)\} &= \{x \in V : x_2 = \gamma_2(x_1), x_3 = g(x_1, x_2)\} \\ &= \{x \in V : x_2 = \gamma_2(x_1), x_3 = f(x_1, x_2)\}.\end{aligned}$$

Using also (2.43), (2.44), (2.46), (2.39), (2.40), Definition 1.2, (iii), (v), and an argument similar to that of the second paragraph of the proof of Claim A of Theorem 2.6, it follows that there exist $\diamond_g, \diamond_f \in \{<, >\}$ such that

$$S_\Omega \cap V_f = \{x \in V : x_2 \diamond_g \gamma_2(x_1), x_3 = g(x_1, x_2)\} \tag{2.51}$$

and

$$S_\Omega \cap V_g = \{x \in V : x_2 \diamond_f \gamma_2(x_1), x_3 = f(x_1, x_2)\}. \tag{2.52}$$

Define

$$\nu(\phi) = \begin{cases} 1, & \text{if } \diamond_\phi = > \\ -1, & \text{if } \diamond_\phi = < \end{cases}. \tag{2.53}$$

Let $\theta_0 : \mathbb{R}^3 \to \mathbb{C}$ be a Borel measurable function satisfying (a)–(c). Define the Borel measurable $\theta_1 : \mathbb{R}^3 \to \mathbb{C}$ by

$$\theta_1(x) = 1 - \theta_0(x) \tag{2.54}$$

and $\mathcal{S}_j : (0, \infty) \to \mathbb{C}$ ($j = 0, 1$) by

$$\mathcal{S}_j(\alpha) = \frac{-\alpha^{(\beta_1+\beta_2+\beta_3)/2}}{2\pi\imath} \int_{\mathbb{R}^3} d\xi \frac{\overline{\hat{\psi}(\alpha^{\beta_1}\xi_1, \alpha^{\beta_2}\xi_2, \alpha^{\beta_3}\xi_3)}}{\xi_1^2 + \xi_2^2 + \xi_3^2} \int_{S_\Omega} dh(x) \theta_j(x) e^{-2\pi\imath\xi\cdot x} \xi \cdot n_\Omega(x). \tag{2.55}$$

Then, (2.32) clearly holds (cf. (2.4)) and (b), (2.54), and Lemma 2.3 imply (2.33). Using (c), (2.48), (2.50), that $f, g \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$ are compactly supported, (a), (i), (2.45), (2.47), (2.51) – (2.53), and the Hausdorff change of variables formula (see §3.2 and §3.3 of [10]), it follows that

$$\begin{aligned}
&\int_{S_\Omega} dh(x)\theta_0(x)e^{-2\pi\imath\xi\cdot x}\xi\cdot n_\Omega(x) \\
&= \int_{S_\Omega\cap V} dh(x)\theta_0(x)e^{-2\pi\imath\xi\cdot x}\xi\cdot n_\Omega(x) \\
&= \sum_{\phi=f,g}\int_{S_\Omega\cap V_\phi} dh(x)\theta_0(x)e^{-2\pi\imath\xi\cdot x}\xi\cdot n_\Omega(x) \\
&= \sum_{\phi=g,f}\mu(\phi)\int_{\mathbb{R}^2} dx\,\theta_0(v_\phi(x))u_\phi(x)e^{-2\pi\imath\xi\cdot v_\phi(x)}w_\phi(x)\cdot\xi,
\end{aligned}$$

where $v_\phi$, $u_\phi$, and $w_\phi$ are defined by (2.35) – (2.37) and (2.42). (2.34) follows from (2.55), the above equality, the change of variable $\eta_1 = \alpha^{\beta_1}\xi_1$, $\eta_2 = \alpha^{\beta_2}\xi_2$, $\eta_3 = \alpha^{\beta_3}\xi_3$, and some algebraic manipulation. This proves the lemma. □

We can now prove Theorem 2.8 and 2.9.

*Proof of Theorem 2.8.* Choose $F$ and $G$ as in Definition 1.2. It follows from Lemma 2.12 that we may assume $i = 1$, $j = 3$, $s = 0$, and $p = 0$. Suppose, then, that

$$\partial_2F(0)\partial_3G(0) - \partial_3F(0)\partial_2G(0) = (\nabla F(0)\times\nabla G(0))_1 \neq 0.$$

We are in the context of Lemma 2.10 with $(i,j,k) = (2,3,1)$. Choose $l,m \in \{2,3\}$ such that $\partial_lF(0), \partial_mG(0) \neq 0$, and $\partial_lG(0) = 0$, if $l \neq m$. All notation used below is as in Lemma 2.10. Assume that $\theta_0 \in C^\infty(\mathbb{R}^3)$. We have the cases $(l,m) = (2,2), (2,3), (3,2), (3,3)$ to consider. By switching $F$ and $G$, we need only consider the cases $(l,m) = (2,2), (2,3)$. We assume $(l,m) = (2,2)$. The case $(l,m) = (2,3)$ is handled similarly. In this case, (2.34) takes the form $S_0(\alpha) = S_0^f(\alpha) + S_0^g(\alpha)$, where

