Taxicab Geometry
another non-Euclidean geometry

Axiomatic Structure:

undefined terms: point, line, plane

axioms:

A1. Given any two distinct points there is exactly one line that contains them.

A2. The Distance Postulate:

To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

A3. The Ruler Postulate:

The points of a line can be placed in a correspondence with the real numbers such that

A. To every point of the line there corresponds exactly one real number.

B. To every real number there corresponds exactly one point of the line,

and

C. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

A4. The Ruler Placement Postulate:

Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.
A5. A. Every plane contains at least three non-collinear points.

B. Space contains at least four non-coplanar points.

A6. If two points line in a plane, then the line containing these points lies in the same plane.

A7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

A8. If two planes intersect, then that intersection is a line.

A9. The Plane Separation Postulate:

Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

A. each of the sets is convex, and

B. if P is in one set and Q is in the other, then segment PQ intersects the line.

A10. The Space Separation Postulate:
The points of space that do not line in a given plane form two sets such that

A. each of the sets is convex, and

B. if P is in one set and Q is in the other, then the segment PQ intersects
   the plane.

A11. The Angle Measurement Postulate:

To every angle there corresponds a real number between $0^\circ$ and $180^\circ$.

A12. The Angle Construction Postulate:

Let AB be a ray on the edge of the half-plane H. For every r between 0 and 180 there is exactly
one ray AP with P in H such that $\angle PAB = r$.

A13. The Angle Addition Postulate:

If D is a point in the interior of $\angle BAC$, then

$m \angle BAC = m \angle BAD + m \angle DAC$.

A14. The Supplement Postulate:

If two angles form a linear pair, then they are supplementary
next in Euclidean Geometry would be SAS, Axiom 15, but we’ll discover that we don’t have this axiom in Taxicab Geometry...it’s NOT true here.
Taxicab distance, TCG coordinates, and angle measurement:

A2. The Distance Postulate:

To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

We will be working in the Cartesian Plane so our points will come with algebraic coordinates (a, b).

The distance between two points $P_1$ and $P_2$ with coordinates similarly subscripted is:

$$\text{TCGD} = |x_1 - x_2| + |y_1 - y_2|$$

*TaxiCab Geometry distance. NOTE this is a different formula from Euclidean Distance!

Does order of subtraction really matter? Could I have written: $|x_2 - x_1| + |y_2 - y_1|$ YES!

Example:

What is the Euclidean distance from $(3, 2)$ to $(-1, 9)$?

The formula: $\sqrt{(3 - (-1))^2 + (2 - 9)^2} = \sqrt{65}$ which rounds to 8.1

What is the TCGD between the same points?

$|3 - (-1)| + |2 - 9| = 11$

These are very different distances between the same two points.
**TCG Distance Exercise:**

Let's begin looking at some distances. Put the points on the graph before you calculate the distances:

<table>
<thead>
<tr>
<th>Points</th>
<th>Euclidean Distance</th>
<th>Taxicab Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0) to (4, 0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0) to (1, 3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0) to (2, 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0) to (0, 4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3, 3) to (6, 7)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3, 3) to (6, 3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3, 3) to (3, 6)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 0) to (8, 0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 0) to (7, 8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5, 0) to (5, 4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Do you see an easy physical description of how to calculate the TCG distance?
Let's graph these lines on a pair of axes:

Do you have a conjecture about when the EG distance and the TCG distance is the same?
A3. The Ruler Postulate:

The points of a line can be placed in a correspondence with the real numbers such that

A. To every point of the line there corresponds exactly one real number.

B. To every real number there corresponds exactly one point of the line, and

C. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

Taxicab Geometry Coordinate Exercise:

We will use the line \( y = x \). Sketch it in here and put on points with \( x \) values: \(-8, -3, 0, 2, 5\).

Fill in the following table that shows the Cartesian coordinates, Euclidean Geometric coordinate, and the TCG coordinates for the distances between the following points:

Recall: the Euclidean geometric coordinate formula is \( x\sqrt{m^2 + 1} \)
Our goal with this exercise is to find the formula for the TCG coordinate. One hint: it’s the x value times something... let’s find “something”

<table>
<thead>
<tr>
<th>point in Cartesian coord</th>
<th>EG coord</th>
<th>TCG coord</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>$2\sqrt{2}$</td>
<td>4</td>
</tr>
<tr>
<td>(5, 5)</td>
<td>$5\sqrt{2}$</td>
<td>10</td>
</tr>
<tr>
<td>(−3, −3)</td>
<td>$−3\sqrt{2}$</td>
<td>−6</td>
</tr>
<tr>
<td>(−8, −8)</td>
<td>$−8\sqrt{2}$</td>
<td>−16</td>
</tr>
</tbody>
</table>

What’s your conjecture about the coordinate formula?

