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An Approach to the Study of Wave Packet Systems

Demetrio Labate, Guido Weiss, and Edward Wilson

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Abstract. The traditional study of reproducing systems involves the Gabor systems, that are generated by the action of translations and modulations on a single or finite family of functions in $L^2(\mathbb{R}^n)$, and the affine systems, where the dilations are used rather than the modulations. In this paper, we show that the interplay of all three operators yield a wide variate of reproducing systems, and we employ the term **wave packet systems**, which has been used by other authors, to describe those function systems generated by the combined action of a class of translations, modulations and dilations on a finite family of functions. We will examine in detail both the continuous and discrete versions of these systems. We shall show that these systems can be studied by using a unified approach that the authors have developed in some of their previous work.

1. Preliminaries

Before embarking in this study, it will be useful to introduce some notation and definitions. We consider three fundamental operators on $L^2(\mathbb{R}^n)$: the *translations* $T_y : (T_y f)(x) = f(x-y)$, where $y \in \mathbb{R}^n$; the *dilations* $D_a : (D_a f)(x) = |\det a|^{1/2} f(ax)$, where $a \in GL_n(\mathbb{R})$; and the *modulations* $M_\nu : (M_\nu f)(x) = e^{2\pi i \nu \cdot x} f(x)$, where $\nu \in \mathbb{R}^n$.

The Fourier transform is defined as

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi \cdot x}\, dx,$$

and the inverse Fourier transform is

$$\check{f}(x) = (\mathcal{F}^{-1} f)(x) = \int_{\mathbb{R}^n} f(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi.$$

The following proposition, which is easily verified, states some basic properties of the translation, modulation and dilation operators.

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PROPOSITION 1.1. *Let*

$$G = \{U = c\, D_a\, M_\nu\, T_y : \, c \in \mathbb{C}, \, |c| = 1, \, (a, \nu, y) \in GL_n(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^n\}. \tag{1.1}$$

G is a subgroup of the group of unitary operators on $L^2(\mathbb{R}^n)$ which is preserved by the action of the operator $U \mapsto \widehat{U}$, where $\widehat{U}\, \hat{f} = (Uf)^\wedge$. In particular, we have:

(i) $T_y\, M_\nu = e^{-2\pi i \nu \cdot y}\, M_\nu\, T_y$;
(ii) $D_a\, T_y = T_{a^{-1}y}\, D_a$;
(iii) $D_a\, M_\nu = M_{a^T \nu}\, D_a$;
(iv) *for $U = c\, D_a\, M_\nu\, T_y$, then $\widehat{U} = c\, D_{a^I}\, T_\nu\, M_{-y}$, where $a^I = (a^T)^{-1}$.*

2. Wave packet systems

In [**CF**], Córdoba and Fefferman introduced “wave packets” as those families of functions obtained by applying certain collections of dilations, modulations and translations to the Gaussian function. In this paper, we will adopt the same expression to describe, more generally, any collections of functions which are obtained by applying a combination of dilations, modulations and translations to a finite family of functions in $L^2(\mathbb{R}^n)$. Unlike the original wave packets of Córdoba and Fefferman, the systems that we will consider will not always be “well localized” in time and frequency.

DEFINITION 2.1. Let $\Psi = \{\psi^\ell : 1 \le \ell \le L\} \subset L^2(\mathbb{R}^n)$, where L is a finite integer, and let $S \subseteq GL_n(\mathbb{R}) \times \mathbb{R}^n$. The **continuous wave packet system** relative to S generated by Ψ is the collection:

$$\mathcal{W}_S(\Psi) = \big\{D_a\, M_\nu\, T_y\, \psi^\ell : \, (a, \nu) \in S, y \in \mathbb{R}^n, 1 \le \ell \le L\big\}. \tag{2.1}$$

Let us make several remarks about this definition.

Note that the map

$$(a, \nu, y) \mapsto U^{(0)}_{(a,\nu,y)} = D_a\, M_\nu\, T_y$$

is a one-to-one function from $S \times \mathbb{R}^n$ into G, where G is the group introduced by (1.1). By changing the oder of the operators, we can also define the following one-to-one functions from $S \times \mathbb{R}^n$ into G:

$$\begin{aligned}
U^{(1)}_{(a,\nu,y)} &= D_a\, T_y\, M_\nu \\
U^{(2)}_{(a,\nu,y)} &= T_y\, D_a\, M_\nu \\
U^{(3)}_{(a,\nu,y)} &= M_\nu\, D_a\, T_y \\
U^{(4)}_{(a,\nu,y)} &= T_y\, M_\nu\, D_a \\
U^{(5)}_{(a,\nu,y)} &= M_\nu\, T_y\, D_a.
\end{aligned}$$

In view of Proposition 1.1, we can generate alternate continuous wave packet systems $\mathcal{W}^{(i)}_S(\Psi)$, with $1 \le i \le 5$, by replacing the operator $U^{(0)}_{(a,\nu,y)} = D_a\, T_y\, M_\nu$ in Definition 2.1 with the operators $U^{(i)}_{(a,\nu,y)}$, $1 \le i \le 5$. The systems $\mathcal{W}^{(0)}_S(\Psi) = \mathcal{W}_S(\Psi)$ and $\mathcal{W}^{(1)}_S(\Psi)$ are equivalent since they only differ by a unimodular scalar. The same is true for the systems $\mathcal{W}^{(4)}_S(\Psi)$ and $\mathcal{W}^{(5)}_S(\Psi)$. The remaining systems, on the other hand, have substantial differences that we will explicitly discuss later.

One could also define more general wave packet systems of the form:

$$\{U_p\,\psi^\ell : p \in \mathcal{P},\, 1 \le \ell \le L\},$$

where L is a finite integer, $\mathcal{P}$ is a parameter set and the map $p \mapsto U_p$ is one-to-one from $\mathcal{P}$ into G, where G is given by (1.1). We will consider some specific examples of this type in the following.

Finally, observe that we will also consider discrete analogs of the wave packet systems given by Definition 2.1. These systems are obtained by letting S in Definition 2.1 be countable and by sampling the translations over a lattice. The reader is probably already familiar with special cases of these systems, including Gabor systems, wavelets and their close variants. Some more general examples will be discussed in Section 3.

A very important class of wave packet systems are those associated with a reproducing formula. More precisely, we introduce the following definition.

Definition 2.2. Let $S \subseteq GL_n(\mathbb{R}) \times \mathbb{R}^n$ and λ be a measure on S. Fix i, where $0 \le i \le 5$. Then the system $\mathcal{W}_S^{(i)}(\Psi)$ is a **continuous Parseval frame wave packet system** relative to (S, λ) for $L^2(\mathbb{R}^n)$, provided that the functions

$$(a, \nu, y) \longmapsto \langle f, U^{(i)}_{(a,\nu,y)}\,\psi\rangle$$

are $\lambda-$measurable for all $f,\, \psi \in L^2(\mathbb{R}^n)$, and

$$\|f\|^2 = \sum_{\ell=1}^{L} \int_{S\times\mathbb{R}^n} |\langle f, U^{(i)}_{(a,\nu,y)}\,\psi^\ell\rangle|^2 \, d\lambda(a,\nu)\, dy$$

for all $f \in L^2(\mathbb{R}^n)$.

The following theorem gives a characterization of all those families $\Psi \subset L^2(\mathbb{R}^n)$ such that the wave packet system $\mathcal{W}_S(\Psi) = \mathcal{W}_S^{(0)}(\Psi)$ is a continuous Parseval frame wave packet system for $L^2(\mathbb{R}^n)$.

Theorem 2.1. *Let $\Psi = \{\psi^\ell : 1 \le \ell \le L\} \subset L^2(\mathbb{R}^n)$. The system $\mathcal{W}_S(\Psi)$, given by (2.1), is a continuous Parseval frame wave packet system relative to (S, λ) for $L^2(\mathbb{R}^n)$ if and only if*

$$\Delta_\Psi(\xi) = \sum_{\ell=1}^{L} \int_S |\hat\psi^\ell(a^I\xi - z)|^2 \, d\lambda(a,z) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.2}$$

Proof. Observe first that, for each $\psi^\ell \in L^2(\mathbb{R}^n)$, $\ell = 1, \cdots, L$, using Proposition 1.1 we obtain that $D_a\, M_\nu\, T_y\, \psi^\ell = e^{2\pi i y\cdot\nu}\, T_{a^{-1}y}\, D_a\, M_\nu\, \psi^\ell$. Hence, using again Proposition 1.1, we have that $\big(D_a\, M_\nu\, T_y\, \psi^\ell\big)^\wedge = e^{2\pi i y\cdot\nu}\, M_{-a^{-1}y}\, D_{a^I}\, T_\nu\, \hat\psi^\ell$, and, thus,

$$|\langle f, D_a\, M_\nu\, T_y\, \psi^\ell\rangle| = |\langle \hat f, M_{-a^{-1}y}\, D_{a^I}\, T_\nu\, \hat\psi^\ell\rangle|. \tag{2.3}$$

Let $u = a^{-1}y$, so that $|\det a|\,du = dy$, and, then, using (2.3) and the Plancherel Theorem, we have that for every $f \in L^2(\mathbb{R}^n)$:

$$\begin{aligned}
\int_{\mathbb{R}^n} |\langle f, D_a\, M_\nu\, T_y\, \psi^\ell\rangle|^2\, dy &= \int_{\mathbb{R}^n} |\langle \hat f, M_{-u}\, D_{a^I}\, T_\nu\, \hat\psi^\ell\rangle|^2\, |\det a|\, du \\
&= \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \hat f(\xi)\, \overline{\hat\psi^\ell(a^I\xi - \nu)}\, e^{2\pi i u\cdot\xi} \Big|^2\, du \\
&= \int_{\mathbb{R}^n} \Big| \big\{ \hat f\, \overline{\hat\psi^\ell(a^I\cdot - \nu)} \big\}^\vee(u)\, d\xi \Big|^2\, du \\
&= \int_{\mathbb{R}^n} |\hat f(\xi)|^2\, |\hat\psi^\ell(a^I\xi - \nu)|^2\, d\xi. \qquad (2.4)
\end{aligned}$$

