A Unified Characterization of Reproducing Systems Generated by a Finite Family

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Abstract

This paper presents a general result from the study of shift-invariant spaces that characterizes tight frame and dual frame generators for shift-invariant subspaces of $L^2(\mathbb{R}^n)$. A number of applications of this general result are then obtained, among which are the characterization of tight frames and dual frames for Gabor and wavelet systems.

1 Introduction

The aim of this paper is to provide an unified approach to the characterization of a large class of systems satisfying a reproducing formula of the form

$$f = \sum_{i \in \mathcal{I}} \langle f, \psi_i \rangle \, \psi_i, \tag{1}$$

or, more generally,

$$f = \sum_{i \in \mathcal{I}} \langle f, \phi_i \rangle \, \psi_i, \tag{2}$$

where $f, \psi_i, \phi_i, i \in \mathcal{I}$, belong to $L^2(\mathbb{R}^n)$. The kind of reproducing systems that we will consider are generated by the action of translations, dilations and modulations on a finite family of functions. To keep the notation to a minimum and focus on the main ideas that we will present in this paper, let us restrict our attention, for the moment, to one-dimensional systems generated by a single function. The Gabor system, for example, is generated by the action of the translations T_{ck}, $k \in \mathbb{Z}$, $c > 0$, and modulations M_{bm}, $m \in \mathbb{Z}$, $b > 0$, on a function $\psi \in L^2(\mathbb{R})$, where $T_{ck}\,\psi(x) = \psi(x - ck)$ and $M_{bm}\,\psi(x) = e^{2\pi i bmx}\,\psi(x)$. The system thus obtained is $\{\psi_i\}_{i\in\mathcal{I}} = \{M_{bm}\,T_{ck}\,\psi\}_{m,k\in\mathbb{Z}}$, where the indexing set, in this case, is $\mathcal{I} = \{(m,k) :\ m,k \in \mathbb{Z}\}$. Wavelets are obtained in a similar way, with the dilations D_{2^j}, where $D_{2^j}\,\psi(x) = 2^{j/2}\,\psi(2^j x)$, $j \in \mathbb{Z}$, replacing the modulations. The system thus obtained has the form $\{\psi_i\}_{i\in\mathcal{I}} = \{D_{2^j}\,T_{ck}\,\psi\}_{j,k\in\mathbb{Z}}$, where $\mathcal{I} = \{(j,k) :\ j,k \in \mathbb{Z}\}$, and is often referred to as an affine system. In the case of the Gabor system, the order of the modulation and translation operators can be reversed; however, this is not the case for the affine system, if we want to preserve the reproducing property (cf. Section 3 for more details).

In the Gabor case, the following simple result, which was originally found in [11] with some mild decay assumptions on ψ, characterizes all the Gabor systems for which the reproducing formula (1) holds (cf. Section 3 for more references about this and related results).

Theorem 1.1. *Let $\psi \in L^2(\mathbb{R})$, $b,\ c > 0$. Then*

$$f = \sum_{k,m\in\mathbb{Z}} \langle f, M_{bm}\,T_{ck}\,\psi\rangle\, M_{bm}T_{ck}\ \psi \qquad \text{for all } f \in L^2(\mathbb{R})$$

if and only if the following two equations are satisfied:

$$\sum_{m\in\mathbb{Z}} |\psi(\xi - cm)|^2 \;=\; b \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{3}$$

$$\sum_{m\in\mathbb{Z}} \psi(\xi - cm)\,\overline{\psi(\xi - cm + b^{-1}u)} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R},\ \ u \in \mathbb{Z}\setminus\{0\}. \tag{4}$$

The study of the characterization of affine systems is more complex. The program of characterizing orthonormal wavelets and affine tight frames in terms of simple equations has been carried out by G. Weiss and his collaborators, starting with the characterization of band-limited orthonormal wavelets [1]. Similarly to the case of Gabor systems, all the affine systems for which the reproducing formula (1) holds can be characterized in terms of two simple equations (cf. Section 3 for references about this and related results). In the one-dimensional case, with dyadic dilation, we have:

Theorem 1.2. *Let $\psi \in L^2(\mathbb{R})$, $c > 0$. Then*

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, D_{2^j}\,T_{ck}\,\psi\rangle\, D_{2^j}\,T_{ck}\,\psi \qquad \text{for all } f \in L^2(\mathbb{R})$$

if and only if the following two equations are satisfied:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\,\xi)|^2 \;=\; c \qquad \textit{for a.e. } \xi \in \mathbb{R}, \tag{5}$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\,\xi)\,\overline{\hat{\psi}(2^j(\xi-q))} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R},\;\; q \in 2\mathbb{Z}+1. \tag{6}$$

Observe that equations (3) and (4) in Theorem 1.1 can be replaced by the following equations in terms of $\hat{\psi}$ instead of ψ (cf. Corollary 3.3):

$$\sum_{m\in\mathbb{Z}} |\hat{\psi}(\xi - bm)|^2 \;=\; c \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{7}$$

$$\sum_{m\in\mathbb{Z}} \hat{\psi}(\xi - bm)\,\overline{\hat{\psi}(\xi - bm + c^{-1}u)} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R},\;\; u \in \mathbb{Z}\setminus\{0\}. \tag{8}$$

A comparison of these equations with equations (5) and (6) shows that the equations that characterize an affine reproducing system are in a certain sense the analog of the equations that characterize a Gabor reproducing system, where the modulation operator is replaced by the dilation operator. One of the motivations of our work is to better understand the analogy between these two systems.

Related to this, is the research of Ron and Shen, who have shown that, if the affine system $\{D_{2^j}\,T_{ck}\,\psi\}_{j,k}$ is modified so that, for $j<0$, $D_{2^j}\,T_{ck}\,\psi$ is replaced by $2^{j/2}\,T_{ck}\,D_{2^j}\,\psi$, then this new collection, which they called a *quasi-affine system*, is a reproducing system whenever the affine system has this property [27]. The importance of the quasi-affine system is that, unlike the affine system, but similarly to the Gabor system, it is shift-invariant, i.e., any integer translation of the functions in the system leaves the system unchanged. Ron and Shen have used this observation to study both wavelets and Gabor systems using techniques from the theory of shift-invariant spaces (cf. [25, 26, 27]). These considerations indicate the special role of the translation operator in the study of Gabor and affine systems. For example, it was recently pointed out by Weiss and Wilson [WW01], that the translations, more than the dilations, play a critical role in the discretization of continuous wavelets.

Let us consider now the general n-dimensional case. Let g_i, $i\in\mathcal{I}$, be elements of $L^2(\mathbb{R}^n)$, and $X(g)=\{T_k\,g_i : k\in\mathbb{Z}^n, i\in\mathcal{I}\}$. Since any multi-integer shifts of the functions in $X(g)$ leave the system unchanged, $X(g)$ is clearly a shift-invariant system, and the space generated as the closure of its span is a called a *shift-invariant* subspace of $L^2(\mathbb{R}^n)$. The general properties of these spaces have been investigated by a number of authors, including [19, 13, 25] and [3]. In this paper, we will present a new approach to the characterization of all the shift-invariant systems $X(g)$ that realize a reproducing system, and will show that such systems are characterized by two simple equations, similar to the equations of Theorem 1.1 and 1.2. With respect to [25], where similar results were obtained, our approach is more direct and more focused on the characterization of functional systems in terms of equations. As an application of our general result, we will deduce the equations that characterize a large class of reproducing systems, including Gabor systems, wavelets and related systems.

The paper is organized as follows. In Section 2 we establish some notation and definitions that will be useful throughout the paper. In Section 3 we describe the main result and its implications. We develop our approach to the characterization of shift-invariant systems for $L^2(\mathbb{R}^n)$ in Section 4, and then apply these results to the study of affine and Gabor systems in Sections 5 and 6, respectively.

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2 Preliminaries

We consider three fundamental operators on $L^2(\mathbb{R}^n)$: the *translations* $T_y : (T_y f)(x) = f(x-y)$, where $y \in \mathbb{R}^n$; the *dilations* $D_A : (D_A f)(x) = |\det A|^{\frac{1}{2}} f(Ax)$, where A is an *expanding* $n \times n$ *dilation* matrix preserving $\mathbb{Z}^n$, i.e., A is an integer matrix and all eigenvalues λ of A satisfy $|\lambda| > 1$; and the *modulations* $M_z : (M_z f)(x) = e^{2\pi i z \cdot x} f(x)$, where $z \in \mathbb{R}^n$.

We use the standard notation $\|f\|$ for the norm of $f \in L^2(\mathbb{R}^n)$, and $\langle f, g\rangle$ for the usual inner product of $f, g \in L^2(\mathbb{R}^n)$. The Fourier transform is defined as $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi \cdot x}\, dx$; the inverse Fourier transform is $\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi$. Throughout the paper, the space $\mathbb{T}^n$ will be identified with $[0,1]^n$.

We will need the following facts from the theory of frames. Additional information on the subject can be found in [14, 18, 20]. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{I}$ a countable indexing set.

Definition 2.1. A sequence $g = \{g_i\}_{i \in \mathcal{I}}$ of elements of $\mathcal{H}$ is a *Bessel sequence* if there exists a constant $B_g > 0$ so that

$$\sum_{i \in \mathcal{I}} |\langle f, g_i \rangle|^2 \;\leq\; B_g \, \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If, in addition, there is a constant $0 < A_g \leq B_g$ so that

$$A_g \, \|f\|^2 \;\leq\; \sum_{i \in \mathcal{I}} |\langle f, g_i \rangle|^2 \;\leq\; B_g \, \|f\|^2 \qquad \text{for all } f \in \mathcal{H},$$

then $\{g_i\}_{i \in \mathcal{I}}$ is a *frame* for $\mathcal{H}$.

The numbers A_g, B_g are called the *lower* and *upper frame bounds*, respectively. The frame is a *tight frame* if A_g and B_g can be chosen so that $A_g = B_g$, and is a *normalized tight frame* if $A_g = B_g = 1$.