$$\begin{aligned}
\mathcal{S}_0^\phi(\alpha) = -\frac{\alpha^{\beta_0}}{2\pi\imath}\mu(\phi)\int_{\mathbb{R}^3} d\xi\,&\frac{\overline{\hat\psi(\xi)}}{\xi_1^2+\alpha^{2(\beta_1-\beta_2)}\xi_2^2+\alpha^{2(\beta_1-\beta_3)}\xi_3^2} \\
&\times\int_{\mathbb{R}^2} dx_1\otimes dx_3\,\theta_0(x_1,\phi(x_1,x_3),x_3)\chi_{(0,\infty)}(\nu(\phi)(x_3-\gamma_3(x_1))) \\
&\times e^{-2\pi\imath(\alpha^{-\beta_1}\xi_1,\alpha^{-\beta_2}\xi_2,\alpha^{-\beta_3}\xi_3)\cdot(x_1,\phi(x_1,x_3),x_3)} \\
&\times(-\partial_1\phi(x_1,x_3),1,-\partial_2\phi(x_1,x_3))\cdot(\xi_1,\alpha^{\beta_1-\beta_2}\xi_2,\alpha^{\beta_1-\beta_3}\xi_3).
\end{aligned}$$

Applying the change of variable $x_1 = y_1$ and $x_3 = y_3 + \gamma_3(y_1)$ to the inner integral of the above equality, it follows that

$$
\begin{aligned}
\mathcal{S}_0^{\phi}(\alpha) = -\frac{\alpha^{\beta_0}}{2\pi\imath}\mu(\phi)\int_{\mathbb{R}^3} d\xi\, &\frac{\overline{\hat{\psi}(\xi)}}{\xi_1^2+\alpha^{2(\beta_1-\beta_2)}\xi_2^2+\alpha^{2(\beta_1-\beta_3)}\xi_3^2} \\
\times \int_{\mathbb{R}^2} &dy_1\otimes dy_3\,\theta_0(y_1,\phi(y_1,y_3+\gamma_3(y_1)),y_3+\gamma_3(y_1))\chi_{(0,\infty)}(\nu(\phi)y_3) \\
\times\, &e^{-2\pi\imath(\alpha^{-\beta_1}\xi_1,\alpha^{-\beta_2}\xi_2,\alpha^{-\beta_3}\xi_3)\cdot(y_1,\phi(y_1,y_3+\gamma_3(y_1)),y_3+\gamma_3(y_1))} \\
\times\, &(-\partial_1\phi(y_1,y_3+\gamma_3(y_1)),1,-\partial_2\phi(y_1,y_3+\gamma_3(y_1)))\cdot(\xi_1,\alpha^{\beta_1-\beta_2}\xi_2,\alpha^{\beta_1-\beta_3}\xi_3).
\end{aligned}
$$

The remainder of the proof uses Lemma 2.11 and proceeds in a similar fashion to the last paragraph of the proof of Theorem 2.6. □

*Proof of Theorem 2.9.* Choose $F$ and $G$ as in Definition 1.2. It follows from Lemma 2.12 that we may assume $i=1$, $j=3$, $s=0$, and $p=0$. Also, we only prove the case $r=0$ from Definition 1.3 (the case $r=1$ is similar). Suppose, then, that

$$\partial_1F(0)\partial_2G(0)-\partial_2F(0)\partial_1G(0)=(\nabla F(0)\times\nabla G(0))_3\neq 0. \tag{2.56}$$

and that

$$\partial_2F(0)\partial_3G(0)-\partial_3F(0)\partial_2G(0)=(\nabla F(0)\times\nabla G(0))_1=0. \tag{2.57}$$

It follows that $\psi$ is $(K_1,1)$-admissible. We are thus in the context of Lemma 2.10 with $(i,j,k)=(1,2,3)$. Unless specified otherwise, all notation below is as in Lemma 2.10. We divide the proof into three parts: $\partial_2F(0)\neq 0\neq\partial_2G(0)$, $\partial_2G(0)=0$, and $\partial_2F(0)=0$. We only consider the first two cases, as the third is completely similar to the second.

***Part I***. Suppose that $\partial_2F(0)\neq 0\neq\partial_2G(0)$. We thus may choose $l=m=2$. Examining the proof of Lemma 2.10, it follows, in this case, that

$$\mu(f)\nu(f)=-\mu(g)\nu(g). \tag{2.58}$$

Let $\tilde{f}$ and $\tilde{g}$ be the first order Taylor approximations to $f$ and $g$ at 0 and define $\tilde{\gamma}_1:\mathbb{R}\to\mathbb{R}$ by $\tilde{\gamma}_1=0$. Note that

$$f(0)=\tilde{f}(0)=g(0)=\tilde{g}(0)=\gamma_1(0)=0,$$

and, by (2.41) and (2.57), that

$$\gamma_1'(0)=0. \tag{2.59}$$

Choose open intervals $J_1,J_3\subset\mathbb{R}$ and $\theta_3\in C^\infty(\mathbb{R},[0,1])$ such that

(i) $0\in J_1\cap J_3$ and $J_q\subset I_q$ $(q=1,3)$;

(ii) $\mathrm{supp}(\theta_3) \subset J_3$ and $\theta_3(x) = 1$, for all $x$ in some open set containing the origin;

(iii) $\phi(J_1 \times J_3) \subset I_2$, for $\phi = f, g, \tilde{f}, \tilde{g}$;

(iv) $\omega(J_3) \subset J_1$, for $\omega = \gamma_1, \tilde{\gamma}_1$.