Thus, using (2.4) and Fubini's theorem, we see that the system $\mathcal{W}_S(\Psi)$ is a continuous Parseval frame if and only if

$$\begin{aligned}
\|f\|^2 &= \sum_{\ell=1}^{L} \int_S \int_{\mathbb{R}^n} |\langle f, D_a\, M_\nu\, T_y\, \psi^\ell\rangle|^2\, dy\, d\lambda(a,\nu) \\
&= \sum_{\ell=1}^{L} \int_S \int_{\mathbb{R}^n} |\hat f(\xi)|^2\, |\hat\psi^\ell(a^I\xi - \nu)|^2\, d\xi\, d\lambda(a,\nu) \\
&= \int_{\mathbb{R}^n} |\hat f(\xi)|^2\, \Delta_\Psi(\xi)\, d\xi,
\end{aligned}$$

for all $f \in L^2(\mathbb{R}^n)$, and (2.2) follows easily. □

Notice that the main step in the proof of Theorem 2.1 is the application of the Plancherel Theorem in (2.4). This computation shows the essential role played by the translation variable y (or $u = a^{-1}y$) in the proof of Theorem 2.1 and, also, explains the reason for using *all* translations T_y, $y \in \mathbb{R}^n$, in Definition 2.1. Later we will consider a similar situation in the case of discrete wave packet systems, which will involve *all* the translations arising from a lattice. Some authors (e.g., [**ACM**], [**CDH**], [**CKS**], [**HK**], [**W**]) have considered the Beurling density and several variants of this notion in order to understand the restrictions on the dispersion of elements in situations where a Parseval frame is obtained from an irregular translation set, as well as irregular dilation and/or modulation sets. To our knowledge, however, there are no results to date on irregular translation sets with an abstract measure replacing the Lebesgue measure. In this paper, we restrict ourselves to lattices ($\mathbb{Z}^n$) or continuous ($\mathbb{R}^n$) translations.

It is clear that, by choosing a different order of the operators, one obtains other versions of Theorem 2.1 for the systems $\mathcal{W}_S^{(i)}(\Psi)$, $1 \le i \le 5$. Indeed, using arguments similar to the ones for Theorem 2.1, we obtain the following characterizing equalities.

Corollary 2.2. *Let $\Psi = \{\psi^\ell : 1 \le \ell \le L\} \subset L^2(\mathbb{R}^n)$. For $1 \le i \le 5$, the system $\mathcal{W}_S^{(i)}(\Psi)$ is a continuous Parseval frame wave packet system relative to (S,λ) for $L^2(\mathbb{R}^n)$ if and only if*

$$\Delta_\Psi^{(i)}(\xi) = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.5}$$

where

$$\Delta_\Psi^{(1)}(\xi) = \Delta_\Psi(\xi),$$

$$\Delta_\Psi^{(2)}(\xi) = \sum_{\ell=1}^{L} \int_S |\hat{\psi}^\ell(a^I \xi - \nu)|^2 \, |\det a|^{-1} \, d\lambda(a,\nu),$$

$$\Delta_\Psi^{(3)}(\xi) = \sum_{\ell=1}^{L} \int_S |\hat{\psi}^\ell(a^I (\xi - \nu))|^2 \, d\lambda(a,\nu),$$

$$\Delta_\Psi^{(4)}(\xi) = \Delta_\Psi^{(5)}(\xi) = \sum_{\ell=1}^{L} \int_S |\hat{\psi}^\ell(a^I (\xi - \nu))|^2 \, |\det a|^{-1} \, d\lambda(a,\nu).$$

Thus we have seen that the changes in the order in which the operators are taken give us four different characterizations corresponding to the four functions $\Delta_\Psi = \Delta_\Psi^{(1)}, \Delta_\Psi^{(2)}, \Delta_\Psi^{(3)}, \Delta_\Psi^{(4)} = \Delta_\Psi^{(5)}$.

2.1. Examples. We will describe a number of examples of continuous wave packet systems, according to Definition 2.1, for different choices of the set S.

Example (a). Let $S = \{0\} \times \mathbb{R}^n \simeq \mathbb{R}^n$ and $\psi \in L^2(\mathbb{R}^n)$, then

$$\mathcal{W}_S(\psi) = \mathcal{G}_S(\psi) = \big\{ M_\nu \, T_y \, \psi : \, \nu \in S, y \in \mathbb{R}^n \big\}$$

is the **continuous Gabor system** (also known as the **Weyl–Heisenberg system**) associated with S. Furthermore, if one chooses λ to be Lebesgue measure, then Theorem 2.1 shows that $\mathcal{W}_S(\psi)$ is a continuous Parseval frame relative to $(S, d\nu)$ for any function $\psi \in L^2(\mathbb{R}^n)$ with $\|\psi\| = 1$ (this property of Gabor systems is well known; see, for example, [**G**, Ch. 3]).

Example (b). Let $S = \{0\} \times \mathbb{Z}^n \simeq \mathbb{Z}^n$ and $\psi \in L^2(\mathbb{R}^n)$, then

$$\mathcal{W}_S(\psi) = \mathcal{G}_S(\psi) = \big\{ M_k \, T_y \, \psi : \, k \in S, y \in \mathbb{R}^n \big\}$$

is the continuous Gabor system associated with S. If one chooses λ to be the counting measure, then it follows from Theorem 2.1 that $\mathcal{W}_S(\psi)$ is a continuous Parseval frame relative to (S, λ) iff

$$\sum_{k \in \mathbb{Z}^n} |\hat{\psi}(\xi - k)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.6}$$

Observe that equation (2.6) is equivalent to the property that $\{T_k \, \psi : \, k \in \mathbb{Z}\}$ is an orthonormal system (cf. [**HW**, Prop. 2.1.11]).

Example (c). Let $S \subseteq GL_n(\mathbb{R}) \times \{0\} \simeq GL_n(\mathbb{R})$, then

$$\mathcal{W}_S(\Psi) = \mathcal{A}_S(\Psi) = \big\{ D_a \, T_y \, \psi^\ell : \, a \in S, y \in \mathbb{R}^n, 1 \le \ell \le L \big\}$$

is the **continuous affine system** associated with S. If S is a closed subgroup of $GL_n(\mathbb{R})$ or, more generally, a topological group continuously embedded in $GL_n(\mathbb{R})$, then it is natural to take λ to be the right Haar measure on S. In this case, the function $\Delta_\Psi(\xi)$ in (2.2) is S-invariant, in the sense that $\Delta_\Psi(a^I \, \xi) = \Delta_\Psi(\xi)$ for all $a \in S$, $\xi \in \mathbb{R}^n$, and equation (2.2) is the classical Calderón condition (see [**C**], where this equation was originally obtained, or [**WW**], for the generalizations). In [**LWWW**], it is shown that the affine group $G = S \ltimes \mathbb{R}^n$ must be non–unimodular if there exists a continuous Parseval frame $\mathcal{A}_S(\Psi)$. Conversely, if G is non–unimodular and almost every $\xi \in \mathbb{R}^n$ has a compact ε–stabilizer $K_\xi^\varepsilon = \{a \in S : \, |a^I \, \xi - \xi| \le \varepsilon\}$, then one can find $\psi \in L^2(\mathbb{R}^n)$ for which $\mathcal{A}_S(\Psi)$ is a continuous Parseval frame relative to (S, λ). In [**LWWW**], the authors have considered other examples involving either integration over a coset space of a group or over products of subgroups. Also observe that the continuous Gabor systems in (a) may be viewed as obtained by integration over a coset space of the Weyl–Heisenberg group (see, for example, [**F**] or [**G**] for more details on this approach to the continuous Gabor systems).

The following two examples, unlike the previous ones, are not continuous Parseval frames. However, they can still be described according to Definition 2.1.

Example (d). Let $S = \{((1+\nu^2)^{1/4} I_n, \nu) : \nu \in \mathbb{R}^n\} \subset GL_n(\mathbb{R}) \times \mathbb{R}^n$, where I_n is the $n \times n$ identity matrix, and $\phi(x) = 2^{n/4}\, e^{-\pi x^2}$, $x \in \mathbb{R}^n$. Then the functions

$$\mathcal{W}_S(\phi) = \big\{T_y\, M_\nu\, D_a\, \phi :\ (a,\nu) \in S, y \in \mathbb{R}^n\big\}$$

are the wave packets introduced by Còrdoba and Fefferman in [**CF**]. In the same paper, Còrdoba and Fefferman show that their wave packets form, "approximately", a continuous Parseval frame, in the sense that, for each $f \in \mathcal{S}(\mathbb{R}^n)$, the system $\mathcal{W}_S(\phi)$ is reproducing up to a small error that is controlled by a Sobolev norm of f.

Example (e). The following example, which is a generalization of the wave packets of Còrdoba and Fefferman, is due to Hogan and Lakey [**HL**]. Let $S = \{(|\nu|^{1/2}, \nu) : \nu \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$ and define the collections

$$\mathcal{W}_S(\psi) = \big\{\psi_{(y,\nu)} = T_y\, M_\nu\, D_{a(\nu)}\, \psi :\ (a(\nu),\nu) \in S, y \in \mathbb{R}\big\}.$$

In [**HL**] it is shown that, if $\psi \in L^1 \bigcap L^2(\mathbb{R})$ is real and if $|\xi|\,|\hat{\psi}(\xi)| \in L^1(\mathbb{R})$, then there are constants $0 < A < B < \infty$ such that

$$A\,\|f\|^2 \;\le\; \iint_{\mathbb{R}^2} \big|\langle f, \psi_{(y,\nu)}\rangle\big|^2\, dy\, d\nu \;\le\; B\,\|f\|^2,$$

for all $f \in L^2(\mathbb{R})$.

3. Discrete wave packet systems

In this section, we will examine the case of discrete wave packet systems. These systems are discrete analogs of the continuous wave packets considered in the previous section, and are obtained by restricting the ordered pairs of dilations and modulations to a countable set and by sampling the translations over a regular lattice (while, in the continuous case, the translations involve all $y \in \mathbb{R}^n$).