Given a frame $g = \{g_i\}_{i \in \mathcal{I}}$ of $\mathcal{H}$ with lower and upper frame bounds A_g and B_g, respectively, the *frame operator* S_g, defined by $S_g \, f \;=\; \sum_{i \in \mathcal{I}} \langle f, g_i \rangle \, g_i$, is a bounded, invertible and positive mapping of $\mathcal{H}$ onto itself. This provides the *frame decomposition*:

$$f \;=\; \sum_{i \in \mathcal{I}} \langle f, S_g^{-1} g_i \rangle \, g_i \;=\; \sum_{i \in \mathcal{I}} \langle f, g_i \rangle \, S_g^{-1} g_i, \qquad \text{for all } f \in \mathcal{H}, \tag{9}$$

with convergence in $\mathcal{H}$. The sequence $\{S_g^{-1} g_i\}_{i \in \mathcal{I}}$ is also a frame for $\mathcal{H}$, called the *canonical dual frame* of $\{g_i\}_{i \in \mathcal{I}}$, and has upper and lower frame bounds B_g^{-1} and A_g^{-1}, respectively. If the frame is tight, i.e. $A_g = B_g$, then $S_g^{-1} = A_g^{-1} \, I$, where I is the identity operator, and the frame decomposition becomes:

$$f \;=\; \frac{1}{A_g} \sum_{i \in \mathcal{I}} \langle f, g_i \rangle \, g_i \qquad \text{for all } f \in \mathcal{H}, \tag{10}$$

with convergence in $\mathcal{H}$.

Equations (9) and (10) show that a frame provides a basis-like representation. In general, however, a frame need not be a basis. If the frame $\{g_i\}_{i \in \mathcal{I}}$ happens to form a basis, then the only way to write

$$f = \sum_{i \in \mathcal{I}} c_i \, g_i, \qquad f \in \mathcal{H}, \tag{11}$$

is with $c_i = \langle f, S_g^{-1} g_i \rangle$. If the frame is not a basis, then there will be other choices of $\{c_i\}_{i \in \mathcal{I}}$ so that equation (11) is satisfied. However, among all these (non-canonical) choices, the canonical dual frame $c_i \;=\; \langle f, S_g^{-1} g_i \rangle$ satisfies the following "minimal" property: the sequence $\{c_i\}_{i\in\mathcal{I}}$ that minimizes the quantity $\sum_{i\in\mathcal{I}} |c_i|^2$ over all $\{c_i\}_{i\in\mathcal{I}}$ satisfying (11) is uniquely given by $c_i = \langle f, S_g^{-1} g_i\rangle$ (cf. [10, p.62]). Furthermore, the elements of a frame $\{g_i\}_{i\in\mathcal{I}}$ must satisfy $\|g_i\| \le \sqrt{B_g}$ for all $i \in \mathcal{I}$, as can easily be seen from

$$\|g_k\|^4 = |\langle g_k, g_k\rangle|^2 \le \sum_{i\in\mathcal{I}} |\langle g_k, g_i\rangle|^2 \le B_g \, \|g_k\|^2.$$

In particular, if $\{g_i\}_{i\in\mathcal{I}}$ is a normalized tight frame, then $\|g_i\| \le 1$ for all $i \in \mathcal{I}$, and the frame is an orthonormal basis for $\mathcal{H}$ if and only if $\|g_i\| = 1$ for all $i \in \mathcal{I}$ (cf. Chapter 8 in [20]).

The following terminology will also be used. Let $g = \{g_i\}_{i\in\mathcal{I}}$ and $\gamma = \{\gamma_i\}_{i\in\mathcal{I}}$ be Bessel sequences for $\mathcal{H}$. Then the operator

$$K_{g,\gamma}(f,h) \;=\; \sum_{i\in\mathcal{I}} \langle f, g_i\rangle \, \langle \gamma_i, h\rangle$$

defines a bounded sesquilinear operator on $\mathcal{H} \times \mathcal{H}$. We have the following definition.

Definition 2.2. Let $g = \{g_i\}_{i\in\mathcal{I}}$ and $\gamma = \{\gamma_i\}_{i\in\mathcal{I}}$ be Bessel sequences for $\mathcal{H}$. Then $\{\gamma_i\}_{i\in\mathcal{I}}$ is called a *dual frame* of $\{g_i\}_{i\in\mathcal{I}}$, if

$$K_{g,\gamma}(f,h) \;=\; \langle f, h\rangle, \qquad \text{for all } f,h \in \mathcal{H}. \tag{12}$$

If this is the case, then we have:

$$f \;=\; \sum_{i\in\mathcal{I}} \langle f, \gamma_i\rangle \, g_i \;=\; \sum_{i\in\mathcal{I}} \langle f, g_i\rangle \, \gamma_i, \qquad \text{for all } f \in \mathcal{H},$$

with convergence in $\mathcal{H}$. Note that, by the polarization identity for sesquilinear forms, we have $K_{g,\gamma}(f,h) \;=\; \frac{1}{4} \sum_{n=0}^{3} i^n \, K_{g,\gamma}(f + i^n h, f + i^n h)$. Therefore, (12) holds if and only if it holds for all $f = h \in \mathcal{H}$.

We complete this section by recalling the following useful properties of tight frames and dual frames for $L^2(\mathbb{R}^n)$.

Lemma 2.1 ([20]). *Let $\{g_i\}_{i\in\mathcal{I}} \subset L^2(\mathbb{R}^n)$. Then the following are equivalent.*

$$\|f\|^2 \;=\; \sum_{i\in\mathcal{I}} |\langle f, g_i\rangle|^2, \qquad \textit{for all } f \in L^2(\mathbb{R}^n),$$

$$\|f\|^2 \;=\; \sum_{i\in\mathcal{I}} |\langle f, g_i\rangle|^2, \qquad \textit{for all } f \in \mathcal{D}, \textit{ where } \mathcal{D} \textit{ is dense in } L^2(\mathbb{R}^n).$$

Lemma 2.2 ([15]). *Suppose that* $\{g_i\}_{i\in\mathcal{I}}$, $\{\gamma_i\}_{i\in\mathcal{I}} \subset L^2(\mathbb{R}^n)$ *are Bessel sequences. Then the following are equivalent.*

$$\langle f, h\rangle = \sum_{i\in\mathcal{I}} \langle f, g_i\rangle \, \langle \gamma_i, h\rangle, \qquad \text{for all } f \in L^2(\mathbb{R}^n),$$
$$\langle f, h\rangle = \sum_{i\in\mathcal{I}} \langle f, g_i\rangle \, \langle \gamma_i, h\rangle, \qquad \text{for all } f \in \mathcal{D}, \text{ where } \mathcal{D} \text{ is dense in } L^2(\mathbb{R}^n).$$

3 Main Results

In this section, we will present the main results of this paper and show that the characterizations of a large class of reproducing systems, including wavelets and Gabor systems, can be obtained as corollaries to a general result from the theory of shift-invariant systems. To focus on this main idea, most of the proofs, together with some additional results, will be discussed in Sections 4, 5 and 6.

Let $\mathcal{P}$ be a countable indexing set. We have the following characterization of tight frame generators for $L^2(\mathbb{R}^n)$ under multi-integer shifts, which is due to Ron and Shen [25, Cor.3.3.6] (cf. also [23, Th.1.2.5]). As described in Section 1, we will present in Section 4 a new proof of this result.

Theorem 3.1. *Let* $\{g_p\}_{p\in\mathcal{P}} \subset L^2(\mathbb{R}^n)$ *and let* C *be a non-singular* $n \times n$ *matrix with real entries. Then,*

$$\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, T_{Ck}\, g_p\rangle\right|^2 = \|f\|^2 \qquad \text{for all } f \in L^2(\mathbb{R}^n) \tag{13}$$

if and only if

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{g}_p(\xi + (C^T)^{-1}u)} = |\det C| \;\; \delta_{u,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{14}$$

for each $u \in \mathbb{Z}^n$, *where* δ *is the product Kronecker delta in* $\mathbb{Z}^n$.

Equation (13) asserts that the system $\{T_{Ck}\, g_p, k \in \mathbb{Z}^n,\ p \in \mathcal{P}\}$ is a normalized tight frame for $L^2(\mathbb{R}^n)$. Observe that, by the properties of tight frames discussed in in Section 2, equation (14) with the assumption that $\|g_p\| = 1$, for every $p \in \mathcal{P}$, is necessary and sufficient for $\{T_{Ck}\, g_p\}_{k\in\mathbb{Z}^n, p\in\mathcal{P}}$ to form an orthonormal basis. A similar observation will hold for all the characterizations of tight frames that we will discuss in this section as applications of Theorem 3.1.

For our first application of Theorem 3.1, we will consider $\{g_p\}_{p\in\mathcal{P}}$, where $\mathcal{P}$ is described by

$$\mathcal{P} = \big\{(\ell, m) : \ell = 1, 2, \ldots L \text{ and } m \in \mathbb{Z}^n\big\},$$

and g_p, $p = (\ell, m)$, is defined by

$$g_p(x) = g_{(\ell,m)}(x) = (M_{Bm}\, g^\ell)(x) = e^{2\pi i Bm\cdot x} g^\ell(x), \tag{15}$$

where B is a fixed non-singular $n \times n$ matrix with real entries. Under multi-integer translations in $\mathbb{Z}^n$, the collection $\{g_p\}_{p\in\mathcal{P}}$ generates the *Gabor* or *Weyl–Heisenberg system* $\big\{T_{Ck}\, M_{Bm}\, g^\ell : k, m \in \mathbb{Z}^n, \ell = 1, \ldots, L\big\}$.

Observe that the order of the modulation and translation operators in the Gabor system can be reversed without affecting the reproducing property of the system. In fact, $M_{Bm}\, T_{Ck} = e^{2\pi i Bm\cdot Ck}\, T_{Ck}\, M_{Bm}$, and so $\sum_{k,m}\langle f, T_{Ck}\, M_{Bm}\, g\rangle\, T_{Ck}\, M_{Bm}\, \gamma = \sum_{k,m}\langle f, M_{Bm}\, T_{Ck}\, g\rangle\, M_{Bm}\, T_{Ck}\, \gamma$, for any $f, g, \gamma \in L^2(\mathbb{R}^n)$ (provided that the sum converges).
Using Theorem 3.1, we then obtain the following known characterization of Gabor tight frames. This result (or the equivalent one stated in Corollary 3.3) can be found in the literature in [26, 6, 9], while it also follows from the developments in [23].