Define $\theta_0 : \mathbb{R}^3 \to \mathbb{C}$ by $\theta_0(x) = \chi_{J_1 \times I_2}(x_1, x_2)\theta_3(x_3)$ and note that $\theta_0$ is Borel measurable and satisfies (a)–(c) in the statement of Lemma 2.10.

If $h \in \{f, g\}$, denote the summand of (2.34) corresponding to $h$ by $\mathcal{S}_0^h(\alpha)$. If $E \subset \mathbb{R}^2$ is measurable, $\phi \in C^{K_2}(\mathbb{R}^2, \mathbb{R})$, $\omega \in C^{K_2}(\mathbb{R}, \mathbb{R})$, $\phi_1, \phi_3 \in C^{K_2 - 1}(\mathbb{R}^2, \mathbb{R})$, $c \in \mathbb{R}$, and $0 < d \leq \infty$, define

$$\begin{aligned}
&\mathcal{R}_h(\alpha, \phi, \phi_1, \phi_3, \omega, E) \\
&\quad = -\frac{\alpha^{\beta_0}}{2\pi\imath}\mu(h)\int_{\mathbb{R}^3} d\xi \, \frac{\overline{\hat{\psi}(\xi)}}{\xi_1^2 + \alpha^{2(\beta_1-\beta_2)}\xi_2^2 + \alpha^{2(\beta_1-\beta_3)}\xi_3^2}\mathcal{I}_h(\xi, \alpha, \phi, \phi_1, \phi_3, \omega, E),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{T}_h(\alpha, \phi, \phi_1, \phi_3, \omega, d, c) \\
&\quad = -\frac{\alpha^{\beta_0}}{2\pi\imath}\mu(h)\int_{\mathbb{R}^3} d\xi \, \frac{\overline{\hat{\psi}(\xi)}}{\xi_1^2 + \alpha^{2(\beta_1-\beta_2)}\xi_2^2 + \alpha^{2(\beta_1-\beta_3)}\xi_3^2}\mathcal{J}_h(\xi, \alpha, \phi, \phi_1, \phi_3, \omega, d, c),
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{I}_h(\xi, \alpha, \phi, \phi_1, \phi_3, \omega, E) = \int_E dx_1 \otimes dx_3 \, \theta_0(x_1, \phi(x_1, x_3), x_3)\chi_{(0,\infty)}(\nu(h)(x_1 - \omega(x_3))) \\
&\times e^{-2\pi\imath(\alpha^{-\beta_1}\xi_1, \alpha^{-\beta_2}\xi_2, \alpha^{-\beta_3}\xi_3)\cdot(x_1, \phi(x_1,x_3), x_3)}(\phi_1(x_1, x_3), 1, \phi_3(x_1, x_3)) \cdot (\xi_1, \alpha^{\beta_1-\beta_2}\xi_2, \alpha^{\beta_1-\beta_3}\xi_3),
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{J}_h(\xi, \alpha, \phi, \phi_1, \phi_2, \omega, d, c) = \nu(h)\int_{-d}^{d} dx_3 \, \theta_3(x_3)e^{-2\pi\imath\alpha^{-\beta_3}\xi_3 x_3} \\
&\times \int_{\omega(x_3)}^{c} dx_1 \, e^{-2\pi\imath(\alpha^{-\beta_1}\xi_1 x_1 + \alpha^{-\beta_2}\xi_2\phi(x_1,x_3))} \\
&\times (\phi_1(x_1, x_3), 1, \phi_3(x_1, x_3)) \cdot (\xi_1, \alpha^{\beta_1-\beta_2}\xi_2, \alpha^{\beta_1-\beta_3}\xi_3).
\end{aligned}$$

Write $J_1 = (a, b)$. Using (2.31), choose $0 < \delta_q < \beta_q$ ($q = 1, 3$) satisfying $\delta_3 < \delta_1 < 2\delta_3$ and

$$\beta_1 + \beta_3 < \min\{K_1(\beta_1 - \delta_1), K_1(\beta_3 - \delta_3), 3\delta_3, \delta_1 + 3\delta_3 - \beta_2\}. \tag{2.60}$$

Write

$$E_{\delta_1,\delta_3}(\alpha) = \{x \in \mathbb{R}^2 : |x_1| \leq \alpha^{\delta_1}, |x_2| \leq \alpha^{\delta_3}\}.$$

We require several claims in the proof of part I, the first two of which are below. The proofs of many of these claims are straightforward and therefore omitted.