More precisely, let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $\mathcal{P}$ be a countable set and, corresponding to each $p \in \mathcal{P}$, let $a_p \in GL_n(\mathbb{R})$ and $\nu_p \in \mathbb{R}^n$. We define the (discrete) **wave packet systems** generated by Ψ relative to $\mathcal{P}$ as the collections

$$\mathcal{W}_{\mathcal{P}}(\Psi) = \big\{D_{a_p}\, M_{\nu_p}\, T_k\, \psi^\ell :\ k \in \mathbb{Z}^n,\ p \in \mathcal{P},\ \ell = 1, \ldots, L\big\}. \tag{3.1}$$

Special cases of the set $\mathcal{W}_{\mathcal{P}}(\Psi)$ are the classical (discrete) Gabor and affine systems. Indeed, if $\mathcal{P} = \mathbb{Z}^n$, $\nu_p = b\,p$, where $b \in GL_n(\mathbb{R})$, $p \in \mathcal{P}$, and $a_p = I_n$, for each $p \in \mathcal{P}$, where I_n is the $n \times n$ identity matrix, then

$$\mathcal{W}_{\mathcal{P}}(\Psi) = \mathcal{G}_b(\Psi) = \big\{M_{bp}\, T_k\, \psi^\ell :\ p, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\big\}$$

is the Gabor system generated by Ψ. Alternatively, if $\mathcal{P} = \mathbb{Z}$, $a_p = a^p$, where $a \in GL_n(\mathbb{R})$, and $\nu_p = 0$, for each $p \in \mathcal{P}$, then

$$\mathcal{W}_{\mathcal{P}}(\Psi) = \mathcal{A}_a(\Psi) = \big\{D_a^p\, T_k\, \psi^\ell :\ p \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\big\}$$

is the affine system generated by Ψ and the integer powers of the dilation matrix $a \in GL_n(\mathbb{R})$.

We will also consider variants of the (discrete) wave packet systems obtained by using a larger collection of translations and an appropriate renormalization of the generators. More precisely, we define the (discrete) **oversampled wave packet systems** generated by Ψ relative to the set $\mathcal{P}$ and the matrices $\{\gamma_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$ as the collections

$$\mathcal{W}_{\mathcal{P}}^{\gamma_p}(\Psi) = \big\{D_{a_p}\, M_{\nu_p}\, T_{\gamma_p^{-1}k}\Big(\frac{1}{\sqrt{|\det \gamma_p|}}\,\psi^\ell\Big) : k \in \mathbb{Z}^n,\, p \in \mathcal{P},\, \ell = 1,\dots,L\big\}. \tag{3.2}$$

The notion of oversampling plays a very important role, for example, in the case of the affine systems, and Definition (3.2) includes, as special cases, the oversampled affine systems of Chui and Shi [**CS**] and the quasi affine systems of Ron and Shen [**RSb**] (cf. [**HLWW**] for a more detailed study of the notion of oversampling in relation to affine systems and other reproducing systems). For example, let $\mathcal{P} = \mathbb{Z}$, $a_p = a^p$, where $a \in GL_n(\mathbb{R})$ and $p \in \mathcal{P}$, $\nu_p = 0$, as in the case of affine systems, and, in addition, let $\gamma_p = I_n$, if $p \ge 0$, $\gamma_p = a^{-p}$, if $p < 0$. Under these assumptions, the collections $\mathcal{W}_{\mathcal{P}}(\psi)$ are the quasi affine systems

$$\mathcal{W}_{\mathcal{P}}(\psi) = \mathcal{A}_a^q(\psi) = \{\psi^q_{p,k} : p \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where

$$\psi^q_{p,k} = \begin{cases} \sqrt{|\det a^p|}\, D_a^p\, T_{a^p k}\,\psi, & k \in \mathbb{Z}^n,\ p < 0 \\ D_a^p\, T_k\,\psi, & k \in \mathbb{Z}^n,\ p \ge 0. \end{cases}$$

This example also illustrates the role of the normalization factor $\frac{1}{\sqrt{|\det\gamma_p|}}$ in Definition 3.2.

More general examples of oversampled wave packet systems will be considered in Section 3.2.

3.1. Unified theory of discrete reproducing systems. In order to describe the properties of the discrete wave packet systems that we will consider in this paper, it is useful to recall the notion of a frame.

A countable family $\{e_j : j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathcal{H}$ is a **frame** if there exist constants $0 < A \le B < \infty$ satisfying

$$A\,\|v\|^2 \le \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2 \le B\,\|v\|^2$$

for all $v \in \mathcal{H}$. The constants A and B are called the **lower** and **upper frame bounds**, respectively. If only the right hand side inequality holds, we say that $\{e_j : j \in \mathcal{J}\}$ is a **Bessel system** with constant B. A frame is a **tight frame** if A and B can be chosen so that $A = B$, and is a **Parseval frame** (also called **normalized tight frame**) if $A = B = 1$. Thus, if $\{e_j : j \in \mathcal{J}\}$ is a Parseval frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2 \tag{3.3}$$

for each $v \in \mathcal{H}$. This is equivalent to the reproducing formula

$$v = \sum_{j\in\mathcal{J}} \langle v, e_j\rangle\, e_j \tag{3.4}$$

for all $v \in \mathcal{H}$, where the series in (3.4) converges in the norm of $\mathcal{H}$ (we refer the reader to [**HW**, Ch. 8] for the basic properties of frames).

One main problem in the study of wavelets, Gabor systems and related systems is to determine conditions on the families $\Psi \subset L^2(\mathbb{R}^n)$ such that these systems form a basis or a Parseval frame or, more generally, a frame. In this section, we will examine this problem in relation to the wave packet systems.

The following simple observation, which is adapted from [**HLWW**, Prop. 2.1] shows that in order for the system $\mathcal{W}_{\mathcal{P}}^{\gamma_p}(\Psi)$, given by (3.2), to be a frame (or even a Bessel system), there are some restrictions on the choice of the oversampling matrices $\{\gamma_p\}_{p\in\mathcal{P}}$.

PROPOSITION 3.1. *If the oversampled wave packet system $\mathcal{W}_{\mathcal{P}}^{\gamma_p}(\Psi)$, given by (3.2), is a Bessel system with constant $B > 0$, then, for each $\ell = 1, \cdots, L$,*

$$|\det \gamma_p| \geq \frac{1}{B}\, \|\psi^\ell\|^2 \qquad \text{for each } p \in \mathcal{P}.$$

PROOF. Since $\mathcal{W}_{\mathcal{P}}^{\gamma_p}(\Psi)$ is a Bessel system with constant β, then

$$\sum_{\ell=1}^{L} \sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{p,k}^\ell \rangle|^2 \leq \beta\, \|f\|^2 \tag{3.5}$$

for all $f \in L^2(\mathbb{R}^n)$, where $\psi_{p,k}^\ell = |\det \gamma_p|^{-1/2}\, D_{a_p}\, M_{\nu_p}\, T_{\gamma_p^{-1} k}\, \psi^\ell$. Equation (3.5) implies that, for any $p_0 \in \mathcal{P}$, $k_0 \in \mathbb{Z}^n$, $1 \leq \ell_0 \leq L$, we have:

$$|\langle \psi_{p_0,k_0}^{\ell_0}, \psi_{p_0,k_0}^{\ell_0} \rangle|^2 \leq \beta\, \|\psi_{p_0,k_0}^{\ell_0}\|^2. \tag{3.6}$$

Since $\|\psi_{p_0,k_0}^{\ell_0}\|^2 = |\det \gamma_{p_0}|^{-1}\, \|\psi^{\ell_0}\|^2$, from (3.6) we deduce:

$$|\det \gamma_{p_0}|^{-2}\, \|\psi^{\ell_0}\|^4 \leq \beta\, |\det \gamma_{p_0}|^{-1}\, \|\psi^{\ell_0}\|^2,$$

and, thus, $|\det \gamma_{p_0}| \geq \beta^{-1}\, \|\psi^{\ell_0}\|^2$, for all $p_0 \in \mathcal{P}$, $1 \leq \ell_0 \leq L$. □

In [**HLW**] and [**HLWW**], the authors and their collaborators have developed a general approach to the study of reproducing systems generated by a finite family $\Psi \subset L^2(\mathbb{R}^n)$. After recalling some basic results from these papers, and making some additional observations, we will apply this approach to the study of the discrete wave packet systems $\mathcal{W}_{\mathcal{P}}(\Psi)$, given by (3.1).

Let $\mathcal{P}$ be a countable collection of indices, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(\mathbb{R}^n)$ and $\{C_p : p \in \mathcal{P}\}$ a corresponding collection of matrices in $GL_n(\mathbb{R})$ and consider the families of the form

$$\Phi_{\{C_p\}}^{\{g_p\}} = \big\{ T_{C_p k}\, g_p : k \in \mathbb{Z}^n, p \in \mathcal{P} \big\}. \tag{3.7}$$

We obtained a characterization of all those $\{g_p\}_{p\in\mathcal{P}}$ such that $\Phi_{\{C_p\}}^{\{g_p\}}$ is a Parseval frames for $L^2(\mathbb{R}^n)$. In order to state the main result, we need to introduce the following notation. Define:

$$\Lambda = \bigcup_{p\in\mathcal{P}} C_p^I\,(\mathbb{Z}^n)\,, \tag{3.8}$$

where $C_p^I = (C_p^T)^{-1}$, and, for $\alpha \in \Lambda$, let

$$\mathcal{P}_\alpha = \{p \in \mathcal{P} :\ \alpha \in C_p^I\, \mathbb{Z}^n\}. \tag{3.9}$$

If $\alpha = 0 \in \Lambda$, then $\mathcal{P}_0 = \mathcal{P}$ (since $C_p^T 0 = 0$ for all $p \in \mathcal{P}$); otherwise the best we can say is that $\mathcal{P}_\alpha \subset \mathcal{P}$. Also, let

$$\mathcal{D} = \mathcal{D}_E = \big\{ f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat{f} \text{ is compact in } \mathbb{R}^n \setminus E \big\}, \tag{3.10}$$

where E is a subspace of $\mathbb{R}^n$ of dimension smaller than n to be specified later, and appropriately chosen for each of the systems defined by (3.7) (for example, we use $E = \{0\}$ in Proposition 3.8 and a nontrivial set E in Section 3.3). It is clear that $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^n)$. We have the following result.

Theorem 3.2 ([**HLW**]). *Let $\{g_p\}_{p\in\mathcal{P}}$ be a collection of functions in $L^2(\mathbb{R}^n)$, where $\mathcal{P}$ is a countable indexing set, and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Assume the local integrability condition (L.I.C.):*

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + C_p^I m)|^2 \, \frac{1}{|\det C_p|} \, |\hat{g}_p(\xi)|^2 \, d\xi < \infty \tag{3.11}$$

for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10) and $C_p^I = (C_p^T)^{-1}$. Then the system $\Phi_{\{C_p\}}^{\{g_p\}}$, given by (3.7), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \frac{1}{|\det C_p|} \, \hat{g}_p(\xi) \, \overline{\hat{g}_p(\xi+\alpha)} = \delta_{\alpha,0} \quad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{3.12}$$

for each $\alpha \in \Lambda$, where δ is the Kronecker delta for $\mathbb{R}^n$.