Corollary 3.2. *Let $g^1, \ldots, g^L \in L^2(\mathbb{R}^n)$ and let C,B be non-singular matrices with real entries. Then,*

$$\sum_{\ell=1}^{L} \sum_{k,m\in\mathbb{Z}^n} \left|\langle f, T_{Ck}\, M_{Bm}\, g^\ell\rangle\right|^2 = \|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}^n)$$

if and only if

$$\sum_{\ell=1}^{L} \sum_{m\in\mathbb{Z}^n} \hat{g}^\ell(\xi - Bm)\, \overline{\hat{g}^\ell(\xi - Bm + (C^T)^{-1}u)} = |\det C|\, \delta_{u,0}, \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{16}$$

where $u \in \mathbb{Z}^n$ and δ is the product Kronecker delta in $\mathbb{Z}^n$.

The following alternative characterization of Gabor tight frames, stated in terms of g instead of $\hat{g}$, is more common in the Gabor literature.

Corollary 3.3. *Let $g^1, \ldots, g^L \in L^2(\mathbb{R}^n)$ and let C,B be non-singular matrices with real entries. Then,*

$$\sum_{\ell=1}^{L} \sum_{k,m\in\mathbb{Z}^n} \left|\langle f, T_{Ck}\, M_{Bm}\, g^\ell\rangle\right|^2 = \|f\|^2 \qquad \text{for all } f \in L^2(\mathbb{R}^n)$$

if and only if

$$\sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} g^\ell(x-Ck)\,\overline{g^\ell(x-Ck+(B^T)^{-1}u)} = |\det B|\,\delta_{u,0}, \qquad \text{for a.e. } x\in\mathbb{R}^n, \quad (17)$$

where $u\in\mathbb{Z}^n$ and δ is the product Kronecker delta in $\mathbb{Z}^n$.

As a second application of Theorem 3.1, it is natural to consider the systems that are obtained by replacing the modulation operator in (15) with the dilation operator. Let $\mathcal{P}$ be defined by

$$\mathcal{P} = \big\{(j,\ell) : j\in\mathbb{Z}, \text{ and } \ell = 1,\dots,L\big\},$$

and let $\tilde{\tilde{\theta}}_p$, $p=(j,\ell)$, be defined by

$$\tilde{\tilde{\theta}}_p(x) = \tilde{\tilde{\theta}}_{(j,\ell)}(x) := (|\det A|^{j/2} D_{A^j}\,\psi^\ell)(x) = |\det A|^j\,\psi^\ell(A^j x),$$

where A is a $n\times n$ nonsingular matrix with real entries, and $\Psi=\{\psi^1,\dots,\psi^L\}$ is a finite collection of functions in $L^2(\mathbb{R}^n)$. We observe that with this choice for $\mathcal{P}$ and $\tilde{\tilde{\theta}}_p$, the system $\{T_k\,\tilde{\tilde{\theta}}_p : k\in\mathbb{Z}^n, p\in\mathcal{P}\}$ is the system $\tilde{\tilde{X}}(\Psi) = \{|\det A|^{j/2}\,T_k\,D_{A^j}\,\psi^\ell : j\in\mathbb{Z}, k\in\mathbb{Z}^n, \ell=1,\dots,L\}$, which will be called a *co-affine system.* We refer to [16] for more details about this system and its relation to the affine group.

We obtain the following characterization of the co-affine system $\tilde{\tilde{X}}(\Psi)$.

Corollary 3.4. $\tilde{\tilde{X}}(\Psi)$ *is a normalized tight frame if and only if*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \big|\hat{\psi}^\ell\big((A^T)^j\xi\big)\big|^2 = 1 \qquad \text{for a.e. } \xi\in\mathbb{R}^n,$$

and

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \hat{\psi}^\ell\big((A^T)^j\xi\big)\,\overline{\hat{\psi}^\ell\big((A^T)^j(\xi+s)\big)} = 0 \qquad \text{for a.e. } \xi\in\mathbb{R}^n,$$

where $s\in\mathbb{Z}^n\setminus\{0\}$.

A comparison of Corollary 3.4 with Corollaries 3.2 shows that the equations that characterize the co-affine system are the analog of the equations that characterize the Gabor system, where the modulations are replaced by dilations. On the other hand, a comparison with the affine system, that we will discuss next (Corollary 3.8), shows that the conditions that characterize the co-affine system are more difficult to meet than the conditions of the affine system. In fact, there are no (nontrivial) co-affine normalized tight frames. This can be observed trivially from the fact that the norms $\| \det A^{j/2} T_k D_{A^j} \psi^\ell \| = |\det A|^{j/2} \|\psi^\ell\|$ are unbounded when $j \to \infty$, and therefore $\widetilde{\widetilde{X}}(\Psi)$ cannot be a Bessel system. The following surprising result shows that this situation cannot be improved by changing the normalization factor in the definition of the co-affine system.

Theorem 3.5 ([16]). *Let $\{\psi^1, \ldots, \psi^L\}$ be a finite collection of functions in $L^2(\mathbb{R}^n)$, and $\{c_j\}_{j\in\mathbb{Z}}$ be a sequence of constants. Then the system $\big\{c_j\, T_k\, D_{A^j}\, \psi^\ell : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots L\big\}$ cannot be a frame for $L^2(\mathbb{R}^n)$.*

The third application of Theorem 3.1 of interest here regards the characterization of affine (wavelet) tight frames for $L^2(\mathbb{R}^n)$. Let $\Psi = \{\psi^1, \ldots, \psi^L\}$ be a finite collection of functions in $L^2(\mathbb{R}^n)$. The *affine system* generated by Ψ, denoted as $X(\Psi)$, is defined by

$$X(\Psi) = \big\{\psi^\ell_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\big\}, \tag{18}$$

where $\psi^\ell_{j,k} = D_{A^j}\, T_k\, \psi^\ell$ and A is an *expanding* $n \times n$ dilation matrix with *integer* entries. Observe that the affine system is not shift-invariant, in general, and thus Theorem 3.1 does not apply here directly.

In order to study the affine system $X(\Psi)$, we will make use of the *quasi-affine system* generated by Ψ, denoted by $\widetilde{X}^{(N)}(\Psi)$, which is defined by

$$\widetilde{X}^{(N)}(\Psi) = \big\{\widetilde{\psi}^\ell_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\big\}, \tag{19}$$

where $N \geq 0$ is fixed, and

$$\widetilde{\psi}^\ell_{j,k} = \begin{cases} |\det A|^{\frac{j}{2}}\, T_k\, D_{A^j}\, \psi^\ell, & j \leq N \\ \psi^\ell_{j,k}, & j > N. \end{cases}$$

Observe that this definition generalizes the definition of quasi-affine systems introduced by Ron and Shen where $N = 0$ [27], and, as we will show later, the reproducing properties of the quasi-affine system depend on N. Also notice that the quasi-affine system, unlike the affine system is shift-invariant. In fact, if $j \leq N$, then $\widetilde{\psi}^\ell_{j,k}$ is shift-invariant by construction, while, if $j > N$, then $\widetilde{\psi}^\ell_{j,k} = \psi^\ell_{j,k}$ is shift-invariant since A is integer-valued and $N \geq 0$.

The utility of the quasi-affine system is demonstrated by the following result, which shows that there is some sort of equivalence between affine and quasi-affine systems when $N = 0$. This result was discovered originally by Ron and Shen [27] under a mild decay assumption on ψ, and proved in full generality by Chui, Shi, and Stöckler [8, Th. 2, Th. 3]. Remark that we always assume that A is an *expanding* $n \times n$ dilation matrix with *integer coefficients*, and the proof of the following theorem requires this assumption.

Theorem 3.6 ([8]). *Let Ψ, Φ be finite subsets of $L^2(\mathbb{R}^n)$ with the same cardinality. Then,*

(1) $X(\Psi)$ is a Bessel sequence if and only if $\widetilde{X}^{(0)}(\Psi)$ is a Bessel sequence. Furthermore, the upper frame bounds are equal.

(2) $X(\Psi)$ is a frame if and only if $\widetilde{X}^{(0)}(\Psi)$ is a frame. Furthermore, their lower and upper frame bounds are equal.

(3) $X(\Phi)$ is a (affine) dual sequence of $X(\Psi)$ if and only if $\widetilde{X}^{(0)}(\Phi)$ is a (quasi-affine) dual sequence of $\widetilde{X}(\Psi)$.

Observe the statement (3) in Theorem 3.6 is not stated and proved in [8], but it follows from the techniques developed in their paper, as observed by Bownik [2, Theorem 4.1].

Since the quasi-affine system $\widetilde{X}^{(N)}(\Psi)$ is invariant under translations by $k \in \mathbb{Z}^n$, we can realize $\widetilde{X}^{(N)}(\Psi)$ by exhibiting a generating set under $\mathbb{Z}^n$-translations. We have

$$\widetilde{X}^{(N)}(\Psi) = \big\{T_k\, \widetilde{\psi}^{\ell}_{j,d} : k \in \mathbb{Z}^n, j \in \mathbb{Z}, d \in \mathcal{D}_j, \ell = 1, \dots, L\big\},$$

where, for $j > N$, $\mathcal{D}_j$ denotes a complete set of representatives of distinct cosets of $\mathbb{Z}^n/(A^j\mathbb{Z}^n)$ and, for $j \leq N$, $\mathcal{D}_j = \{0\}$. This description of $\widetilde{X}^{(N)}(\Psi)$, when $N = 0$, can be also found in [4].

In accordance with the above observation, we now define the collection $\{\theta_p\}_{p \in \mathcal{P}}$ to which Theorem 3.1 will be applied for the characterization of quasi-affine normalized tight frames. Let $\mathcal{P}$ be defined by

$$\mathcal{P} = \big\{(j, d, \ell) : j \in \mathbb{Z}, d \in \mathcal{D}_j, \text{ and } \ell = 1, \dots, L\big\},$$

and let θ_p be

$$\theta_p(x) = \theta_{(j,d,\ell)}(x) := (D_{A^j}\, T_d\, \widetilde{\psi}^{\ell})(x) = |\det A|^{j/2}\, \widetilde{\psi}^{\ell}(A^j x - d).$$

We observe that with this choice for $\mathcal{P}$ and θ_p, the system $\{T_k\,\theta_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the quasi-affine system $\widetilde{X}^{(N)}(\Psi)$.

We have the following characterization of the quasi-affine systems $\widetilde{X}^{(N)}(\Psi)$.

Corollary 3.7. *Let $N \geq 0$. $\widetilde{X}^{(N)}(\Psi)$ is a normalized tight frame if and only if*

$$\sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left|\hat{\psi}^\ell\big((A^T)^j \xi\big)\right|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{20}$$

and

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \hat{\psi}^\ell\big((A^T)^j \xi\big)\, \overline{\hat{\psi}^\ell\big((A^T)^j(\xi + (A^T)^r s)\big)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{21}$$

for $r = 0, -1, \ldots, -N$, where $s \in \mathbb{Z}^n \setminus A^T\mathbb{Z}^n$.