**Claim A**. If $\phi(J_1 \times J_3) \subset I_2$, $\omega(J_3) \subset J_1$, and $\omega(0) = \omega'(0) = 0$, then there exists $0 < K < \infty$ such that

$$\mathcal{R}_h(\alpha, \phi, \phi_1, \phi_3, \omega, \mathbb{R}^2) = \mathcal{T}_h(\alpha, \phi, \phi_1, \phi_3, \omega, \infty, c(h)),$$

where

$$c(h) = \begin{cases} b, & \text{if } \nu(h) = 1; \\ a, & \text{if } \nu(h) = -1; \end{cases}$$

and

$$\mathcal{R}_h(\alpha, \phi, \phi_1, \phi_3, \omega, E_{\delta_1,\delta_3}(\alpha)) = \mathcal{T}_h(\alpha, \phi, \phi_1, \phi_3, \omega, \alpha^{\delta_3}, \nu(h)\alpha^{\delta_1})$$

for all $\alpha \leq K$. $\square$

**Claim B**. There exists $0 < K < \infty$ such that

$$|\mathcal{R}_h(\alpha, \phi, \phi_1, \phi_3, \omega, \mathbb{R}^2 \setminus E_{\delta_1,\delta_3}(\alpha))| \leq K\alpha^{\beta_0 + K_1 \min\{\beta_1 - \delta_1, \beta_3 - \delta_3\}},$$

for all $\alpha$. $\square$

Let $\widetilde{\partial_q h}$ be the zero order Taylor approximation to $\partial_q h$ at 0 ($q = 1, 2$). Eq. (2.59), (iii), (iv), and Claim A imply that

$$\begin{aligned}
S_0^h(\alpha) &= \mathcal{R}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \mathbb{R}^2) \\
&+ \mathcal{R}_h(\alpha, h, -\partial_1 h, -\partial_2 h, \gamma_1, E_{\delta_1,\delta_3}(\alpha)) - \mathcal{R}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, E_{\delta_1,\delta_3}(\alpha)) \\
&+ \mathcal{R}_h(\alpha, h, -\partial_1 h, -\partial_2 h, \gamma_1, \mathbb{R}^2 \setminus E_{\delta_1,\delta_3}(\alpha)) - \mathcal{R}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \mathbb{R}^2 \setminus E_{\delta_1,\delta_3}(\alpha)) \\
&= \mathcal{T}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \infty, c(h)) \qquad (2.61) \\
&+ \mathcal{T}_h(\alpha, h, -\partial_1 h, -\partial_2 h, \gamma_1, \alpha^{\delta_3}, \nu(h)\alpha^{\delta_1}) - \mathcal{T}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \alpha^{\delta_3}, \nu(h)\alpha^{\delta_1}) \\
&+ \mathcal{R}_h(\alpha, h, -\partial_1 h, -\partial_2 h, \gamma_1, \mathbb{R}^2 \setminus E_{\delta_1,\delta_3}(\alpha)) - \mathcal{R}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \mathbb{R}^2 \setminus E_{\delta_1,\delta_3}(\alpha)),
\end{aligned}$$

for all small enough $\alpha$.

**Claim C**. There exists $0 < K, M < \infty$ such that

$$\begin{aligned}
&|\mathcal{T}_h(\alpha, h, -\partial_1 h, -\partial_2 h, \gamma_1, \alpha^{\delta_3}, \nu(h)\alpha^{\delta_1}) - \mathcal{T}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \alpha^{\delta_3}, \nu(h)\alpha^{\delta_1})| \\
&\quad \leq M\alpha^{\beta_0 + \min\{3\delta_3, \delta_1 + 3\delta_3 - \beta_2\}},
\end{aligned}$$

for all $\alpha \leq K$. $\square$

Write $\tilde{h}(x_1, x_3) = Ax_1 + Bx_3$, $\widetilde{\partial_1 h}(x_1, x_3) = A$, $\widetilde{\partial_2 h}(x_1, x_3) = B$, $\mathcal{T}_h(\alpha) = \mathcal{T}_h(\alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \infty, c(h))$ and $\mathcal{J}_h(\xi, \alpha) = \mathcal{J}_h(\xi, \alpha, \tilde{h}, -\widetilde{\partial_1 h}, -\widetilde{\partial_2 h}, \tilde{\gamma}_1, \infty, c(h))$. Choose $\epsilon, M > 0$ such that $\mathrm{supp}(\hat{\psi}_1) \subset \mathbb{R} \setminus (-\epsilon, \epsilon)$ and $\mathrm{supp}(\hat{\psi}_2) \subset [-M, M]$. For the remainder of part I, we assume that $\alpha \leq 1$ and, if $A \neq 0$, that

$$\alpha \leq \left(\frac{\epsilon}{2|A|M}\right)^{1/(\beta_1 - \beta_2)}.$$

Write
$$D(\alpha,\xi_1,\xi_2)=\xi_1+A\alpha^{\beta_1-\beta_2}\xi_2,$$
and note that $|D(\alpha,\xi_1,\xi_2)|\geq\epsilon/2$ and $\operatorname{sgn}(D(\alpha,\xi_1,\xi_2))=\operatorname{sgn}(\xi_1)$, for all $(\xi_1,\xi_2)$ with $|\xi_1|\geq\epsilon$ and $|\xi_2|\leq M$, for all $\alpha$.