In the same paper, we also obtained a similar theorem for the characterization of dual systems (cf. [**HLW**, Thm.9.1]), as well as the following result about Bessel families (cf. [**HLW**, Prop.4.1]) which will be useful in the next section.

Proposition 3.3 ([**HLW**]). *Let $\{g_p\}_{p\in\mathcal{P}} \subset L^2(\mathbb{R}^n)$, where $\mathcal{P}$ is a countable set, and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. If the system $\Phi_{\{C_p\}}^{\{g_p\}}$, given by (3.7), is Bessel with constant $B > 0$, then*

$$\sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} \, |\hat{g}_p(\xi)|^2 \le B \quad \textit{for a.e. } \xi \in \mathbb{R}^n. \tag{3.13}$$

In the case of general frames (not just Parseval frames) no characterization results in the spirit of Theorem 3.2 are known for the collections $\Phi_{\{C_p\}}^{\{g_p\}}$. Nevertheless, the following theorem, which uses an idea from [**D**, Sec. 3.3], gives sufficient conditions for $\Phi_{\{C_p\}}^{\{g_p\}}$ to be a frame for $L^2(\mathbb{R}^n)$.

Theorem 3.4. *Let $\{g_p\}_{p\in\mathcal{P}} \subset L^2(\mathbb{R}^n)$, where $\mathcal{P}$ is countable, and $\{C_p\}_{p\in\mathcal{P}} \subset GL_n(\mathbb{R})$. Suppose that there are $0 < \alpha \le \beta < \infty$ such that*

$$\alpha \le \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} \, |\hat{g}_p(\xi)|^2 \le \beta, \qquad \textit{for all } \xi \in \mathbb{R}^n \tag{3.14}$$

and that

$$\sum_{m\ne 0} \sup_{\xi\in\mathbb{R}^n} \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} \, |\overline{\hat{g}_p(\xi)} \, \hat{g}_p(\xi + C_p^I m)| < \alpha. \tag{3.15}$$

Then the family $\Phi_{\{C_p\}}^{\{g_p\}}$, given by (3.7), is a frame for $L^2(\mathbb{R}^n)$ with lower and upper frame bounds

$$A = \alpha - \sum_{m\neq 0} \sup_{\xi\in\mathbb{R}^n} \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|$$

and

$$B = \beta + \sum_{m\neq 0} \sup_{\xi\in\mathbb{R}^n} \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|} |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|,$$

respectively.

Proof. It suffices to prove the theorem for $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10).

We fix $p \in \mathcal{P}$ and estimate first the quantity $\sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2$. Since $(T_{C_p k}\, g_p)^\wedge(\xi) = e^{-2\pi i C_p k\cdot\xi}\, \hat{g}(\xi)$, it follows from the Plancherel theorem that

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 = \sum_{k\in\mathbb{Z}^n} \Big| \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}_p(\xi)}\, e^{2\pi i C_p k\cdot\xi}\, d\xi \Big|^2. \tag{3.16}$$

Let $\mathbb{T}^n = [0,1)^n$. Since $\mathbb{R}^n = \bigcup_{l\in\mathbb{Z}^n} \{C^I(\mathbb{T}^n - l)\}$ is a disjoint union, the integral in (3.16) can be written in the form

$$\sum_{l\in\mathbb{Z}^n} \int_{C_p^I(\mathbb{T}^n)} \hat{f}(\xi - C_p^I l)\, \overline{\hat{g}_p(\xi - C_p^I l)}\, e^{2\pi i C_p k\cdot\xi}\, d\xi =$$
$$= \int_{C_p^I(\mathbb{T}^n)} \Big(\sum_{l\in\mathbb{Z}^n} \hat{f}(\xi - C_p^I l)\, \overline{\hat{g}_p(\xi - C_p^I l)} \Big) e^{2\pi i C_p k\cdot\xi}\, d\xi.$$

Since the function $\sum_{l\in\mathbb{Z}^n} \hat{f}(\xi - C_p^I l)\, \overline{\hat{g}_p(\xi - C_p^I l)}$ is $C_p^I\, \mathbb{Z}^n$-periodic and belongs to $L^2(C_p^I\mathbb{T}^n)$ (recall that $f \in \mathcal{D}$), it follows that the right hand side of (3.16) is, up to a constant, the square of the l^2-norm of the Fourier coefficients of this $C_p^I\, \mathbb{Z}^n$-periodic function, with respect to the orthonormal basis

$$\{\sqrt{|\det C_p|}\, e^{2\pi i C_p k\cdot\xi} : k \in \mathbb{Z}^n\}$$

of $L^2(C_p^I\mathbb{T}^n)$. Therefore, from this observation we obtain:

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 = \frac{1}{|\det C_p|} \int_{C_p^I\mathbb{T}^n} |\sum_{l\in\mathbb{Z}^n} \hat{f}(\xi - C_p^I l)\, \overline{\hat{g}_p(\xi - C_p^I l)}|^2\, d\,\xi, \tag{3.17}$$

From equation (3.17), making the change of indices $u = l + m$ we obtain:

$$|\det C_p| \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 =$$
$$= \int_{C_p^I\mathbb{T}^n} \sum_{l,u\in\mathbb{Z}^n} \hat{f}(\xi + C_p^I l)\, \overline{\hat{g}_p(\xi + C_p^I l)}\, \overline{\hat{f}(\xi + C_p^I u)}\, \hat{g}_p(\xi + C_p^I u)\, d\xi$$
$$= \int_{C_p^I\mathbb{T}^n} \sum_{l,m\in\mathbb{Z}^n} \hat{f}(\xi + C_p^I l)\, \overline{\hat{g}_p(\xi + C_p^I l)}\, \overline{\hat{f}(\xi + C_p^I(l+m))}\, \hat{g}_p(\xi + C_p^I(l+m))\, d\xi$$
$$= \sum_{m\in\mathbb{Z}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}_p(\xi)}\, \overline{\hat{f}(\xi + C_p^I m)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi.$$

From this expression, summing over $p \in \mathcal{P}$ and splitting the sum in m into the case $m = 0$ and $m \neq 0$ we obtain:

$$\sum_{p\in\mathcal{P}} \sum_{k\in\mathbb{Z}^n} |\langle f, T_{C_p k}\, g_p\rangle|^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\hat{g}_p(\xi)|^2\, d\xi + R(f), \tag{3.18}$$

where

$$R(f) = \sum_{p\in\mathcal{P}} \sum_{m\neq 0} \frac{1}{|\det C_p|} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}_p(\xi)}\, \overline{\hat{f}(\xi + C_p^I m)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi. \tag{3.19}$$

Using the Cauchy-Schwarz inequality and the change of variables $\eta = \xi + C_p^I m$, for each $p \in \mathcal{P}$ we have:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}_p(\xi)}\, \overline{\hat{f}(\xi + C_p^I m)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi \leq \\
&\quad \leq \Big(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|\, d\xi\Big)^{1/2} \Big(\int_{\mathbb{R}^n} |\hat{f}(\xi + C_p^I m)|^2 \times \\
&\qquad \times |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|\, d\xi\Big)^{1/2} \\
&\quad = \Big(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|\, d\xi\Big)^{1/2} \Big(\int_{\mathbb{R}^n} |\hat{f}(\eta)|^2\, |\overline{\hat{g}_p(\eta - C_p^I m)}\, \hat{g}_p(\eta)|\, d\eta\Big)^{1/2}.
\end{aligned}$$

Thus, using the inequality $2\,ab \leq a^2 + b^2$, and summing over m, from the above expression we obtain that

$$\begin{aligned}
&\sum_{m\neq 0} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \overline{\hat{g}_p(\xi)}\, \overline{\hat{f}(\xi + C_p^I m)}\, \hat{g}_p(\xi + C_p^I m)\, d\xi \leq \\
&\qquad\qquad \leq \sum_{m\neq 0} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|\, d\xi.
\end{aligned}$$

Using this observation into (3.19) we obtain:

$$\begin{aligned}
R(f) &\leq \sum_{m\neq 0} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|\, d\xi \\
&\leq \|f\|^2 \sum_{m\neq 0} \sup_{\xi\in\mathbb{R}^n} \sum_{p\in\mathcal{P}} \frac{1}{|\det C_p|}\, |\overline{\hat{g}_p(\xi)}\, \hat{g}_p(\xi + C_p^I m)|. \qquad (3.20)
\end{aligned}$$

The proof now follows from (3.18) and (3.20), and by making appropriate choices of $f \in \mathcal{D}$. □

3.2. Discrete wave packet Parseval frames. We will now apply the unified theory presented in the previous section to the study of the discrete wave packet systems. As a first application, we will use Theorem 3.2 to deduce a characterization of the functions that generate a discrete wave packet system Parseval frame, in the same spirit as the results which are known in the case of affine and Gabor systems (cf. [**HLW**] and the references there).

We obtain the following characterization of all families $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, for which the wave packet systems $\mathcal{W}_{\mathcal{P}}^{\gamma_p}(\Psi)$, given by (3.2) form a Parseval frame for $L^2(\mathbb{R}^n)$, provided the L.I.C. is satisfied. It is clear that the corresponding characterization for the systems $\mathcal{W}_{\mathcal{P}}(\Psi)$, given by (3.1), follows by considering the case $\gamma_p = I_n$, for each $p \in \mathcal{P}$.

THEOREM 3.5. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $\mathcal{P}$ be countable and, corresponding to each $p \in \mathcal{P}$, let $a_p, \gamma_p \in GL_n(\mathbb{R})$ and $\nu_p \in \mathbb{R}^n$. Assume the L.I.C.:*

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\, \hat{f}} |\hat{f}(\xi + a_p^T\, \gamma_p^T\, m)|^2\, |\hat{\psi}(a_p^I\, \xi - \nu_p)|^2\, d\xi < \infty, \tag{3.21}$$

for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10). Then the system $\mathcal{W}_{\mathcal{P}}(\Psi)$, given by (3.1), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \hat{\psi}(a_p^I\, \xi - \nu_p)\, \overline{\hat{\psi}(a_p^I(\xi + \alpha) - \nu_p)} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{3.22}$$

for each $\alpha \in \Lambda = \bigcup_{p\in\mathcal{P}} a_p^T\, \gamma_p^T\, \mathbb{Z}^n$, where $\mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in a_p^T\, \gamma_p^T\, \mathbb{Z}^n\}$.