Observe that (21), with $r = 0, -1, \ldots, -N$, is a set of $N+1$ equations. In case $N = 0$, then (21) reduces to one equation, and, by Theorem 3.6, equations (20) and (21) also characterize the affine system $X(\Psi)$. It is easy to construct examples of quasi-affine normalized tight frames with $N > 0$. Consider, for instance, the one-dimensional quasi-affine system generated by $\hat{\psi}(\xi) = \chi_{[-1/2,-1/4]\cup[1/4,1/2]}(\xi)$, with dyadic dilation, $L = 1$ and $N = 1$. Then, the quasi-affine system $\widetilde{X}^{(1)}(\Psi)$ satisfies the assumptions of Corollary 3.7, and thus it forms a (quasi-affine) normalized tight frame. In general, it is clear that, if $\widetilde{X}^{(N_1)}(\Psi)$ is a normalized tight frame, then $\widetilde{X}^{(N_2)}(\Psi)$, with $N_2 < N_1$ has the same property. On the other hand, as N increases, and more (affine) functions $\psi^\ell_{j,k}$ are replaced by (co-affine) functions $|\det A|^{1/2}\, T_k\, D_{A^j}\, \psi^\ell$, there are fewer quasi-affine normalized tight frames.

As we mentioned in the preceding paragraph, by Theorem 3.6, an affine system $X(\Psi)$ is a normalized tight frame if and only if the quasi-affine system $\widetilde{X}^{(0)}(\Psi)$ has the same property. Therefore, using Corollary 3.7, with $N = 0$, we obtain the following known characterization of multiwavelets. This result can be found in the literature in [2, 4, 5]. This result is also obtained in [27] under some decay conditions on ψ, and for special dilations in [20] and [15]. A more general result, with arbitrary (non-integer) expanding dilation matrices, has recently been found in [7].

Corollary 3.8. *$X(\Psi)$ is a normalized tight frame if and only if (20) and*

$$\sum_{\ell=1}^{L} \sum_{j \geq 0} \hat{\psi}^\ell\big((A^T)^j \xi\big)\, \overline{\hat{\psi}^\ell\big((A^T)^j(\xi + s)\big)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{22}$$

where $s \in \mathbb{Z}^n \setminus A^T\mathbb{Z}^n$, hold.

4 Shift-invariant systems

In this section, we present the proof of Theorem 3.1 and discuss some additional properties of multivariate shift-invariant systems, including the characterization of dual frames for $L^2(\mathbb{R}^n)$.

The following useful facts are easy to verify.

Lemma 4.1. *Let A be nonsingular a $n \times n$ matrix with real entries, and $y, z \in \mathbb{R}^n$.*

(1) $(T_y f)^\wedge = M_{-y}\hat{f}, \quad (M_z f)^\wedge = T_z\hat{f}, \quad (D_{A^j} f)^\wedge = D_{(A^T)^{-j}}\hat{f};$

(2) $T_y M_z f = e^{-2\pi i z\cdot y} M_z T_y f;$

(3) $(T_y M_z f)^\wedge = e^{-2\pi i z\cdot y} T_z M_{-y}\hat{f};$

(4) $(D_{A^j} T_y f)^\wedge(\xi) = M_{-A^{-j}y} D_{(A^T)^{-j}}\hat{f}(\xi) = |\det A|^{-j/2}\hat{f}((A^T)^{-j}\xi)\, e^{-2\pi i A^{-j}y\cdot\xi};$

(5) $(T_y D_{A^j} f)^\wedge(\xi) = M_{-y} D_{(A^T)^{-j}}\hat{f}(\xi) = |\det A|^{-j/2}\hat{f}((A^T)^{-j}\xi)\, e^{-2\pi i y\cdot\xi}.$

In order to prove Theorem 3.1, we need to discuss some useful properties of shift-invariant systems. From now on, let

$$\mathcal{D} = \left\{f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \hat{f} \text{ has compact support in } \mathbb{R}^n\right\}.$$

It is clear that $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^n)$. For simplicity, we will use the notation $g_{p,y} = T_y\, g_p$.

Let C be a non-singular $n\times n$ matrix with real entries. Then the *C-bracket product* of f and g is defined as

$$[f,g](x;C) \;=\; \sum_{k\in\mathbb{Z}^n} f(x-Ck)\,\overline{g(x-Ck)}. \tag{23}$$

This is an extension of the notion and and notation introduced in [13] when $C = I$. It is easy to verify that for $f, g \in L^2(\mathbb{R}^n)$ the series (23) converges absolutely a.e. to a function in $L^1(C\,\mathbb{T}^n)$. This follows from the following: by the Monotone Convergence Theorem,

$$\begin{aligned}\int_{C\,\mathbb{T}^n}\sum_{k\in\mathbb{Z}^n}|f(x-Ck)\,\overline{g(x-Ck)}|\,dx &= \sum_{k\in\mathbb{Z}^n}\int_{C\,\mathbb{T}^n}|f(x-Ck)\,\overline{g(x-Ck)}|\,dx\\ &= \int_{\mathbb{R}^n}|f(x)\,\overline{g(x)}|\,dx \le \infty.\end{aligned}$$

The following observation is easy to verify:

Lemma 4.2. *Let C be a non-singular $n\times n$ matrix with real entries. Then,*

$$\int_{C\mathbb{T}^n}[f,h](x,C)\,dx \;=\; \langle f,h\rangle, \qquad \textit{for all } f,\, h\in L^2(\mathbb{R}^n).$$

Observe that, by Lemma 4.2, $\int_{C\mathbb{T}^n}[f,h](x,C)\,dx = \int_{C\mathbb{T}^n}[\hat f,\hat h](\xi,C)\,d\xi$. It also follows from Lemma 4.2 that the integral $\int_{C\mathbb{T}^n}[f,h](x,C)\,dx$ is independent of the choice of the matrix C.

We make the following useful observation:

Lemma 4.3. *Let C be a non-singular $n\times n$ matrix with real entries, and let $C^I = (C^T)^{-1}$. For all $g,\gamma\in L^2(\mathbb{R}^n)$ and $f\in\mathcal{D}$, we have*

$$\sum_{k\in\mathbb{Z}^n}\langle f,T_{Ck}\,g\rangle\,\langle T_{Ck}\,\gamma,f\rangle = \frac{1}{|\det C|}\int_{C^I\mathbb{T}^n}[\hat f,\hat g](\xi;C^I)\,[\hat\gamma,\hat f](\xi;C^I)\,d\,\xi,$$

with absolute convergence of the integral.

Proof. Since $[\hat f,\hat g](\xi;C^I)\in L^1(C^I\mathbb{T}^n)$, we have that, for every $k\in\mathbb{Z}^n$,

$$\langle f,T_{Ck}\,g\rangle = \int_{\mathbb{R}^n}\hat f(\xi)\,\overline{\hat g(\xi)}\,e^{2\pi iCk\cdot\xi}\,d\xi = \int_{C^I\mathbb{T}^n}[\hat f,\hat g](\xi;C^I)\,e^{2\pi iCk\cdot\xi}\,d\xi, \tag{24}$$

which presents us with the Fourier coefficients of $[\hat f,\hat g](\xi;C^I)$. From (24), using the Plancherel theorem for Fourier series, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n}&\langle f,T_{Ck}\,g\rangle\,\langle T_{Ck}\,\gamma,f\rangle = \\ &= \sum_{k\in\mathbb{Z}^n}\int_{C^I\mathbb{T}^n}[\hat f,\hat g](\xi;C^I)\,e^{2\pi iCk\cdot\xi}\,d\xi\;\overline{\int_{C^I\mathbb{T}^n}[\hat f,\hat\gamma](\xi;C^I)\,e^{2\pi iCk\cdot\xi}\,d\xi} \\ &= \frac{1}{|\det C|}\int_{C^I\mathbb{T}^n}[\hat f,\hat g](\xi;C^I)\,\overline{[\hat f,\hat\gamma](\xi;C^I)}\,d\,\xi. \qquad \square\end{aligned}$$

The following result will be the main ingredient in the proof of Theorem 3.1 and the successive characterization of dual frames. The proof that we will present contains some ideas from [23] adapted to our setting.

Proposition 4.4. *Let C be a $n\times n$ nonsingular matrix with real entries, and let $C^I = (C^T)^{-1}$. Assume that $g_p,\gamma_p\in L^2(\mathbb{R}^n)$ for every $p\in\mathcal{P}$, with $\sum_{p\in\mathcal{P}}|\hat g_p(\xi)|^2\le B_g$, $\sum_{p\in\mathcal{P}}|\hat\gamma_p(\xi)|^2\le B_\gamma$, $B_g,B_\gamma>0$. Then, for every $f\in\mathcal{D}$, the function*

$$H(x) \;=\; \sum_{p,k}\langle T_x\,f,g_{p,Ck}\rangle\,\langle\gamma_{p,Ck},T_x\,f\rangle \tag{25}$$

is continuous and $C\mathbb{Z}^n$-periodic, i.e., $H(x + C\ell) = H(x)$ for every $\ell \in \mathbb{Z}^n$, and coincides pointwise with its Fourier series $\sum_{u\in\mathbb{Z}^n} \hat{H}(u)\, e^{2\pi i C^I u\cdot x}$, with

$$\hat{H}(u) \;=\; \frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I u)} \Big(\sum_p \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + C^I u) \Big)\, d\xi, \tag{26}$$

where $u \in \mathbb{Z}^n$ and the integral in (26) converges absolutely.

Remark. The hypotheses $\sum_{p\in\mathcal{P}} |\hat{g}_p(\xi)|^2 \le B_g$ and $\sum_{p\in\mathcal{P}} |\hat{\gamma}_p(\xi)|^2 \le B_\gamma$ are natural. We will show later (see Proposition 4.5), for example, that this follows from the better known assumption that $\{g_{p,Ck}\}_{p,k}$ and $\{\gamma_{p,Ck}\}_{p,k}$ are Bessel systems.