**Claim D**. We have
$$\lim_{\alpha\to 0^+}\alpha^{-\beta_0-\beta_1-\beta_3}\mathcal{T}_h(\alpha)=\frac{-A\mu(h)\nu(h)\overline{\psi_2(0)}\;\overline{\hat{\psi}_3(0)}}{4\pi^2}\int_{\mathbb{R}}d\eta\,\frac{\overline{\hat{\psi}_1(\eta)}}{\eta^2}.$$
□

*Proof of Claim D.* Integration and the change of variable $\eta_q=\xi_q$ $(q=1,2)$, $\eta_3=\alpha^{-\beta_3}\xi_3+\alpha^{-\beta_2}B\xi_2$ allow us to write $\mathcal{T}_h(\alpha)=\mathcal{T}_h^0(\alpha)+\mathcal{T}_h^1(\alpha)$, where
$$\mathcal{T}_h^q(\alpha)=\frac{\mu(h)\nu(h)\alpha^{\beta_0+\beta_1+\beta_3}}{4\pi^2}\int_{\mathbb{R}^3}d\eta\,\Phi(\eta,\alpha)\mathcal{J}_h^q(\eta,\alpha),$$
$$\Phi(\eta,\alpha)=\frac{\overline{\hat{\psi}_1(\eta_1)}\;\overline{\hat{\psi}_2(\eta_2)}\;\overline{\hat{\psi}_3(\alpha^{\beta_3}\eta_3-\alpha^{\beta_3-\beta_2}B\eta_2)}}{\eta_1^2+\alpha^{2(\beta_1-\beta_2)}\eta_2^2+(\alpha^{\beta_1}\eta_3-\alpha^{\beta_1-\beta_2}B\eta_2)^2},$$
$$\mathcal{J}_h^q(\eta,\alpha)=(-A,1,-B)\cdot(\eta_1,\alpha^{\beta_1-\beta_2}\eta_2,\alpha^{\beta_1}\eta_3-\alpha^{\beta_1-\beta_2}B\eta_2)\hat{\theta}_3(\eta_3)\frac{\phi_q(\eta)}{D(\alpha,\eta_1,\eta_2)},$$
and
$$\phi_q(\xi)=\begin{cases}1, & \text{if } q=0;\\ -e^{-2\pi\imath(\alpha^{-\beta_1}\eta_1+\alpha^{-\beta_2}A\eta_2)c(h)}, & \text{if } q=1.\end{cases}$$

It follows that
$$|\Phi(\eta,\alpha)\mathcal{J}_h^0(\eta,\alpha)|\leq\frac{2\|(-A,1,-B)\|\|\hat{\psi}_3\|_\infty|\hat{\psi}_1(\eta_1)||\hat{\psi}_2(\eta_2)||\hat{\theta}_3(\eta_3)|}{\epsilon^2},\tag{2.62}$$
for a.e. $\eta$, for all $\alpha$ and
$$\lim_{\alpha\to 0^+}\Phi(\eta,\alpha)\mathcal{J}_h^0(\eta,\alpha)=\frac{-A\overline{\hat{\psi}_3(0)}\;\overline{\hat{\psi}_1(\eta_1)}\;\overline{\hat{\psi}_2(\eta_2)}\;\hat{\theta}_3(\eta_3)}{\eta_1^2},$$
for a.e. $\eta$. Moreover, arguments similar to those used in the proof of Lemma 2.3 imply that
$$\lim_{\alpha\to 0^+}\int_{\mathbb{R}^3}d\eta\,\Phi(\eta,\alpha)\mathcal{J}_h^1(\eta,\alpha)=0.\tag{2.63}$$
The claim follows from (2.62), (2.63) and the Lebesgue Dominated Convergence theorem. □

Using (2.31), (2.32) – (2.34), (2.58), (2.60), (2.61), and Claims B, C, and D, it follows that

$$\lim_{\alpha\to 0^+} \alpha^{-\beta_0-\beta_1-\beta_3}\mathcal{S}(\Omega)(\alpha,0,0) = \pm\frac{\overline{\psi_2(0)}\ \overline{\hat{\psi}_3(0)}(\nabla F(0)\times\nabla G(0))_3}{4\pi^2\partial_2 F(0)\partial_2 G(0)}\int_{\mathbb{R}} d\eta\, \frac{\overline{\hat{\psi}_1(\eta)}}{\eta^2},$$

which is non-zero. This completes the proof of part I. □

***Part II***. Suppose that $\partial_2 G(0) = 0$. Then, by (2.56), $\partial_2 F(0) \neq 0 \neq \partial_1 G(0)$. We thus may choose $l = 2$ and $m = 1$.