PROOF. Since

$$D_{a_p}\, M_{\nu_p}\, T_{\gamma_p^{-1}k}\, \psi = e^{2\pi i \nu_p \cdot \gamma_p^{-1} k}\, T_{a_p^{-1}\gamma_p^{-1}k}\, D_{a_p}\, M_{\nu_p}\, \psi,$$

using the notation introduced before Theorem 3.2 we can write equation (3.1) as

$$\mathcal{W}_{\mathcal{P}}(\psi) = \big\{T_{C_p\, k}\, g_p : \, k \in \mathbb{Z},\, p \in \mathcal{P}\big\},$$

where

$$C_p = a_p^{-1}\gamma_p^{-1} \quad \text{and} \quad g_p = \frac{e^{2\pi i \nu_p \cdot \gamma_p^{-1} k}}{\sqrt{|\det \gamma_p|}}\, D_{a_p}\, M_{\nu_p}\, \psi.$$

Thus, we have:

$$\Lambda = \bigcup_{p\in\mathcal{P}} a_p^T\, \gamma_p^T\, \mathbb{Z}^n, \quad \text{and, for } \alpha \in \Lambda, \quad \mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in a_p^T\, \gamma_p^T\, \mathbb{Z}\}.$$

With the assumptions that we made for $\mathcal{P}$ and g_p, the functional $L(f)$, given by the left hand side of (3.11), takes the form

$$L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{\text{supp}\, \hat{f}} |\hat{f}(\xi + a_p^T\, \gamma_p^T\, m)|^2\, |\hat{\psi}(a_p^I\, \xi - \nu_p)|^2\, d\xi,$$

and, by (3.21), we have that $L(f) < \infty$ for all $f \in \mathcal{D}$. It now follows from Theorem 3.2 that the system $\mathcal{W}_{\mathcal{P}}(\Psi)$ is a Parseval frame if and only if (3.22) holds. $\square$

As in the case of continuous wave packet systems, we have different versions of Theorem 3.5 if we change the order of the operators in Definition 3.1. Let

$$\begin{aligned}
U^{(1)}_{(p,k)} &= D_{a_p}\, T_{\gamma_p^{-1}k}\, M_{\nu_p},\\
U^{(2)}_{(p,k)} &= T_{\gamma_p^{-1}k}\, D_{a_p}\, M_{\nu_p},\\
U^{(3)}_{(p,k)} &= M_{\nu_p}\, D_{a_p}\, T_{\gamma_p^{-1}k},\\
U^{(4)}_{(p,k)} &= T_{\gamma_p^{-1}k}\, M_{\nu_p}\, D_{a_p},\\
U^{(5)}_{(p,k)} &= M_{\nu_p}\, T_{\gamma_p^{-1}k}\, D_{a_p}.
\end{aligned}$$

We obtain alternate versions of the discrete wave packet systems $\mathcal{W}^{(i)}_{\mathcal{P}}(\Psi)$, with $1 \le i \le 5$, by replacing the operator $U^{(0)}_{(p,k)} = D_{a_p}\, T_{\gamma_p^{-1}k}\, M_{\nu_p}$ in Definition 3.1 with the operators $U^{(i)}_{(a,\nu,y)}$, $1 \le i \le 5$. We have the following characterization result.

COROLLARY 3.6. *Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$, $\mathcal{P}$ be countable and, corresponding to each $p \in \mathcal{P}$, let $a_p, \gamma_p \in GL_n(\mathbb{R})$ and $\nu_p \in \mathbb{R}^n$. Assume $L^{(i)}(f) < \infty$, $1 \le i \le 5$, for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10) and*

$$\begin{aligned}
L^{(1)}(f) \quad &= L(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + a_p^T\,\gamma_p^T\, m)|^2\, |\hat{\psi}(a_p^I\,\xi - \nu_p)|^2\, d\xi,\\
L^{(2)}(f) \quad &= \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + \gamma_p^T\, m)|^2\, |\hat{\psi}(a_p^I\,\xi - \nu_p)|^2\, d\xi,\\
L^{(3)}(f) \quad &= \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + a_p^T\,\gamma_p^T\, m)|^2\, |\hat{\psi}(a_p^I(\xi - \nu_p))|^2\, d\xi,\\
L^{(4)}(f) \quad &= L^{(5)}(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}^n} \int_{supp\,\hat{f}} |\hat{f}(\xi + \gamma_p^T\, m)|^2\, |\hat{\psi}(a_p^I(\xi - \nu_p))|^2\, d\xi.
\end{aligned}$$

Then the system $\mathcal{W}_{\mathcal{P}}^{(i)}(\Psi)$, $1 \le i \le 5$, is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if $E_\alpha^{(i)}(\xi) = \delta_{\alpha,0}$, for a.e. $\xi \in \mathbb{R}^n$, for each $\alpha \in \Lambda^{(i)}$, where

$$\begin{aligned}
E_\alpha^{(1)}(\xi) \quad &= E_\alpha^{(2)}(\xi) = E_\alpha(\xi) = \sum_{p\in\mathcal{P}_\alpha} \hat{\psi}(a_p^I\,\xi - \nu_p)\, \overline{\hat{\psi}(a_p^I(\xi+\alpha) - \nu_p)},\\
E_\alpha^{(3)}(\xi) \quad &= E_\alpha^{(4)}(\xi) = E_\alpha^{(5)}(\xi) = \sum_{p\in\mathcal{P}_\alpha} \hat{\psi}(a_p^I(\xi - \nu_p))\, \overline{\hat{\psi}(a_p^I(\xi+\alpha - \nu_p))},
\end{aligned}$$

and $\Lambda^{(1)} = \Lambda^{(3)} = \Lambda = \bigcup_{p\in\mathcal{P}} a_p^T\,\gamma_p^T\,\mathbb{Z}^n$, $\quad \Lambda^{(2)} = \Lambda^{(4)} = \Lambda^{(5)} = \bigcup_{p\in\mathcal{P}} \gamma_p^T\,\mathbb{Z}^n$, $\mathcal{P}_\alpha^{(1)} = \mathcal{P}_\alpha^{(3)} = \mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in a_p^T\,\gamma_p^T\,\mathbb{Z}^n\}$, $\quad \mathcal{P}_\alpha^{(2)} = \mathcal{P}_\alpha^{(4)} = \mathcal{P}_\alpha^{(5)} = \{p \in \mathcal{P} : \alpha \in \gamma_p^T\,\mathbb{Z}^n\}$.

There are special classes of wave packet systems $\mathcal{W}_{\mathcal{P}}(\Psi)$ for which the L.I.C., given by (3.21), is satisfied, and, as a consequence, one obtains versions of Theorem 3.5 which do not contain this hypothesis. For simplicity, in the following, we will restrict our attention to one dimensional systems with one generator. However, the same ideas can be extended to higher dimensions.

The following theorem deals with the case of (one-dimensional) wave packet systems where the dilations are generated by the integer powers of a real number. We have the following simple characterization result.

THEOREM 3.7. *Let $\psi \in L^2(\mathbb{R})$, $\mathcal{P}$ be countable and*

$$\mathcal{W}_{\mathcal{P}}(\psi) = \big\{D_{a^{j_p}}\, M_{\nu_p}\, T_k\, \psi :\ k \in \mathbb{Z},\ p \in \mathcal{P}\big\}, \tag{3.23}$$

where $j_p \in \mathbb{Z}$, $\nu_p, a \in \mathbb{R}$, $a > 1$, and, for each $j \in \mathbb{Z}$, there is a finite number K_j of indices $p \in \mathcal{P}$ for which $j_p = j$. Then the system $\mathcal{W}_{\mathcal{P}}(\psi)$, given by (3.23), is a Parseval frame for $L^2(\mathbb{R})$ if and only if

$$\sum_{p\in\mathcal{P}_\alpha} \hat{\psi}(a^{-j_p}\xi - \nu_p)\, \overline{\hat{\psi}(a^{-j_p}(\xi+\alpha) - \nu_p)} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{3.24}$$

for each $\alpha \in \Lambda = \bigcup_{p\in\mathcal{P}} a^{j_p}\,\mathbb{Z}$, where $\mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in a^{j_p}\,\mathbb{Z}\}$.

In order to prove this theorem, we need the following result which uses an idea from [**HLW**, Prop.5.2] (where a similar result is proved in the case of affine systems).

PROPOSITION 3.8. *Let $\psi \in L^2(\mathbb{R})$, $\mathcal{P}$ be countable, ν_p, $a \in \mathbb{R}$, $a > 1$, and, for each $j \in \mathbb{Z}$, assume that there is a finite number K_j of indices $p \in \mathcal{P}$ for which $j_p = j$. Also assume that*

$$\sum_{p\in\mathcal{P}} |\hat{\psi}(a^{-j_p}\xi - \nu_p)|^2 \le \beta \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{3.25}$$

for some $\beta > 0$. Then

$$L(f) = \sum_{p\in\mathcal{P}}\sum_{m\in\mathbb{Z}} \int_{supp\, \hat{f}} |\hat{f}(\xi + a^{j_p}m)|^2\, |\hat{\psi}(a^{-j_p}\xi - \nu_p)|^2\, d\xi < \infty, \tag{3.26}$$

for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10) with $E = \{0\}$.

To prove this proposition, we need the following two lemmas that are special cases of Lemma 5.10 and Lemma 5.11 from [**HLW**] (it suffices to set $E = \{0\}$). For $r \in \mathbb{R}$, $r > 0$, define:

$$\mathcal{Q}(r) = \{x \in \mathbb{R} : \frac{1}{r} < |x| < r\}.$$

LEMMA 3.9. *Let $a \in \mathbb{R}$, $a > 1$, and $r \in \mathbb{R}$. There exists $N = N(a, r) \in \mathbb{N}$ such that*

$$\#\{j \in \mathbb{Z} : a^j\, \eta \in Q(r)\} \le N$$

for all $\eta \in \mathbb{R}^n$.

LEMMA 3.10. *Let $a \in \mathbb{R}$, $a > 1$, and $r \in \mathbb{R}$. There exists $C = C(a, r) \in \mathbb{R}$ such that*

$$\#\{m \in \mathbb{Z}^n \setminus \{0\} : a^j\, m \in Q(r)\} \le C\, a^{-j}$$

for all $j \in \mathbb{Z}$.