Proof of Proposistion 4.4. Choose $f \in \mathcal{D}$. For $p \in \mathcal{P}$ fixed, let

$$H_p(x) = \sum_k \langle T_x\, f, g_{p,Ck}\rangle\, \langle \gamma_{p,Ck}, T_x\, f\rangle \;=\; \sum_k \langle f, g_{p,Ck-x}\rangle\, \langle \gamma_{p,Ck-x}, f\rangle. \tag{27}$$

Equality (27) shows that $H_p(x)$ is $C\mathbb{Z}^n$-periodic. Using Lemma 4.3 and a change of index, we have

$$\begin{aligned}
&|\det C|\, H_p(x) = \\
&= \int_{C^I\mathbb{T}^n} [(T_x\, f)^\wedge, \hat{g}_p](\xi; C^I)\, [\hat{\gamma}_p, (T_x\, f)^\wedge](\xi; C^I)\, d\,\xi \\
&= \int_{C^I\mathbb{T}^n} \sum_{l,m\in\mathbb{Z}^n} \hat{f}(\xi + C^I l)\, \overline{\hat{g}_p(\xi + C^I l)}\, \hat{\gamma}_p(\xi + C^I m)\, \overline{\hat{f}(\xi + C^I m)}\, e^{2\pi i C^I (m-l)\cdot x}\, d\xi \\
&= \int_{C^I\mathbb{T}^n} \sum_{l,u\in\mathbb{Z}^n} \hat{f}(\xi + C^I l)\, \overline{\hat{f}(\xi + C^I(l+u))}\, \overline{\hat{g}_p(\xi + C^I l)}\, \hat{\gamma}_p(\xi + C^I(l+u))\, e^{2\pi i C^I u\cdot x}\, d\xi \\
&= \sum_{u\in\mathbb{Z}^n} \Big(\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + C^I u)}\, \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + C^I u)\, d\xi \Big)\, e^{2\pi i C^I u\cdot x}.
\end{aligned} \tag{28}$$

The interchange of sum and integral in (28) is justified since $f \in \mathcal{D}$, which implies that the sum in (28) is finite.

By the Cauchy-Schwarz inequality, we have

$$|\sum_{p\in\mathcal{P}} \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + Cl)| \le (\sum_{p\in\mathcal{P}} |\hat{g}_p(\xi)|^2)^{1/2}\, (\sum_{p\in\mathcal{P}} |\hat{\gamma}_p(\xi + Cl)|^2)^{1/2} \le B_g^{1/2}\, B_\gamma^{1/2}. \tag{29}$$

Therefore, since $f \in \mathcal{D}$ and $g_p, \gamma_p \in L^2(\mathbb{R}^n)$, using the Lebesgue Dominated Convergence Theorem, we have

$$H(x) = \lim_{P\to\infty} \sum_{|p|\le P} H_p(x)$$

$$= \sum_{u\in\mathbb{Z}^n} \Big(\frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat f(\xi)\, \overline{\hat f(\xi + C^I u)} \sum_{p\in\mathcal{P}} \overline{\hat g_p(\xi)}\, \hat\gamma_p(\xi + C^I u)\, d\xi \Big)\, e^{2\pi i C^I u\cdot x}, \quad (30)$$

with absolute convergence of the integral. The function $H(x)$ is certainly continuous since the sum over $u \in \mathbb{Z}^n$ is finite. By the uniqueness of the Fourier series, (30) is the Fourier series of $H(x)$. $\square$

From the proof of Proposition 4.4, we deduce the following observation, that can also be found in [Jan97, Prop.1.2.1].

Proposition 4.5. *Let C be a $n \times n$ nonsingular matrix with real entries and assume that the sequence $\{g_{p,Ck} : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ of elements of $L^2(\mathbb{R}^n)$ is a Bessel sequence with upper frame bound B_g. Then*

$$\sum_{p\in\mathcal{P}} |\hat g_p(\xi)|^2 \le |\det C|\, B_g, \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \quad (31)$$

Proof. It is sufficient to prove the statement for f in a dense subspace of $L^2(\mathbb{R}^n)$. Let $f \in \mathcal{D}$. Since $\{g_{p,Ck}\}_{p,k}$ is a Bessel sequence, then

$$\int_{C\mathbb{T}^n} \sum_{p\in\mathcal{P}, k\in\mathbb{Z}^n} |\langle T_x f, g_{p,Ck}\rangle|^2\, dx \le |\det C|\, B_g\, \|f\|^2, \quad (32)$$

for every $x \in \mathbb{R}^n$ and every $f \in \mathcal{D}$. On the other hand, by direct calculation, using the fact that $\{g_{p,Ck}\}_{p,k}$ is a Bessel sequence, we find that

$$\begin{aligned}
\int_{C\mathbb{T}^n} \sum_{p,k} |\langle T_x f, g_{p,Ck}\rangle|^2\, dx &= \sum_{p,k} \int_{C\mathbb{T}^n} |\langle f, g_{p,Ck-x}\rangle|^2\, dx \\
&= \sum_p \int_{\mathbb{R}^n} |\langle f, g_{p,x}\rangle|^2\, dx \\
&= \sum_p \int_{\mathbb{R}^n} |\langle \hat f, (g_{p,x})^\wedge\rangle|^2\, dx \\
&= \sum_p \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \hat f(\xi)\, \overline{\hat g_p(\xi)}\, e^{2\pi i x\cdot\xi}\, d\xi \Big|^2 dx \\
&= \sum_p \int_{\mathbb{R}^n} \Big| \big(\hat f\, \overline{\hat g_p}\big)^\vee \Big|^2 dx \\
&= \sum_p \int_{\mathbb{R}^n} |\hat f(\xi)|^2\, |\overline{\hat g_p(\xi)}|^2\, d\xi \\
&= \int_{\mathbb{R}^n} |\hat f(\xi)|^2 \sum_{p\in\mathcal{P}} |\hat g(\xi)|^2\, d\xi \qquad (33)
\end{aligned}$$

for every $f \in \mathcal{D}$. Equation (31) then follows by comparing (32) and (33). $\square$

Now we can prove Theorem 3.1.

Proof of Theorem 3.1

By Lemma 2.1, it is sufficient to prove the theorem when $f \in \mathcal{D}$. Define (formally)

$$N(x) = \sum_{p,k} |\langle T_x f, g_{p,Ck} \rangle|^2.$$

Observe that this function coincides with the function $H(x)$ of Proposition 4.4 when $g_p = \gamma_p$ for every $p \in \mathcal{P}$ (see (25)).

If equation (13) holds, then $N(x) = \langle T_x f, T_x f \rangle = \|f\|^2$ for every $f \in \mathcal{D}$. Furthermore, $\{g_{p,Ck} : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a Bessel sequence with upper frame bound $B_g = 1$, and, by Proposition 4.5,

$$\sum_{p \in \mathcal{P}} |\hat{g}_p(\xi)|^2 \leq |\det C| \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{34}$$

Since $N(x)$ is a constant function, then $N(x)$ equals its constant Fourier coefficient $\hat{N}(0) = \|f\|^2$. By Proposition 4.4 with $g_p = \gamma_p$ for every $p \in \mathcal{P}$, we then have

$$\frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + (C^T)^{-1}u)} \left(\sum_p \overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + (C^T)^{-1}u) \right) d\xi = \delta_{u,0}\, \|f\|^2, \tag{35}$$

with $u \in \mathbb{Z}^n$, for every $f \in \mathcal{D}$. By (34) the function $\sum_{p \in \mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{g}_p(\xi - (C^T)^{-1}u)}$ is locally integrable, and so, making appropriate choices of f in (35), we establish (14).

Conversely, assume that equation (14) holds. This implies that $\sum_{p \in \mathcal{P}} |\hat{g}_p(\xi)|^2 \leq |\det C|$ for a.e. $\xi \in \mathbb{R}^n$ and, therefore, we can apply Proposition 4.4 with $g_p = \gamma_p$ for every $p \in \mathcal{P}$. By Proposition 4.4 and equation (14), the function $N(x)$ is continuous and its Fourier series is

$$\sum_{u \in \mathbb{Z}^n} \hat{N}(u)\, e^{2\pi i (C^T)^{-1} u \cdot x} \;=\; N(x),$$

for every $x \in \mathbb{R}^n$, with

$$\hat{N}(u) \;=\; \delta_{u,0} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + (C^T)^{-1}u)}\, d\xi.$$

Therefore, $N(x)$ is a constant function and $N(x) = \|f\|^2$ for every $x \in \mathbb{R}^n$. Taking $x = 0$, this establishes equation (13). $\square$

From Proposition 4.4, we will now obtain a characterization of the shift-invariant dual frame generators for $L^2(\mathbb{R}^n)$. A similar characterization can also be found in $[25, Cor.4.2]$, and, for $n = 1$, in $[23, Th.1.2.2]$. Some applications of Theorem 4.6, including the characterization of wavelet and Gabor dual frames, will be discussed in Sections 5 and 6

Theorem 4.6. *Let C be a $n \times n$ nonsingular matrix with real entries and $\{g_{p,Ck}\}_{p,k}$, $\{\gamma_{p,Ck}\}_{p,k}$, $k \in \mathbb{Z}^n, p \in \mathcal{P}$, be Bessel sequences with upper frame bounds B_g and B_γ, respectively. Then*

$$\sum_{p,k} \langle f, g_{p,Ck}\rangle \, \langle \gamma_{p,Ck}, h\rangle \;=\; \langle f, h\rangle \qquad \text{for all } f, h \in L^2(\mathbb{R}^n) \tag{36}$$

if and only if

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{\gamma}_p(\xi + (C^T)^{-1}u)} = |\det C|\, \delta_{u,0}, \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{37}$$

for each $u \in \mathbb{Z}^n$, where δ is the product Kronecker delta in $\mathbb{Z}^n$.

Proof. By Lemma 2.2, it is sufficient to prove the theorem when $f, h \in \mathcal{D}$. Furthermore, by the polarization identity, we can set $f = h$.

Since $\{g_{p,Ck}\}_{k,p}$ and $\{\gamma_{p,Ck}\}_{k,p}$ are Bessel sequences, it follows from Proposition 4.5 that

$$\sum_{p\in\mathcal{P}} |\hat{g}_p(\xi)|^2 \le |\det C|\, B_g, \qquad \sum_{p} |\hat{\gamma}_p(\xi)|^2 \le |\det C|\, B_\gamma, \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{38}$$

Therefore, we can apply Proposition 4.4, which involves the function H, where $H(x) = \sum_{p,k} \langle T_x f, g_{p,Ck}\rangle \, \langle \gamma_{p,Ck}, T_x f\rangle$.