Let $\tilde{f}$ be the first order Taylor approximation to $f$ at 0 and define $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ and $\tilde{\gamma}_1, \tilde{\gamma}_2 : \mathbb{R} \to \mathbb{R}$ by $\tilde{g} = 0$ and $\tilde{\gamma}_1 = \tilde{\gamma}_2 = 0$. Note, by (2.40), (2.41), and (2.57), that

$$f(0) = \tilde{f}(0) = g(0) = \gamma_1(0) = \gamma_2(0) = 0 \tag{2.64}$$

and

$$\gamma_1'(0) = \partial_1 g(0) = \partial_2 g(0) = 0. \tag{2.65}$$

Using (2.64) and (2.65), choose open intervals $J_1, J_2, J_3 \subset \mathbb{R}$ and $\theta_3 \in C^\infty(\mathbb{R}, [0,1])$ such that

(i) $0 \in J_1 \cap J_2 \cap J_3$ and $J_q \subset I_q$ $(q = 1,2,3)$;

(ii) $\text{supp}(\theta_3) \subset J_3$ and $\theta_3(x) = 1$, for all $x$ in some open set containing the origin;

(iii) $\phi(J_1 \times J_3) \subset J_2$, for $\phi = f, \tilde{f}$;

(iv) $\phi(J_2 \times J_3) \subset J_1$, for $\phi = g, \tilde{g}$;

(v) $\omega(J_3) \subset J_1$, for $\omega = \gamma_1, \tilde{\gamma}_1$.

(vi) $\omega(J_3) \subset J_2$, for $\omega = \gamma_2, \tilde{\gamma}_2$.

Define $\theta_0 : \mathbb{R}^3 \to \mathbb{C}$ by

$$\theta_0(x) = \chi_{J_1\times J_2}(x_1,x_2)\theta_3(x_3)$$

and note that $\theta_0$ is Borel measurable and satisfies (a) – (c) in the statement of Lemma 2.10.

If $h \in \{f, g\}$, denote the summand of (2.34) corresponding to $h$ by $\mathcal{S}_0^h(\alpha)$. It follows from the proof of part I that

$$\lim_{\alpha\to 0^+} \alpha^{-\beta_0-\beta_2-\beta_3} S_0^f(\alpha) = 0.$$

Using arguments similar to those in Part I, we find that

$$\lim_{\alpha\to 0^+} \alpha^{-\beta_0-\beta_2-\beta_3} S_1(\alpha) = 0$$

and

$$\lim_{\alpha\to 0^+} \alpha^{-\beta_0-\beta_2-\beta_3} S_0^f(\alpha) = \frac{\mu(g)\nu(g)\ \overline{\hat{\psi}_3(0)}}{4\pi^2}\int_{\mathbb{R}} d\eta\, \frac{\overline{\hat{\psi}_1(\eta)}}{\eta}\int_{\mathbb{R}} d\eta\, \frac{\overline{\hat{\psi}_2(\eta)}}{\eta}.$$

It follows from the above three equalities that

$$\lim_{\alpha\to 0^+} \alpha^{-\beta_0-\beta_2-\beta_3}\mathcal{S}(\Omega)(\alpha,0,0) = \frac{\mu(g)\nu(g)\,\overline{\hat{\psi}_3(0)}}{4\pi^2}\int_{\mathbb{R}} d\eta\,\frac{\overline{\hat{\psi}_1(\eta)}}{\eta}\int_{\mathbb{R}} d\eta\,\frac{\overline{\hat{\psi}_2(\eta)}}{\eta} \neq 0.$$

This completes the proof of part II. □

This also completes the proof of the theorem. □

*2.5. Changes of variables, etc.*

In this subsection, we collect several results, that, in particular, allow us reduce to the case $i = 1$, $j = 3$, $s = 0$, and $p = 0$ in the main results of the previous subsections. Parts (i) – (iii) of the below lemma follow from results in §2.6 and §11.2 of [12]. The other results are straightforward.

**Lemma 2.12.** *Let $c \in GL_3(\mathbb{R})$ and $y \in \mathbb{R}^3$ and define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $Tx = cx + y$. We have the following:*

- *$T\Omega$ is a bounded Lebesgue measurable subset of $\mathbb{R}^3$.*
- *$S_{T\Omega} = TS_\Omega$ and $h(S_{T\Omega}) < \infty$.*
- $$n_{T\Omega}(x) = (c^*)^{-1}n_\Omega(T^{-1}x)/\|(c^*)^{-1}n_\Omega(T^{-1}x)\|,$$
  *for a.e. $x \in S_{T\Omega}$, where $c^*$ denotes the transpose of $c$.*
- *If $E \subset \mathbb{R}^3$ and $p \notin \overline{E}$, then $Tp \notin \overline{TE}$.*
- *If $S_\Omega$ is $C^K$ at $p$, with $F$ and $U$ as in Definition 1.1, then $S_{T\Omega}$ is $C^K$ at $Tp$, with $F \circ T^{-1}$ and $TU$ as in Definition 1.1.*
- *If $S_\Omega$ is piecewise $C^K$ at $p$, with $F$, $G$, and $U$ as in Definition 1.2, then $S_{T\Omega}$ is $C^K$ at $Tp$, with $F \circ T^{-1}$, $G \circ T^{-1}$, and $TU$ as in Definition 1.2.*
- *If $z, w \in \mathbb{R}^3$, then $(zc) \times (wc) = (\det c)(z \times w)(c^*)^{-1}$.*
- *We have*
  $$S(\Omega)(\alpha, s, p) = S(b(s)^{-1}\Omega - b(s)^{-1}p)(\alpha, 0, 0)$$
  *and*
  $$S^{ij}(\Omega)(\alpha, s, p) = S((\sigma^{ij})^{-1}\Omega)(\alpha, s, (\sigma^{ij})^{-1}p),$$
  *for all $i$, $j$, $\alpha$, $s$, and $p$.*

## 3. Extensions and Generalizations

In this section, we briefly discuss some extensions and generalizations of the results presented above.