We can now prove the Proposition 3.8.

PROOF OF PROPOSITION 3.8.

Without loss of generality, we can assume that, for each $f \in \mathcal{D}$, there exists $r \in \mathbb{R}$ such that $\operatorname{supp} \hat{f} \subset \mathcal{Q}(r)$.

Write $L_0(f)$ for the sum of the terms in (3.26) for which $m = 0$ and $L_1(f)$ for the sum of the terms in (3.26) for which $m \neq 0$. Then $L(f) = L_0(f) + L_1(f)$.

We first estimate $L_0(f)$. If $m = 0$, then, using (3.25), we have:

$$L_0(f) = \sum_{p\in\mathcal{P}} \int_{\mathcal{Q}(r)} |\hat{f}(\xi)|^2\, |\hat{\psi}(a^{-j_p}\xi - \nu_p)|^2\, d\xi \;\le\; \beta\, \|f\|^2 \;<\; \infty. \tag{3.27}$$

We now estimate $L_1(f)$. Using the change of variable $a^{-j_p}\xi = \eta$, from (3.26) we have:

$$L_1(f) = \sum_{p\in\mathcal{P}}\sum_{m\neq 0} \int_{a^{j_p}\eta\in \operatorname{supp} \hat{f}} |\hat{f}(a^{j_p}(\eta + m)|^2\, |\hat{\psi}(\eta - \nu_p)|^2\, a^{j_p}\, d\eta. \tag{3.28}$$

Suppose $m \in \mathbb{Z} \setminus \{0\}$, $a^{j_p}\, \eta \in Q(r)$ and $a^{j_p}(\eta + m) \in Q(r)$; then, for $j_p \in \mathbb{Z}$, we have:

$$|a^{j_p}m| \le |a^{j_p}(\eta + m)| + |a^{j_p}\eta| < r + r = 2r.$$

Thus, with the notation we introduced before Lemma 3.9, we have that

$$\{m \in \mathbb{Z}\setminus\{0\} : a^{j_p}\eta \in Q(r) \text{ and } a^{j_p}(\eta+m) \in Q(r)\} \subset \{m \in \mathbb{Z}\setminus\{0\} : a^{j_p}m \in Q(2r)\},$$

for every $j_p \in \mathbb{Z}$. By Lemma 3.10, the number of elements in the last set does not exceed $C(a, 2r)\, a^{-j_p}$, for each $j_p \in \mathbb{Z}$. Thus, it follows that

$$L_1(f) \le C(a, 2r) \|\hat{f}\|_\infty^2 \sum_{p\in\mathcal{P}} \int_{a^{j_p}\eta\in Q(r)} |\hat{\psi}(\eta - \nu_p)|^2\, d\eta.$$

Next, observe that, by Lemma 3.9, the number of distinct $j_p \in \mathbb{Z}$ such that $a^{j_p}\eta \in Q(r)$ does not exceed a fixed number $N(a, r)$, independently of $\eta \in \mathbb{R}$. Furthermore, by hypothesis, for any (of these finitely many distinct) j_p there is a finite number K_{j_p} indices $p' \in \mathcal{P}$ such that $j_p = j_{p'}$. Hence, letting $K(a, r)$ to be the maximum of the (finitely many) K_{j_p}, we have:

$$L_1(f) \le K(a, r)\, C(a, 2r)\, N(a, r)\, \|\hat{f}\|_\infty^2\, \|\hat{\psi}\|_2^2 < \infty. \tag{3.29}$$

Finally, from (3.27) and (3.29), we deduce that, if $f \in \mathcal{D}$, then $L(f) < \infty$. □

We can now prove Theorem 3.7.

Proof of Theorem 3.7.

We apply Theorem 3.5 with $a_p = a^{j_p}$ and $\gamma_p = 1$, for each $p \in \mathcal{P}$. Under these assumptions, equation (3.22) becomes equation (3.24) and, thus, in order to complete the proof, we only need to show that the L.I.C. (3.21) is satisfied. In order to show that this is the case, we apply Proposition 3.8. Indeed, if the system $\mathcal{W}_\mathcal{P}(\psi)$, given by (3.23), is a Parseval frame, then, by Proposition 3.3 applied to $\mathcal{W}_\mathcal{P}(\psi)$, we obtain (3.25). Conversely, if the equalities (3.24) are satisfied, then also in this case (3.25) is true (it suffices to take $\alpha = 0$ in (3.24)). In both cases, we can apply Proposition 3.8 and, thus, we obtain condition (3.26) (which is exactly the L.I.C. (3.21) under our assumptions for a_p, γ_p). □

As we mentioned at the beginning of Section 3, the oversampled wave packet systems $\mathcal{W}_\mathcal{P}^{\gamma_p}(\psi)$, given by (3.2), are obtained from the corresponding wave packet systems by using a larger collection of translations and an appropriate renormalization of the generators. For example, corresponding to the wave packet system $\mathcal{W}_\mathcal{P}(\psi)$, given by (3.23), we define the oversampled wave packet system generated by $\psi \in L^2(\mathbb{R})$, relative to the oversampling coefficient $\gamma \in \mathbb{Z}\setminus\{0\}$, as the collections of the form

$$\mathcal{W}_\mathcal{P}^{\gamma}(\psi) = \big\{\gamma^{-1/2}\, D_{a^{j_p}}\, M_{\nu_p}\, T_{\gamma^{-1}k}\, \psi :\ k \in \mathbb{Z},\, p \in \mathcal{P}\big\}, \tag{3.30}$$

where $j_p \in \mathbb{Z}$, ν_p, $a \in \mathbb{R}$. Observe that these systems include as special cases the classical oversampled affine systems introduced by Chui and Shi [**CS**], where $\nu_p = 0$ for each $p \in \mathcal{P}$.

Another interesting class of oversampled systems is obtained by considering the following collection of oversampling coefficients γ_p, $p \in \mathcal{P}$:

$$\gamma_p = \begin{cases} a^{-j_p} & \text{if } j_p < 0, \\ 1 & \text{if } j_p \ge 0. \end{cases} \tag{3.31}$$

Corresponding to the wave packet system $\mathcal{W}_\mathcal{P}(\psi)$, given by (3.23), we then define the **quasi affine wave packet system** generated by $\psi \in L^2(\mathbb{R})$ as the collections of the form

$$\mathcal{W}_\mathcal{P}^{q}(\psi) = \big\{\gamma_p^{-1/2}\, D_{a^{j_p}}\, M_{\nu_p}\, T_{\gamma_p^{-1}k}\, \psi :\ k \in \mathbb{Z},\, p \in \mathcal{P}\big\}, \tag{3.32}$$

where $j_p \in \mathbb{Z}$, ν_p, $a \in \mathbb{R}$ and γ_p is given by (3.31). The reason for the terminology is that these systems include, as special cases, the quasi affine systems of Ron and

Shen [**RSb**], where $\nu_p = 0$ for each $p \in \mathcal{P}$. As we will later show, these systems also inherit the basic fundamental properties of the quasi affine systems.

In particular, it is simple to verify that the quasi affine wave packet systems $\mathcal{W}^q_{\mathcal{P}}(\psi)$, unlike the corresponding wave packet systems $\mathcal{W}_{\mathcal{P}}(\psi)$, are **shift-invariant**, that is, they are invariant with respect to integer translations, when $a \in \mathbb{Z}$. Indeed, to see that this is the case, let $\psi^q_{p,k} = \gamma_p^{-1/2} \, D_{a^{j_p}} \, M_{\nu_p} \, T_{\gamma_p^{-1} k} \, \psi$, where γ_p is given by (3.31). It is clear that, using this notation, we can write $\mathcal{W}^q(\psi) = \{\psi^q_{p,k} : p \in \mathcal{P}\}$. Next consider the effect of a shifts T_ℓ, $\ell \in \mathbb{Z}$, on the system $\mathcal{W}^q(\psi)$. If $j < 0$, by changing the order of the operators, we have:

$$T_\ell \, \psi^q_{p,k} = a^{j_p/2} \, T_\ell \, D_a^{j_p} \, M_{\nu_p} \, T_{a^{j_p} k} \, \psi = a^{j_p/2} \, e^{-2\pi i \nu_p \cdot a^{j_p} \ell} \, D_a^{j_p} \, M_{\nu_p} \, T_{a^{j_p}(k+\ell)} \, \psi = \\ = e^{-2\pi i \nu_p \cdot a^{j_p} \ell} \, \psi^q_{p,k+\ell}.$$

Similarly, if $j \geq 0$, by changing again the order of the operators we have:

$$T_\ell \, \psi^q_{p,k} = T_\ell \, D_a^{j_p} \, M_{\nu_p} \, T_k \, \psi = e^{-2\pi i \nu_p \cdot a^{j_p} \ell} \, D_a^{j_p} \, M_{\nu_p} \, T_{k + a^{j_p} \ell} \, \psi = e^{-2\pi i \nu_p \cdot a^{j_p} \ell} \, \psi^q_{p,k+a^{j_p} \ell},$$

and $k + a^{j_p} \ell \in \mathbb{Z}$, since $a \in \mathbb{Z}$ and $j_p \geq 0$. Observe that the scalars $e^{-2\pi i \nu_p \cdot a^{j_p} \ell}$ play no role in the wave packets frame expansions.

Another basic property of the oversampled wave packet systems that we have just described is that they preserve the tight frame property of the corresponding wave packet system. More precisely, we have the following result.

THEOREM 3.11. *Let $\psi \in L^2(\mathbb{R})$ and assume that the wave packet system $\mathcal{W}_{\mathcal{P}}(\psi)$, given by (3.23), where $a \in \mathbb{Z} \setminus \{0\}$, is a Parseval frame for $L^2(\mathbb{R})$. Also assume that, in (3.23), for each $j \in \mathbb{Z}$, there is only a finite number K_j of indices $p \in \mathcal{P}$ for which $j_p = j$. We then have the following:*

(a) *If $\gamma \in \mathbb{Z} \setminus a\mathbb{Z}$, then the corresponding oversampled wave packet system $\mathcal{W}^\gamma_{\mathcal{P}}(\psi)$, given by (3.30), is also a Parseval frame for $L^2(\mathbb{R})$.*
(b) *The corresponding quasi affine wave packet system $\mathcal{W}^q_{\mathcal{P}}(\psi)$, given by (3.32), is also a Parseval frame for $L^2(\mathbb{R})$.*

This theorem extends similar results for the affine and Gabor systems (cf. [**HLWW**] and the references in there). In particular, if $\mathcal{P} = \mathbb{Z}$ and $\nu_p = 0$ for each $p \in \mathcal{P}$, then, as we observed before, $\mathcal{W}_{\mathcal{P}}(\psi) = \mathcal{A}_a(\psi)$ is the affine system, $\mathcal{W}^\gamma_{\mathcal{P}}(\psi) = \mathcal{A}^\gamma_a(\psi)$ is the corresponding oversampled affine system, and, in this case, Theorem 3.11 gives the Second Oversampling Theorem originally discovered by Chui and Shi [**CS**].