If (36) holds, then $H(x) = \langle T_x f, T_x f\rangle = \|f\|^2$ for every $x \in \mathbb{R}^n$. Since $H(x)$ is a constant function, then $H(x)$ equals its Fourier coefficient $\hat{H}(0) = \|f\|^2$. By Proposition 4.4, we then have that for every $f \in \mathcal{D}$

$$\frac{1}{|\det C|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi + (C^T)^{-1}u)} \Big(\sum_p \overline{\hat{g}_p(\xi)}\, \hat{\gamma}_p(\xi + (C^T)^{-1}u) \Big)\, d\xi = \delta_{u,0}\, \|f\|^2, \tag{39}$$

with $u \in \mathbb{Z}^n$. By (38) and the Cauchy-Schwarz inequality, it follows that the function $\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{\gamma}_p(\xi + (C^T)^{-1}u)}$ is locally integrable, and so, making appropriate choices of f in (39) we establish (37).

Conversely, assume that (37) holds. By Proposition 4.4 the function $H(x)$ coincides pointwise with its Fourier series

$$H(x) \;=\; \sum_{u\in\mathbb{Z}^n} \hat{H}(u)\, e^{2\pi i (C^T)^{-1}u\cdot x},$$

and, by (37),

$$\hat{H}(u) \;=\; \delta_{u,0} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{f}(\xi+(C^T)^{-1}u)}\, d\xi.$$

Therefore, H is a constant function and $H(x) = \|f\|^2$ for every $x \in \mathbb{R}^n$. The choice $x = 0$ gives us equation (36). $\square$

The following characterization of tight frame generators for shift-invariant subspaces of $L^2(\mathbb{R}^n)$ appears to be new.

Theorem 4.7. *Let $\{g_p\}_{p\in\mathcal{P}} \subset L^2(\mathbb{R}^n)$ and let C be a non-singular $n\times n$ matrix with real entries. Then,*

$$\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, g_{p,Ck}\rangle\right|^2 = \|f\|^2 \qquad \text{for all } f \in L^2(\mathbb{R}^n) \tag{40}$$

if and only if the following two equations hold:

$$\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, g_{p,Ck}\rangle\right|^2 \le \|f\|^2 \qquad \text{for all } f \in L^2(\mathbb{R}^n) \tag{41}$$

$$\sum_{p\in\mathcal{P}} |\hat{g}_p(\xi)|^2 = |\det C| \qquad \text{for a.e.} \xi \in \mathbb{R}^n. \tag{42}$$

Proof. By Lemma 2.1, it is sufficient to prove the theorem when $f \in \mathcal{D}$.

Equation (40) clearly implies (41). Furthermore, by Theorem 3.1, (40) implies (42).

For the converse, argue as follows. Let $N(x) = \sum_{p\in\mathcal{P},k\in\mathbb{Z}^n} |\langle T_x f, g_{p,Ck}\rangle|^2$. Since $\{g_{p,Ck}\}_{c,k}$ is a Bessel sequence with upper frame bound $B_g = 1$, then, by Proposition 4.5, $\sum_p |\hat{g}_p(\xi)|^2 \le |\det C|$ for a.e. $\xi \in \mathbb{R}^n$, and therefore Proposition 4.4 can be applied. Using Proposition 4.4 with $g_p = \gamma_p$ for every $p \in \mathcal{P}$, we have that

$$\begin{aligned}\hat{N}(0) &= \frac{1}{|\det C|} \int_{C\mathbb{T}^n} \sum_{p\in\mathcal{P},k\in\mathbb{Z}^n} |\langle T_x f, g_{p,Ck}\rangle|^2\, dx \\ &= \frac{1}{|\det C|} \int_{C\mathbb{T}^n} |\hat{f}(\xi)|^2 \sum_{p\in\mathcal{P}} |\hat{g}(\xi)|^2\, d\xi = \|f\|^2. \end{aligned} \tag{43}$$

By (41), $N(x) = \sum_{p,k} |\langle T_x f, g_{p,Ck}\rangle|^2 \le \|f\|^2$ for all $x \in \mathbb{R}^n$, and $N(x)$ is continuous by Proposition 4.4. Therefore, (43) implies that $N(x) = \|f\|^2$ for every $x \in \mathbb{R}^n$. The proof is completed by taking $x = 0$. $\square$

5 Affine Systems

In this section we prove Corollaries 3.4, 3.7 and 3.8. Next, we present the characterization of affine dual frames for $L^2(\mathbb{R}^n)$.

Proof of Corollary 3.4.

Let $\mathcal{P} = \{(j,\ell) : j \in \mathbb{Z}, \text{ and } \ell = 1,\ldots,L\}$, and $\tilde{\tilde{\theta}}_p = \tilde{\tilde{\theta}}_{j,\ell} = |\det A|^{j/2} D_{A^j}\,\psi^\ell$. With this choice for $\mathcal{P}$ and $\tilde{\tilde{\theta}}_p$, then $\{T_k\,\theta_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the co-affine system $\tilde{\tilde{X}}(\Psi)$.

By Theorem 3.1 with $C = I$, the system $\{T_k\,\tilde{\tilde{\theta}}_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame if and only if

$$\sum_{p\in\mathcal{P}} \hat{\theta}_p(\xi)\,\overline{\hat{\theta}_p(\xi+u)} = \delta_{u,0}, \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{44}$$

with $u \in \mathbb{Z}^n$. Using Lemma 4.1, we compute the left hand side of equation (44) as:

$$\sum_{p\in\mathcal{P}} \hat{\theta}_p(\xi)\,\overline{\hat{\theta}_p(\xi+u)} \;=\; \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \hat{\psi}^\ell((A^T)^{-j}\xi)\,\overline{\hat{\psi}^\ell((A^T)^{-j}(\xi+u))}. \quad \square$$

In order to prove Corollaries 3.7 and 3.8, we need the following result, which can be found in [17, Lemma 5.1].

Lemma 5.1. *Let A be a $n\times n$ dilation matrix with integer coefficients, and $q = |\det A|$. Choose a complete set $\{d_r\}_{r=0}^{q-1}$ of distinct representatives of the group $\mathbb{Z}^n/A\mathbb{Z}^n$ with $d_0 = 0$, that is, $\mathbb{Z}^n = \bigcup_{r=0}^{q-1}(d_r + A\mathbb{Z}^n)$. Then*

$$\frac{1}{q}\sum_{r=0}^{q-1} e^{2\pi i A^{-1}d_r\cdot k} \;=\; \begin{cases} 1 & \text{if } k \in A^T\mathbb{Z}^n \\ 0 & \text{otherwise.} \end{cases}$$

We will now prove Corollary 3.7. The proof that we will present contains some ideas from [4] adapted to our setting.

Proof of Corollary 3.7. Let $\mathcal{P} = \{(j,k,\ell) : j \in \mathbb{Z}, d \in \mathcal{D}_j, \text{ and } \ell = 1,\ldots,L\}$, and $\theta_p = \theta_{(j,d,\ell)} = D_{A^j}\,T_d\,\tilde{\psi}^\ell$, where, for $j > N$, $\mathcal{D}_j$ denotes a complete set of representatives of distinct cosets of $\mathbb{Z}^n/(A^j\mathbb{Z}^n)$ and, for $j \le N$, $\mathcal{D}_j = \{0\}$. With this choice for $\mathcal{P}$ and θ_p, the system $\{T_k\,\theta_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the quasi-affine system $\tilde{X}^{(N)}(\Psi)$.

By Theorem 3.1 with $C = I$, the system $\{T_k\, \theta_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is a normalized tight frame if and only if

$$\sum_{p\in\mathcal{P}} \hat\theta_p(\xi)\, \overline{\hat\theta_p(\xi+u)} = \delta_{u,0}, \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{45}$$

with $u \in \mathbb{Z}^n$. Using Lemma 4.1, we compute the left hand side of equation (45) as:

$$\begin{aligned}\sum_{p\in\mathcal{P}} \hat\theta_p(\xi)\, \overline{\hat\theta_p(\xi+u)} = \sum_{\ell=1}^{L} \sum_{j\le N} \hat\psi^\ell((A^T)^{-j}\xi)\, \overline{\hat\psi^\ell((A^T)^{-j}(\xi+u))} +& \\ + \sum_{\ell=1}^{L} \sum_{j> N} \frac{1}{|\det A|^j}\, \hat\psi^\ell((A^T)^{-j}\xi)\, \overline{\hat\psi^\ell((A^T)^{-j}(\xi+u))} \sum_{d\in\mathcal{D}_j} e^{-2\pi i A^{-j} d\cdot u}&.\end{aligned} \tag{46}$$

Observe that, by Lemma 5.1,

$$\sum_{d\in\mathcal{D}_j} e^{-2\pi i A^{-j} d\cdot u} \;=\; \begin{cases} |detA|^j & \text{if } u \in (A^T)^j\mathbb{Z}^n \\ 0 & \text{otherwise.}\end{cases} \tag{47}$$

If $u = 0$, then $u \in (A^T)^j\mathbb{Z}^n$ for every $j > N$, and thus, by (46) and (47), it follows that

$$\sum_{p\in\mathcal{P}} |\hat\theta_p(\xi)|^2 \;=\; \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat\psi^\ell((A^T)^{-j}\xi)|^2,$$

which proves (20).