*3.1. Other dilation matrices*

In Theorem 2.9 (and hence in Theorem 1.4, which depends on Theorem 2.9), we make the assumption $\beta_1 < 2\beta_2$. Defintion 1.2 necessitates fairly technical geometric arguments in the proof of Theorem 2.9, and we partly made this assumption on $\beta_1$ and $\beta_2$ to avoid additional technical arguments in the evaluation of the integral in (2.34). Note, however, that this assuption excludes several cases of interest, for example, the cases $(\beta_1, \beta_2, \beta_3) = (2, 1, \sqrt{2})$ and $(\beta_1, \beta_2, \beta_3) = (3, 1, 2)$. It is therefore natural to ask whether Theorem 2.9 (and thus Theorem 1.4) can be extended to include these cases. Preliminary investigations indicate that this should be possible.

*3.2. Higher dimensions*

It is interesting to ask whether the framework and results of this paper can be extented to dimensions $n \geq 4$. We highlight some aspects of the case $n = 4$; similar observations can be made about the cases $n \geq 5$. Definition 1.2 has the following interesting and nontrivial generalization to $n = 4$:

**Definition 3.1.** *Assume that $n = 4$. Let $K \in \mathbb{Z}^+ \cup \{\infty\}$ and $p \in S_\Omega$.*

*(i) We say that $S_\Omega$ is piecewise $C^K$ at $p$ with one-dimensional intersection if there exists an open set $U \subset \mathbb{R}^4$ with $p \in U$ and $F, G, H \in C^K(U, \mathbb{R})$ with $F(p) = G(p) = H(p) = 0$ and $\{\nabla F(p), \nabla G(p), \nabla H(p)\}$ linearly independent such that*

$$\Omega \cap U = \{x \in U : F(x) < 0\} \square_1 \{x \in U : G(x) < 0\} \square_2 \{x \in U : H(x) < 0\}$$

*(in the a.e. sense), where each of the symbols $\square_1$ and $\square_2$ can be either $\cap$ or $\cup$. In this case, we call the one-dimensional linear subspace*

$$\mathcal{O}_\Omega(p) = (\operatorname{span}\{\nabla F(p), \nabla G(p), \nabla H(p)\})^\perp$$

*the orientation of $S_\Omega$ at $p$. Note that $\mathcal{O}_\Omega(p)$ is well-defined and spanned by the tangent vector at $p$ to the curve defined by*

$$\{x : F(x) = G(x) = H(x) = 0\}$$

*near $p$.*

*(ii) We say that $S_\Omega$ is piecewise $C^K$ at $p$ with two-dimensional intersection if there exists an open set $U \subset \mathbb{R}^4$ with $p \in U$ and $F, G \in C^K(U, \mathbb{R})$ with $F(p) = G(p) = 0$ and $\{\nabla F(p), \nabla G(p)\}$ linearly independent such that*

$$\Omega \cap U = \{x \in U : F(x) < 0\} \square \{x \in U : G(x) < 0\}$$

*(in the a.e. sense), where the symbol $\square$ can be either $\cap$ or $\cup$. In this case, we call the two-dimensional linear subspace*

$$\mathcal{O}_\Omega(p) = (\operatorname{span}\{\nabla F(p), \nabla G(p)\})^\perp$$

*the orientation of $S_\Omega$ at $p$. Note that $\mathcal{O}_\Omega(p)$ is well-defined and equals the tangent space at $p$ to the two-dimensional surface defined by $\{x : F(x) = G(x) = 0\}$ near $p$.*

*(We omit the case "piecewise $C^K$ at $p$ with three-dimensional intersection" as this case, while interesting, is the generalization to four dimensions of the results in [17].)*

Note that the dilation and shear matrices used in this paper generalize readily to two collections of four-dimensional dilation and shear matrices, one of which is naturally suited to case (i) of the above definition, the other naturally suited to case (ii).

**Acknowledgments.** The authors acknowledge support from NSF grants DMS 1005799 and 1008900; the second author also acknowledges support from NSF grant DMS 1320910.

## References

[1] E. Candès, L. Demanet, D. Donoho, and L. Ying, *Fast discrete curvelet transforms*, Multiscale Modeling & Simulation **5** (2006), no. 3, 861–899.

[2] E. J. Candès and D. L. Donoho, *Ridgelets: the key to high dimensional intermittency*, Philosophical Transactions of the Royal Society of London A **357** (1999), 2495–2509.

[3] E. J. Candès and D. L. Donoho, *New tight frames of curvelets and optimal representations of objects with piecewise $C^2$ singularities*, Commun. Pure Appl. Math. **57** (2004), no. 2, 219–266.

[4] E. J. Candès and D. L. Donoho, *Continuous curvelet transform: I. Resolution of the wavefront set*, Appl. Comput. Harmon. Anal. **19** (2005), no. 2, 162–197.