Before proving Theorem 3.11, we will make a few observations about the proof, which uses the same main idea in both cases (a) and (b) of the theorem. In our proof, we will apply Theorem 3.2 to the systems $\mathcal{W}^\gamma_{\mathcal{P}}(\psi)$ and $\mathcal{W}^q_{\mathcal{P}}(\psi)$ and obtain the corresponding characterization equations, as we did for the system $\mathcal{W}_{\mathcal{P}}(\psi)$. Then we will observe that every characterization equation of the systems $\mathcal{W}^\gamma_{\mathcal{P}}(\psi)$ and $\mathcal{W}^q_{\mathcal{P}}(\psi)$ is also a characterization equation of the corresponding wave packet system $\mathcal{W}_{\mathcal{P}}(\psi)$. The proof of Theorem 3.11 uses similar arguments to those of Theorem 3.7. Here, in order to avoid undue repetitions, we will only sketch the main steps of the proof.

PROOF OF THEOREM 3.11.

If the wave packet system $\mathcal{W}_{\mathcal{P}}(\psi)$ is a Parseval frame for $L^2(\mathbb{R})$, then, by Theorem 3.7, equations (3.24) are satisfied and, as a consequence, the inequality (3.26) is also satisfied. Furthermore, since $a \in \mathbb{Z}$, for each $\alpha_0 \in \Lambda = \bigcup_{p\in\mathcal{P}} a^{j_p}\,\mathbb{Z}$, there are unique $j_0 \in \mathbb{Z}$ and $q_0 \in \mathbb{Z} \setminus a\,\mathbb{Z}$ such that $\alpha_0 = a^{j_0}\, q_0$. Thus,

$$\mathcal{P}_{\alpha_0} = \{p \in \mathcal{P} : a^{-j_p}\,\alpha_0 \in \mathbb{Z}\} = \{p \in \mathcal{P} : a^{-j_p+j_0}\, q_0 \in \mathbb{Z}\} = \{p \in \mathcal{P} : -j_p + j_0 \geq 0\},$$

and, using the change of variable $\eta = a^{-j_0}\xi$, from (3.24) we obtain

$$\sum_{p\in\mathcal{P}_{\alpha_0}} \hat{\psi}(a^{-j_p+j_0}\eta - \nu_p)\, \overline{\hat{\psi}(a^{-j_p+j_0}(\eta + q_0) - \nu_p)} = \delta_{q_0,0} \quad \text{for a.e. } \eta \in \mathbb{R}. \tag{3.33}$$

Case (a). We apply Theorem 3.5 to the system $\mathcal{W}^{\gamma}_{\mathcal{P}}(\psi)$. This gives us that, if the L.I.C.

$$L^{\gamma}(f) = \sum_{p\in\mathcal{P}} \sum_{m\in\mathbb{Z}} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + \gamma\, a^{j_p} m)|^2\, |\hat{\psi}(a^{-j_p}\xi - \nu_p)|^2\, d\xi < \infty \tag{3.34}$$

is satisfied for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10) with $E = \{0\}$, then the system $\mathcal{W}^{\gamma}_{\mathcal{P}}(\psi)$ is a Parseval frame for $L^2(\mathbb{R})$ if and only if

$$\sum_{p\in\mathcal{P}^{\gamma}_{\alpha}} \hat{\psi}(a^{-j_p}\xi - \nu_p)\, \overline{\hat{\psi}(a^{-j_p}(\xi + \alpha) - \nu_p)} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{3.35}$$

for each $\alpha \in \Lambda^{\gamma} = \bigcup_{p\in\mathcal{P}} \gamma\, a^{j_p}\,\mathbb{Z}$, where $\mathcal{P}^{\gamma}_{\alpha} = \{p \in \mathcal{P} : \alpha \in \gamma\, a^{j_p}\,\mathbb{Z}\}$. Observe that, since $\gamma \in \mathbb{Z}$, (3.26) implies (3.34). Also observe that, since $a \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}\setminus a\,\mathbb{Z}$, for each $\alpha_1 \in \Lambda^{\gamma}$, there are are unique $j_1 \in \mathbb{Z}$ and $q_1 \in \mathbb{Z} \setminus a\,\mathbb{Z}$ such that $\alpha_1 = a^{j_1}\,\gamma\, q_1$. Thus, we have that

$$\begin{aligned} \mathcal{P}^{\gamma}_{\alpha_1} &= \{p \in \mathcal{P} : \gamma^{-1}\, a^{-j_p}\,\alpha_1 \in \mathbb{Z}\} = \{p \in \mathcal{P} : a^{-j_p+j_1}\, q_1 \in \mathbb{Z}\} = \\ &= \{p \in \mathcal{P} : -j_p + j_1 \geq 0\}, \end{aligned}$$

and, using the change of variable $\eta = a^{-j_1}\xi$, from (3.35) we obtain

$$\sum_{p\in\mathcal{P}^{\gamma}_{\alpha_1}} \hat{\psi}(a^{-j_p+j_1}\eta - \nu_p)\, \overline{\hat{\psi}(a^{-j_p+j_1}(\eta + \gamma\, q_1) - \nu_p)} = \delta_{q_1,0} \quad \text{for a.e. } \eta \in \mathbb{R}. \tag{3.36}$$

Since $\gamma \in \mathbb{Z}\setminus a\,\mathbb{Z}$, then also $\gamma\, q_1 \in \mathbb{Z}\setminus a\,\mathbb{Z}$, and, consequently, (3.36) holds whenever (3.33) holds. Thus, $\mathcal{W}^{\gamma}_{\mathcal{P}}(\psi)$ is a Parseval frame for $L^2(\mathbb{R})$.

Case (b). Under the assumptions that, for each $p \in \mathcal{P}$,

$$C_p = \begin{cases} 1, & j_p < 0 \\ a^{-j_p} & j_p \geq 0 \end{cases} \qquad g_p = \begin{cases} e^{2\pi i \nu_p \cdot a^{j_p} k}\, a^{j_p/2}\, D_a^{j_p}\, M_{\nu_p}\,\psi, & j_p < 0 \\ e^{2\pi i \nu_p \cdot k}\, D_a^{j_p}\, M_{\nu_p}\,\psi, & j_p \geq 0, \end{cases}$$

the family $\{T_{C_p k}\, g_p : p \in \mathcal{P}\}$ is exactly the system $\mathcal{W}^{q}_{\mathcal{P}}(\psi)$, and so we can apply again Theorem 3.5. We obtain that, if the L.I.C.

$$\begin{aligned} L^{q}(f) = &\sum_{\{p\in\mathcal{P}: j_p<0\}} \sum_{m\in\mathbb{Z}} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + m)|^2\, |\hat{\psi}(a^{-j_p}\xi - \nu_p)|^2\, d\xi + \\ &+ \sum_{\{p\in\mathcal{P}: j_p\geq 0\}} \sum_{m\in\mathbb{Z}} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + a^{j_p} m)|^2\, |\hat{\psi}(a^{-j_p}\xi - \nu_p)|^2\, d\xi < \infty \end{aligned} \tag{3.37}$$

is satisfied for all $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (3.10) with $E = \{0\}$, then the system $\mathcal{W}^q_{\mathcal{P}}(\psi)$ is a Parseval frame for $L^2(\mathbb{R})$ if and only if

$$\sum_{p \in \mathcal{P}^q_m} \hat{\psi}(a^{-j_p}\xi - \nu_p)\,\overline{\hat{\psi}(a^{-j_p}(\xi + m) - \nu_p)} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{3.38}$$

for each $m \in \Lambda^q = \mathbb{Z}$, where $\mathcal{P}^q_m = \{p \in \mathcal{P} : m \in a^{j_p}\,\mathbb{Z}\}$. Observe that, as in part (a), (3.26) implies (3.37). Also observe that, for each $m_1 \in \mathbb{Z}$, there are are unique $j_1 \in \mathbb{Z}$, $j_1 \geq 0$, and $q_1 \in \mathbb{Z} \setminus a\,\mathbb{Z}$ such that $m_1 = a^{j_1}\,q_1$. Thus,

$$\mathcal{P}^q_{m_1} = \{p \in \mathcal{P} : a^{-j_p + j_1}\,q_1 \in \mathbb{Z}\} = \{p \in \mathcal{P} : -j_p + j_1 \geq 0\},$$

and, using the change of variable $\eta = a^{-j_1}\xi$, from (3.38) we obtain

$$\sum_{p \in \mathcal{P}^q_{m_1}} \hat{\psi}(a^{-j_p + j_1}\eta - \nu_p)\,\overline{\hat{\psi}(a^{-j_p + j_1}(\eta + q_1) - \nu_p)} = \delta_{q_1,0} \quad \text{for a.e. } \eta \in \mathbb{R}. \tag{3.39}$$

It is now clear that (3.39) holds whenever (3.33) holds (observe that $j_1 \geq 0$ while, in (3.33), $j_0 \in \mathbb{Z}$ and, thus, the converse implication does not hold). Thus, $\mathcal{W}^q_{\mathcal{P}}(\psi)$ is a Parseval frame for $L^2(\mathbb{R})$. $\square$

Using Theorem 3.11 we can show that the continuous wave packets $\mathcal{W}_S(\Psi)$, given by (2.1), are, in a certain sense, the "ultimate oversampling" over the discrete wave packets $\mathcal{W}_{\mathcal{P}}(\Psi)$, given by (3.1) (a similar observation for the affine systems can be found in [**WW**]) . This also sheds some light into the discretization process that leads from the continuous to the discrete systems. For simplicity, we consider the one-dimensional case only.