On the other hand, if $u \neq 0$, let $s \in \mathbb{Z}^n \setminus A^T\mathbb{Z}^n$, $s \neq 0$. Then any $u \in \mathbb{Z}^n$, $u \neq 0$, is of the form $u = (A^T)^r s$, for some $r \ge 0$. Now, if $u = (A^T)^r s$ for some $0 \le r \le N$, then, by (47), $\sum_{d\in\mathcal{D}_j} e^{-2\pi i A^{-j} d\cdot u} = 0$, for every $j \in \mathbb{Z}$, and thus, from (46) we obtain:

$$\begin{aligned}\sum_{p\in\mathcal{P}} \hat\theta_p(\xi)\, \overline{\hat\theta_p(\xi+u)} \;&=\; \sum_{\ell=1}^{L} \sum_{j\le N} \hat\psi^\ell((A^T)^{-j}\xi)\, \overline{\hat\psi^\ell((A^T)^{-j}(\xi+(A^T)^r s))} \\ &=\; \sum_{\ell=1}^{L} \sum_{j\ge 0} \hat\psi^\ell((A^T)^{(j-N)}\xi)\, \overline{\hat\psi^\ell((A^T)^{(j-N)}(\xi+(A^T)^r s))} \\ &=\; \sum_{\ell=1}^{L} \sum_{j\ge 0} \hat\psi^\ell((A^T)^{j}((A^T)^{-N}\xi))\, \overline{\hat\psi^\ell((A^T)^{j}((A^T)^{-N}\xi+(A^T)^{r-N} s))}.\end{aligned} \tag{48}$$

Finally, if $u = (A^T)^r s$ for some $r > N$, then, by (47), $\sum_{d\in\mathcal{D}_j} e^{-2\pi i A^{-j} d\cdot u} = |\det A|^j$, for every $N < j \le r$. In this case, from (46) we obtain:

$$\begin{aligned}
\sum_{p\in\mathcal{P}} \hat\theta_p(\xi)\,\overline{\hat\theta_p(\xi+u)} &= \sum_{\ell=1}^{L}\sum_{j=-\infty}^{r} \hat\psi^\ell((A^T)^{-j}\xi)\,\overline{\hat\psi^\ell((A^T)^{-j}(\xi+(A^T)^r s))} \\
&= \sum_{\ell=1}^{L}\sum_{j\ge 0} \hat\psi^\ell((A^T)^{(j-r)}\xi)\,\overline{\hat\psi^\ell((A^T)^{(j-r)}(\xi+(A^T)^r s))} \\
&= \sum_{\ell=1}^{L}\sum_{j\ge 0} \hat\psi^\ell((A^T)^{j}((A^T)^{-r}\xi))\,\overline{\hat\psi^\ell((A^T)^{j}((A^T)^{-r}\xi+s))}. \qquad (49)
\end{aligned}$$

Combining (48) and (49), and making a change of variables, we thus obtain (21). $\square$

In order to study the characterization of affine dual system, we will again exploit the relation between affine and quasi-affine systems given by Theorem 3.6. Let $K_{\psi,\phi}(f,h)$ and $\widetilde{K}_{\psi,\phi}(f,h)$ be the operators defined by

$$K_{\psi,\phi}(f,h) \;=\; \sum_{\ell=1}^{l}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f,\psi^\ell_{j,k}\rangle\,\langle \phi^\ell_{j,k},h\rangle$$

and

$$\widetilde{K}_{\psi,\phi}(f,h) \;=\; \sum_{\ell=1}^{l}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f,\widetilde{\psi}^\ell_{j,k}\rangle\,\langle \widetilde{\phi}^\ell_{j,k},h\rangle,$$

where $\psi^\ell_{j,k}$ and $\widetilde{\psi}^\ell_{j,k}$ are defined in (18) and (19), respectively.

We have the following characterization of duality for quasi-affine systems. Since the proof is very similar to the proof of Corollary 3.7, we will only sketch it.

Theorem 5.2. *Let $N \ge 0$, and assume that $\widetilde{X}^{(N)}(\Psi)$ and $\widetilde{X}^{(N)}(\Phi)$ have the same cardinality and generate two affine Bessel sequences. Then*

$$\widetilde{K}_{\psi,\phi}(f,h) \;=\; \langle f,h\rangle \qquad \text{for all } f,h\in L^2(\mathbb{R}^n)$$

if and only if

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \hat\psi^\ell\big((A^T)^j\xi\big)\,\overline{\hat\phi^\ell\big((A^T)^j\xi\big)} = 1 \qquad \text{for a.e. } \xi\in\mathbb{R}^n, \qquad (50)$$

and

$$\sum_{\ell=1}^{L}\sum_{j\ge 0} \hat\psi^\ell\big((A^T)^j\xi\big)\,\overline{\hat\phi^\ell\big((A^T)^j(\xi+(A^T)^r s)\big)} = 0 \qquad \text{for a.e. } \xi\in\mathbb{R}^n, \qquad (51)$$

for $r = 0,-1,\dots,-N$, with $s\in\mathbb{Z}^n\setminus A^T\mathbb{Z}^n$.

Proof. Let $\mathcal{P} = \{(j,k,\ell) : j \in \mathbb{Z}, d \in \mathcal{D}_j, \text{ and } \ell = 1,\dots,L\}$, $g_p = g_{(j,d,\ell)} = D_{A^j}\, T_d\, \widetilde{\psi}^\ell$, and $\gamma_p = \gamma_{(j,d,\ell)} = D_{A^j}\, T_d\, \widetilde{\phi}^\ell$. Under these assumptions, the systems $\{T_k\, g_p\}_{k,p}$ and $\{T_k\, \gamma_p\}_{k,p}$, $k \in \mathbb{Z}^n, p \in \mathcal{P}$, are the quasi-affine system $\widetilde{X}^{(N)}(\Psi)$ and $\widetilde{X}^{(N)}(\Phi)$, respectively. By Proposition 4.6 with $C = I$, the systems $\{T_k\, g_p\}_{k,p}$ and $\{T_k\, \gamma_p\}_{k,p}$, $k \in \mathbb{Z}^n, p \in \mathcal{P}$ are dual systems if and only if

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{\gamma}_p(\xi+u)} = \delta_{u,0}. \tag{52}$$

Using Lemma 4.1, we compute the left hand side of equation (52) as:

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{\gamma}_p(\xi+u)} = \sum_{\ell=1}^{L}\sum_{j\le 0} \hat{\psi}^\ell((A^T)^{-j}\xi)\, \overline{\hat{\phi}^\ell((A^T)^{-j}(\xi+u))} \; + $$
$$+ \sum_{\ell=1}^{L}\sum_{j>0} |\det A|^{-j}\, \hat{\psi}^\ell((A^T)^{-j}\xi)\, \overline{\hat{\phi}^\ell((A^T)^{-j}(\xi+u))} \sum_{d\in\mathcal{D}_j} e^{-2\pi i A^{-j} d\cdot u}.$$

The proof now continues exactly like the proof of Corollary 3.7. $\square$

From Theorem 5.2 with $N = 0$, using Theorem 3.6, we now immediately obtain the characterization of duality for the affine systems.

Corollary 5.3. *Assume that $X(\Psi)$ and $X(\Phi)$ have the same cardinality and generate two affine Bessel sequences. Then*

$$K_{\psi,\phi}(f,h) = \langle f,h\rangle \qquad \text{for all } f,h \in L^2(\mathbb{R}^n)$$

if and only if equations (50) and

$$\sum_{\ell=1}^{L}\sum_{j\ge 0} \hat{\psi}^\ell\big((A^T)^j\xi\big)\, \overline{\hat{\phi}^\ell\big((A^T)^j(\xi+s)\big)} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{53}$$

where $s \in \mathbb{Z}^n \setminus A^T\mathbb{Z}^n$, hold.

6 Gabor systems

In this section, we present the proofs of Corollaries 3.2 and 3.3 and then we discuss the characterization of duality for Gabor systems.

Proof of Corollary 3.2. Let $\mathcal{P} = \{(\ell,m) : \ell = 1,\dots,L \text{ and } m \in \mathbb{Z}^n\}$, and let $g_p = g_{(\ell,m)} = M_{Bm}\, g^\ell$. With this choice for $\mathcal{P}$ and g_p, the system $\{T_{Ck}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the Gabor system $\{T_{Ck}\, M_{Bm}\, g^\ell\}_{k,m,\ell}$.

By Theorem 3.1, the system $\{T_{Ck}\, g_p\}_{k,p}$ is a normalized tight frame if and only if

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{g}_p(\xi + (C^T)^{-1}u)} = |\det C|\, \delta_{u,0} \qquad \text{for a.e } \xi \in \mathbb{R}^n, \tag{54}$$

where $u \in \mathbb{Z}^n$. Using Lemma 4.1, the left hand side of (54) is computed as follows:

$$\sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\, \overline{\hat{g}_p(\xi + (C^T)^{-1}u)} = \sum_{\ell=1}^{L} \sum_{m\in\mathbb{Z}^n} \hat{g}^\ell(\xi - Bm)\, \overline{\hat{g}^\ell(\xi - Bm + (C^T)^{-1}u)}. \quad \square$$

We give a direct proof of Corollary 3.3.

Proof of Corollary 3.3. By Plancherel Theorem and Lemma 4.1 we have

$$\begin{aligned}\sum_{\ell=1}^{L} \sum_{k,m\in\mathbb{Z}^n} \left|\langle f, T_{Ck}\, M_{Bm}\, g^\ell\rangle\right|^2 &= \sum_{\ell=1}^{L} \sum_{k,m\in\mathbb{Z}^n} \left|\langle \check{f}, \left(T_{Ck}\, M_{Bm}\, g^\ell\right)^\vee\rangle\right|^2 \\ &= \sum_{\ell=1}^{L} \sum_{k,m\in\mathbb{Z}^n} \left|\langle \check{f}, T_{-Bm}\, M_{Ck}\, \check{g}^\ell\rangle\right|^2.\end{aligned}$$

This shows that the collection $\{T_{Ck}\, M_{Bm} g^\ell\}_{k,m,\ell}$ is a normalized tight frame if and only if $\{T_{Bm}\, M_{Ck}\, \check{g}^\ell\}_{k,m,\ell}$ is a normalized tight frame.

By Theorem 3.1, with g replaced by $\check{g}$, the system $\{T_{Bm}\, \check{g}_p\}_{m,p}$ is a normalized tight frame if and only if

$$\sum_{p\in\mathcal{P}} g_p(\xi)\, \overline{g_p(\xi + (B^T)^{-1}u)} = |\det B|\, \delta_{u,0} \qquad \text{for a.e } \xi \in \mathbb{R}^n, \tag{55}$$

where $u \in \mathbb{Z}^n$.

Let $\mathcal{P} = \big\{(\ell, k) : \ell = 1, \ldots, L \text{ and } k \in \mathbb{Z}^n\big\}$, and let $\check{g}_p = \check{g}_{(\ell,k)} = M_{Ck}\, \check{g}^\ell$. With this choice for $\mathcal{P}$ and g_p, the system $\{T_{Bm}\, \check{g}_p : m \in \mathbb{Z}^n, p \in \mathcal{P}\}$ is the Gabor system $\{T_{Bm}\, M_{Ck}\, \check{g}^\ell\}_{m,k,\ell}$. Using Lemma 4.1, the left hand side of (55) is thus computed as follows:

$$\sum_{p\in\mathcal{P}} g_p(\xi)\, \overline{g_p(\xi + (B^T)^{-1}u)} = \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \hat{g}^\ell(\xi - Ck)\, \overline{\hat{g}^\ell(\xi - Ck + (B^T)^{-1}u)}. \quad \square$$

There is an alternative argument for Corollary 3.3 which we present as an application of the bracket product introduced in Section 4.