[5] S. Dekel and A. Sherman, *Curvelets: A low-level system for computer vision*, preprint, GE Healthcare, 2008.

[6] D. L. Donoho and X. Huo, *Beamlets and multiscale image analysis*, Multiscale and Multiresolution Methods (T. J. Barth, T. Chan, and R. Haimes, eds.), Lecture Notes in Computational Science and Engineering, vol. 20, Springer Berlin Heidelberg, 2002, pp. 149–196.

[7] D. L. Donoho and G. Kutyniok, *Microlocal analysis of the geometric separation problem*, Communications on Pure and Applied Mathematics **66** (2013), 1–47.

[8] G. Easley, D. Labate, and W.-Q Lim, *Sparse directional image representations using the discrete shearlet transform*, Appl. Comput. Harmon. Anal. **25** (2008), no. 1, 25–46.

[9] G. R. Easley and D. Labate, *Image processing using shearlets*, Shearlets: Multiscale Analysis for Multivariate Data, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2012, pp. 283–325. MR MR2249312 (2007d:42072)

[10] L. Evans and R. Gariepy, *Measure theory and fine properties of functions*, CRC Press, 1992.

[11] W. Fleming, *Functions of several variables*, second ed., Springer-Verlag, New York, 1977.

[12] G. B. Folland, *Real analysis: Modern techniques and their applications*, second ed., John Wiley and Sons, Inc., New York, 1999.

[13] M. Genzel and G. Kutyniok, *Asymptotic analysis of inpainting via universal shearlet systems*, preprint (2014).

[14] K. Guo, R. Houska, and D. Labate, *Microlocal analysis of singularities from directional multiscale representations*, Proceedings from the international conference on Approximation Theory XIV (San Antonio, Texas), Springer Proceedings in Mathematics, April 2014.

[15] K. Guo and D. Labate, *Characterization and analysis of edges using the continuous shearlet transform*, SIAM on Imaging Sciences **2** (2009), 959–986.

[16] ———, *Analysis and detection of surface discontinuities using the 3d continuous shearlet transform*, Appl. Comput. Harmon. Anal **30** (2011), 231–242.

[17] ———, *Characterization of piecewise-smooth surfaces using the 3D continuous shearlet transform*, J. Fourier Anal. Appl. **18** (2012), 488–516.

[18] ———, *Optimally sparse representations of 3d data with $C^2$ surface singularities using parseval frames of shearlets*, SIAM J. Math. Analysis **44** (2012), no. 2, 851–886.

[19] K. Guo, D. Labate, and W.-Q Lim, *Edge analysis and identification using the continuous shearlet transform*, Appl. Comput. Harmon. Anal. **27** (2009), no. 1, 24–46.

[20] E. J. King, G. Kutyniok, and X. Zhuang, *Analysis of inpainting via clustered sparsity and microlocal analysis*, Journal of Mathematical Imaging and Vision **48** (2014), no. 2, 205–234.

[21] G. Kutyniok, *Geometric separation by single-pass alternating thresholding*, Appl. Comput. Harmon. Anal. **36** (2014), 23–50.

[22] G. Kutyniok and D. Labate, *Resolution of the wavefront set using continuous shearlets*, Trans. Amer. Math. Soc. **361** (2009), no. 5, 2719–2754.

[23] D. Labate, F. Laezza, B. Ozcan, P. Negi, and M. Papadakis, *Efficient processing of fluorescence images using directional multiscale representations*, Math. Model. Nat. Phenom. (2014).

[24] D. Labate, W.-Q Lim, G. Kutyniok, and G. Weiss, *Sparse multidimensional representation using shearlets*, Wavelets XI (San Diego, CA, 2005) (A. F. Laine, M. Papadakis, and M. A. Unser, eds.), SPIE, Bellingham, WA, 2009, 2005, p. 59140U.

[25] J. M. Lee, *Introduction to smooth manifolds*, Springer Science+Business Media, Inc., New York, 2003.

[26] D. A. Schug, G. R. Easley, and D. OLeary, *Three-dimensional shearlet edge analysis*, Proc. of SPIE Conference on Independent Component Analyses, Wavelets, Neural Networks, Biosystems, and Nanoengineering IX (Orlando), 2011.

[27] J. Starck, M. Elad, and D. L. Donoho, *Image decomposition via the combination of sparse representations and a variational approach.*, IEEE Transactions on Image Processing **14** (2005), no. 10, 1570–1582.

[28] J. Starck, F. Murtagh, E.. Candés, and D. Donoho, *Gray and color image contrast enhancement by the curvelet transform*, IEEE Trans. Image Process. **12** (2003), no. 6, 706–717.

[29] A. Woiselle, J.-L. Starck, and J. Fadili, *3D curvelet transforms and astronomical data restoration*, Appl. Comput. Harmon. Anal. **28** (2010), 171–188.

[30] S. Yi, D. Labate, G. R. Easley, and H. Krim, *A shearlet approach to edge analysis and detection*, IEEE Trans. Image Process. **18** (2009), no. 5, 929–941.