Let $\psi \in L^2(\mathbb{R})$ and consider the (discrete) wave packet system $\mathcal{W}_{\mathcal{P}}(\psi)$, given by (3.23), with $a \in \mathbb{Z} \setminus \{0\}$. Assume that $\mathcal{W}_{\mathcal{P}}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R})$. Now consider the corresponding oversampled wave packet systems $\mathcal{W}^N_{\mathcal{P}}(\psi)$, given by(3.30), where the $N \in \mathbb{Z} \setminus a\mathbb{Z}$. By Theorem 3.11, the systems $\mathcal{W}^N_{\mathcal{P}}(\psi)$ are also a Parseval frame for $L^2(\mathbb{R})$. Thus, for each $f \in L^2(\mathbb{R})$, we have

$$f = \frac{1}{N} \sum_{p \in \mathcal{P}} \sum_{k \in \mathbb{Z}} \langle f, D_a^{j_p}\,M_{\nu_p}\,T_{N^{-1}k}\,\psi\rangle\,D_a^{j_p}\,M_{\nu_p}\,T_{N^{-1}k}\,\psi,$$

with convergence in $L^2(\mathbb{R})$, for each $N \in \mathbb{Z} \setminus a\,\mathbb{Z}$. We can interpret this sum as a Riemann sum. Thus, choosing a sequence of $N \in \mathbb{Z} \setminus a\,\mathbb{Z}$, and taking the limit for N tending to ∞, we obtain that, for each $f \in L^2(\mathbb{R})$

$$f = \sum_{p \in \mathcal{P}} \int_{\mathbb{R}} \langle f, D_a^{j_p}\,M_{\nu_p}\,T_y\,\psi\rangle\,D_a^{j_p}\,M_{\nu_p}\,T_y\,\psi\,dy,$$

with convergence in $L^2(\mathbb{R})$. This shows that the system $\mathcal{W}_S(\Psi)$, given by (2.1) where $S = \{(a^{j_p}, \nu_p) : p \in \mathcal{P}\}$, is a continuous Parseval frame wave packet system relative to S for $L^2(\mathbb{R})$.

3.3. A very general example. In [**HLWW**], we gave an example of a Parseval frame wave packet system associated with a general disjoint covering of the real line by intervals. In this section, we extend this example to dimension $n = 2$, where the triangles assume the role played by the intervals. This construction presents a very general example of wave packet systems: in fact, the dilations do not have to be powers of a matrix and the modulations do not have to be associated with a lattice.

Consider a triangularization of $\mathbb{R}^2$. More precisely, let τ_1 be the triangle of vertices $(0,0)$, $(1,0)$, $(0,1)$, and consider a tiling of $\mathbb{R}^2$ given by the union of countably many non-degenerate triangles $\{\tau_j\}_{j\in\mathbb{N}}$:

$$\mathbb{R}^2 = \bigcup_{j\in\mathbb{N}} \tau_j, \tag{3.40}$$

where the triangles τ_j have disjoint interiors. For each triangle τ_j, $j \in \mathbb{N}$, with vertices $u_j = (u_j^{(1)}, u_j^{(2)})$, $v_j = (v_j^{(1)}, v_j^{(2)})$ and $w_j = (w_j^{(1)}, w_j^{(2)})$, let the matrix a_j be determined by

$$a_j^T = \begin{pmatrix} v_j^{(1)} - u_j^{(1)} & w_j^{(1)} - u_j^{(1)} \\ v_j^{(2)} - u_j^{(2)} & w_j^{(2)} - u_j^{(2)}. \end{pmatrix}$$

It is clear that, for each $j \in \mathbb{N}$, $\tau_j = a_j^T\,\tau_1 + u_j$, and this shows that, by choosing an appropriate ordering of the vertices, to each triangle τ_j there is a uniquely associated dilation $a_j \in GL_2(\mathbb{R})$ and translation $u_j \in \mathbb{R}^2$ mapping τ_1 into τ_j.

Consider the wave packet systems

$$\mathcal{W}(\psi) = \big\{D_{a_j}\, M_{\nu_j}\, T_k\, \psi :\, k \in \mathbb{Z}^2,\, j \in \mathbb{N}\big\}, \tag{3.41}$$

with $\nu_j = (a_j^T)^{-1} u_j = a_j^I\, u_j$ and $\hat{\psi}(\xi) = \chi_{\tau_1}(\xi)$, $\xi \in \mathbb{R}^2$. We will now apply Theorem 3.5 to show that $\mathcal{W}(\psi)$ is a Parseval frame for $L^2(\mathbb{R}^2)$.

Since $|\hat{\psi}(a_j^I \xi - \nu_j)| = |\hat{\psi}(a_j^I(\xi - u_j))| = \chi_{\tau_j}(\xi)$, the left hand side of the L.I.C., given by (3.21), takes the form:

$$\begin{aligned} L(f) &= \sum_{j\in\mathbb{N}} \sum_{m\in\mathbb{Z}} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + a_j^T m)|^2\, |\hat{\psi}((a_j^T)^{-1}\xi - \nu_j))|^2\, d\xi \\ &= \sum_{j\in\mathbb{N}} \sum_{m\in\mathbb{Z}} \int_{\operatorname{supp} \hat{f} \cap \tau_j} |\hat{f}(\xi + a_j^T m)|^2\, d\xi. \end{aligned} \tag{3.42}$$

We need to show that $L(f) < \infty$ for all $f \in \mathcal{D}$, where $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^2)$. Take $\mathcal{D} = \{f \in L^2(\mathbb{R}^2) : \|\hat{f}\|_\infty < \infty, \operatorname{supp} \hat{f}$ compact and $\operatorname{supp} \hat{f} \subseteq$ finite union of interiors of $\tau_j\}$. This clearly implies that the intersection $\operatorname{supp} \hat{f} \bigcap \tau_j$ is nonzero only for finitely many j. Furthermore, since $a_j \in GL_2(\mathbb{R})$, for each $j \geq 1$ the norm $\|a_j^T\, m\|$ tends to infinity as $\|m\|$ goes to infinity and, as a consequence, there are only finitely many $m \in \mathbb{Z}$ such that the integral in (3.42) is nonzero. Combining these two observations, it follows that $L(f) < \infty$ for all $f \in \mathcal{D}$.

By Theorem 3.5, in order to show that the wave packet system $\mathcal{W}(\psi)$ is a Parseval frame it only remains to show that ψ satisfies the characterizing equations (3.22) which, in this case, are the two equations:

$$\sum_{j\in\mathbb{N}} |\hat{\psi}(a_j^I \xi - \nu_j)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^2, \tag{3.43}$$

$$\sum_{j\in\mathcal{P}_\alpha} \hat{\psi}(a_j^I \xi - \nu_j)\, \overline{\hat{\psi}(a_j^I(\xi + \alpha) - \nu_j)} = 0, \quad \text{for a.e. } \xi \in \mathbb{R}^2, \text{ if } \alpha \neq 0, \tag{3.44}$$

where $\mathcal{P}_\alpha = \{j \in \mathcal{P} :\, a_j^I\, \alpha \in \mathbb{Z}^2\}$ and $\alpha \in \bigcup_{j\geq 1} a_j^T\, \mathbb{Z}^2$. As we observed before, we have that $\hat{\psi}(a_j^I \xi - \nu_j) = \chi_{\tau_j}(\xi)$, and so (3.40) implies equation (3.43). Next, consider equation (3.44) with $\alpha \neq 0$. Fix $j \in \mathcal{P}_\alpha$ and let $\eta = a_j^I \xi - \nu_j$. Under these assumptions, we have

$$\hat{\psi}(a_j^I \xi - \nu_j)\, \overline{\hat{\psi}(a_j^I(\xi + \alpha) - \nu_j)} = \chi_{\tau_1}(\eta)\, \chi_{\tau_1}(\eta + (a_j^T)^{-1}\alpha). \tag{3.45}$$

Since $j \in \mathcal{P}_\alpha$, $\alpha \neq 0$, then $a_j^I\, \alpha \in \mathbb{Z}^2 \setminus \{0\}$ and, because the interiors of the triangles are pairwise disjoint, the expression (3.45) vanishes almost everywhere. This implies that equation (3.44) is also satisfied and, thus, our system is a Parseval frame for $L^2(\mathbb{R}^2)$.

Observe that we can also show directly, without using Theorem 3.5, that the system $\mathcal{W}(\psi)$ is a Parseval frame for $L^2(\mathbb{R}^2)$. In order to do this, observe first that the collection $\{M_k\, \chi_{\tau_1} : k \in \mathbb{Z}^2\}$ is a Parseval frame for $L^2(\tau_1)$. As we observed before, we have the following tiling of $\mathbb{R}^2$:

$$\bigcup_{j \in \mathbb{N}} D_{a_j^I}\, T_{\nu_j}\, \tau_1 = \mathbb{R}^2,$$

where the union is disjoint. This implies that the collection

$$\{D_{a_j^I}\, T_{\nu_j}\, M_{-k}\, \chi_{\tau_1} : k \in \mathbb{Z}^2, j \in \mathbb{N}\} = \{\big(D_{a_j}\, M_{\nu_j}\, T_k\, (\chi_{\tau_1})^\vee\big)^\wedge : k \in \mathbb{Z}^2, j \in \mathbb{N}\}$$

is a Parseval frame for $L^2(\mathbb{R}^2)$.

Also observe that the choice of the triangle τ_1 plays no special role in this example; in fact, the construction can easily be modified by choosing a different initial triangle. Finally, observe that this construction generalizes to higher dimensions. Indeed, if the triangles are replaced by n-dimensional simplices, then all the computations carry over in a similar way.

3.4. Discrete wave packet frames. By applying Theorem 3.4 to the wave packet systems $\mathcal{W}_{\mathcal{P}}(\Psi)$, given in (3.23), one obtains results very much in the spirit of those found in [**D**, Ch. 3] for Gabor and affine systems, or in [**HL**], where one can find examples of wave packet systems that are a frame for $L^2(\mathbb{R})$. In particular, we are able to recover the results in [**D**] and [**HL**]. For the sake of brevity, we will not examine this situation further in this paper.

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Department of Mathematics, Washington University, Saint Louis, Missouri 63130
E-mail address: dlabate@math.wustl.edu

Department of Mathematics, Washington University, Saint Louis, Missouri 63130
E-mail address: guido@math.wustl.edu

Department of Mathematics, Washington University, Saint Louis, Missouri 63130
E-mail address: enwilson@math.wustl.edu