(Second) Proof of Corollary 3.3. Let $C^I = (C^T)^{-1}$. In order to compare (16) and (17), we will compare the Fourier coefficients of the functions $F_u(\xi) = \sum_{m\in\mathbb{Z}^n} \hat{g}(\xi -$

$Bm)\,\overline{\hat{g}(\xi - Bm + C^I u)}$, $u \in \mathbb{Z}^n$, and $G_v(\xi) = \sum_{m\in\mathbb{Z}^n} g(\xi - Cm)\,\overline{g(\xi - Cm + B^I v)}$, $v \in \mathbb{Z}^n$. Since $g \in L^2(\mathbb{R}^n)$, then the $F \in L(B^{-1}[0,1]^n)$ is $B\mathbb{Z}^n-$periodic and the $G \in L(C^{-1}[0,1]^n)$ is $C\mathbb{Z}^n-$periodic. Notice that the index ℓ in (16) and (17) will be disregarded, since it plays no role in the proof. Using Lemma 4.2, we obtain the following relation between the Fourier coefficients of F_u and G_v:

$$\begin{aligned} \hat{F}_u(k) &= \frac{1}{|\det B|}\int_{B\mathbb{T}^n} [\hat{g}, T_{-C^I u}\,\hat{g}](\xi, B)\, e^{-2\pi i B^I k\cdot\xi}\, d\xi \\ &= \frac{1}{|\det B|}\int_{B\mathbb{T}^n} [M_{-B^I k}\,\hat{g}, T_{-C^I u}\,\hat{g}](\xi, B)\, d\xi \\ &= \frac{1}{|\det B|}\int_{C\mathbb{T}^n} [T_{B^I k}\, g, M_{-C^I u}\, g](x, C)\, dx = \frac{|\det C|}{|\det B|}\,\overline{\hat{G}_{-k}(u)}. \end{aligned} \tag{56}$$

If (16) holds, then $F_0(\xi) = |\det C|$, and $F_u(\xi) = 0$, when $u \neq 0$. This implies that $\hat{F}_0(0) = |\det C|$, $\hat{F}_0(k) = 0$, if $k \neq 0$, and $\hat{F}_u(k) = 0$, for each k, when $u \neq 0$. By (56), then $\hat{G}_0(0) = |\det B|$, $\hat{G}_0(u) = 0$, if $u \neq 0$, and $\hat{G}_k(u) = 0$, for each u, when $k \neq 0$. This implies that $G_0(\xi) = |\det B|$, and $G_v(\xi) = 0$, when $v \neq 0$, which is equation (17). The proof that (17) implies (16) is similar. $\square$

As a consequence of Theorem 4.6 we obtain the following proof of the Wexler–Raz theorem which characterizes the duality for Gabor systems. This theorem was found by Wexler and Raz [29], and proved in [21, 22, 12, 25]. Our proof adapts to the n-dimensional case the approach used in [21, Prop. A].

Theorem 6.1 (Wexler-Raz). *Let B, C be $n \times n$ nonsingular matrices with real entries, and let $\{T_{Ck}\, M_{Bm}\, g^\ell\}_{k,m,\ell}$ and $\{T_{Ck}\, M_{Bm}\, \gamma^\ell\}_{k,m,\ell}$, where $k,\ m \in \mathbb{Z}^n$, $\ell \in 1,\ldots,L$, be Bessel sequences for $L^2(\mathbb{R}^n)$. Then*

$$\sum_{\ell=1}^{L}\sum_{k,m\in\mathbb{Z}^n} \langle f, T_{Ck}\, M_{Bm}\, g^\ell\rangle\langle T_{Ck}\, M_{Bm}\, \gamma^\ell, h\rangle \;=\; \langle f, h\rangle \qquad \text{for all } f, h \in L^2(\mathbb{R}^n) \tag{57}$$

if and only if

$$\sum_{\ell=1}^{L}\langle g^\ell, T_{(B^T)^{-1}v}\, M_{(C^T)^{-1}u}\, \gamma^\ell\rangle = |\det B|\,|\det C|\,\delta_{u,0}\,\delta_{v,0} \tag{58}$$

for each $u,\ v \in \mathbb{Z}^n$, where δ is the product Kronecker delta in $\mathbb{Z}^n$.

Proof. Denote $C^I = (C^T)^{-1}$. Let $\mathcal{P} = \big\{(m,\ell) : m \in \mathbb{Z}^n,\ \ell = 1,\ldots,L\big\}$, and let $g_p = g_{m,\ell} = \{M_{Bm}\, g^\ell\}_{m,\ell}$. Under these assumptions, the collection $\{T_{Ck}\, g_p\}_{k,p}$ is the Gabor system $\{T_{Ck}\, M_{Bm}\, g^\ell\}_{k,m,\ell}$. Similarly for γ_p.

By Theorem 4.6, (57) holds if and only if

$$F_u(\xi) \;=\; \sum_{p\in\mathcal{P}} \hat{g}_p(\xi)\,\overline{\hat{\gamma}_p(\xi + C^I u)} = |\det C|\,\delta_{u,0}, \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{59}$$

where $u \in \mathbb{Z}^n$. Using Lemma 4.1, we obtain

$$F_u(\xi) = \sum_{\ell=1}^{L} \sum_{m\in\mathbb{Z}^n} \hat{g}^\ell(\xi - Bm)\,\overline{\hat{\gamma}^\ell(\xi - Bm + C^I u)}. \tag{60}$$

It is clear from (60) that $F \in L^1(B[0,1]^n)$ and is $B\mathbb{Z}^n$ periodic, i.e., $F(\xi + Bl) = F(\xi)$ for every $l \in \mathbb{Z}^n$. Using Lemma 4.2, Lemma 4.1 and Plancherel theorem, we compute the Fourier coefficients $\hat{F}_u(v)$, $v \in \mathbb{Z}^n$, of the function F_u:

$$\begin{aligned}
\hat{F}_u(v) \;&=\; |\det B|^{-1} \int_{B\mathbb{T}^n} \sum_{\ell=1}^{L} [\hat{g}^\ell, T_{-C^I u}\,\hat{\gamma}^\ell](\xi, B)\, e^{-2\pi i B^I v\cdot\xi}\, d\xi \\
&=\; |\det B|^{-1} \sum_{\ell=1}^{L} \int_{B\mathbb{T}^n} [M_{-B^I v}\,\hat{g}^\ell, T_{-C^I u}\,\hat{\gamma}^\ell](\xi, B)\, d\xi \qquad (61)\\
&=\; |\det B|^{-1} \sum_{\ell=1}^{L} \big\langle M_{-B^I v}\,\hat{g}^\ell,\, T_{-C^I u}\,\hat{\gamma}^\ell \big\rangle \\
&=\; |\det B|^{-1} \sum_{\ell=1}^{L} \big\langle T_{(B^T)^{-1} v}\, g^\ell,\, M_{-(C^T)^{-1} u}\,\gamma^\ell \big\rangle \\
&=\; |\det B|^{-1} \sum_{\ell=1}^{L} \big\langle g^\ell,\, T_{-(B^T)^{-1} v}\, M_{-(C^T)^{-1} u}\,\gamma^\ell \big\rangle. \qquad (62)
\end{aligned}$$

Observe that Equation (61) is justified since the sum in (60) is absolutely convergent. By (62), thus equation (59) is equivalent to

$$\sum_{\ell=1}^{L} \big\langle g^\ell,\, T_{-(B^T)^{-1} v}\, M_{-(C^T)^{-1} u}\,\gamma^\ell \big\rangle = |\det B|\,|\det C|\,\delta_{u,0}\,\delta_{v,0},$$

for each $u, v \in \mathbb{Z}^n$. □

Remark. The condition that $\{T_{Ck}\, M_{Bm}\, g^\ell\}_{k,m,\ell}$ and $\{T_{Ck}\, M_{Bm}\,\gamma^\ell\}_{k,m,\ell}$ are Bessel sequences cannot be removed in Theorem 6.1 (cf. [21]).

Theorem 6.1 implies the following result, which is due to Rieffel [24]. Our proof adapts to the n-dimensional case the approach in [21].

Corollary 6.2. *Let $g \in L^2(\mathbb{R}^n)$, and let C, B be $n \times n$ nonsingular matrices with real coefficients. If $|\det C|\,|\det B| > 1$, then the set $\{T_{Ck}\, M_{Bm}\, g : k \in \mathbb{Z}^n, m \in \mathbb{Z}^n\}$ cannot be a frame.*

Proof. Choose $g \in L^2(\mathbb{R}^n)$ so that the set $\{T_{Ck}\, M_{Bm}\, g : k, m \in \mathbb{Z}^n\}$ is a frame. Then we have the frame decomposition

$$f = \langle f, T_{Ck}\, M_{Bm}\, \gamma\rangle\, T_{Ck}\, M_{Bm}\, g \qquad \text{for all } f, h \in L^2(\mathbb{R}^n),$$

where $\gamma = S_g^{-1}\, g$ is the canonical dual of g, and S_g is the frame operator associated with $\{T_{Ck}\, M_{Bm}\, g\}_{k,m}$. By the "minimal" property of the canonical dual frame (cf. Section 2), then $\sum_{k,m} |\langle g, T_{Ck}\, M_{Bm}\, \gamma\rangle|^2 \le \sum_{k,m} |\alpha_{k,m}|^2$ for all vectors $\alpha = \{\alpha_{k,m}\}_{k,m\in\mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n \times \mathbb{Z}^n)$ satisfying

$$f \;=\; \sum_{k,m} \alpha_{k,m}\, T_{Ck}\, M_{Bm}\, g. \tag{63}$$

If we set $f = g$ in (63), then we can write $g = \sum_{k,m} \delta_{k,0}\, \delta_{m,0}\, T_{Ck}\, M_{Bm}\, g$, and, therefore,

$$|\langle g, \gamma\rangle|^2 \le \sum_{k,m} |\langle g, T_{Ck}\, M_{Bm}\, \gamma\rangle|^2 \le \sum_{k,m} |\delta_{k,0}\, \delta_{m,0}|^2 = 1.$$

On the other hand, by (58) with $u = 0, v = 0, L = 1$, we have that $\langle g, \gamma\rangle = |\det C|\,|\det B|$. $\square$

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