Try the same exercise with the line $y = −2x + 1$ Sketch in the line: show the y-intercept and the points with x values: −4, −2, 0, 1, 3, and 4.
<table>
<thead>
<tr>
<th></th>
<th>EG coord -- on your own</th>
<th>TCG coord</th>
</tr>
</thead>
<tbody>
<tr>
<td>y intercept</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>(0, 1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4, −7)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>(−2, 5)</td>
<td></td>
<td>−6</td>
</tr>
<tr>
<td>(1, −1)</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>(−4, 9)</td>
<td></td>
<td>−12</td>
</tr>
<tr>
<td>(3, −5)</td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

What’s your conjecture? Go to Activities and write it in there!
A11. The Angle Measurement Postulate:

To every angle there corresponds a real number between $0^\circ$ and $180^\circ$.

Angles will be measured with protractors just as though you're measuring a Euclidean angle.

Go to ACTIVITIES Taxicab 3 - 4 ... let's talk a bit about this

Does it seem like trigonometry is safe and sound and will work with this system?

    Hint: measure the side lengths in TCGD and talk about the sine being the length of the side
    opposite divided by the length of the hypotenuse.

Did I lead you astray talking about the triangle in Euclidean terms - I bet so. We have to be careful of
the context here. I'll be marking every question on the final so you know WHICH geometry the question
is about.

So what's the same and what's different with TCG and EG:
What about an isosceles right triangle with the base parallel to the x axis?

Do our trig values hold there?

Use the points (7, 7), (2, 3) and (12, 3). What are the 3 angles measures?

What are the Euclidean side lengths?

What are the TCGD side lengths?

Does traditional Trigonometry work? Can you use inverse trig functions freely in Taxicab Geometry.
In actuality there's a TCG trigonometry that we'll look at as soon as we get some geometric shapes established. It's mostly a diversion and not really useful, but it's fun to contemplate and it won't be on the final, promise.

**Geometric Shapes:**

**Lines:** look like lines, no problem there. However, the interaction of points on a line with other points and with other lines is somewhat changed in TCG.

**Betweenness:**

We have a concept in EG called "between" — B is between A and C iff $AB + BC = AC$.

In Euclidean Geometry, if you have points A and C on a line, the points "between" them are also on the line. They are collinear with the segment endpoints.

Now, since we're using a different formula for distance, you should not be startled if different points add up properly to be "between" in a sense particular to TCG.

**Betweenness exercise:** Let's look at TCG betweenness
Point A is (1, 7) and point C is (11, 1). How far is point A from point C?

Well, now, I have to clarify this question --

Which is to say, what is AC in Taxicab Geometry? in Euclidean Geometry?
Mark the midpoint of the line...(6, 4)

How far is the midpoint from each endpoint in TCG?

So the midpoint of the line segment is between A and C. $8 + 8 = 16$. It “adds up” correctly.

I’ll get you started: using point (11, 7) as C

$AC = 10$ and $CB = 6$ so $AC + CB = 16$ for this C. And (11, 7) is between A and C in the TCG sense of between. So is: $(6, 6)$…do you see how?

Find at least 5 additional points that add up correctly according to the formula. The third point will be called C and $AC + CB = 16$. (hint: work on both sides of the line segment…)
Extend this to cover ALL the points that are “between” A and C in a TCG sense.

**Triangles:**

Since we measure angles the same way, the sum of the interior angles is still $180^\circ$.

But let’s check on a few other items:

**Equilateral triangles exercise:**

Sketch an EG equilateral triangle with side measure 4 and the left vertex at the origin. What are the coordinates of the apex angle?

Is it TCG equilateral? nope... the sides are not length 4, they are an irrational length.

Sketch a TCG equilateral triangle side length 4 with the base starting on the left at $(5, 0)$ on top of the x axis... do the angles measure $60^\circ$? What do the angles measure?

What are the coordinates of the apex angle?
Then sketch the one with vertices: (1, 7) (4, 8) and (2, 10). It's TCG equilateral BUT.... EG isosceles.

What can you conclude about TCG equilateral triangles? How are they like the familiar Euclidean ones? How are they different?

Now let's sketch in a triangle with a horizontal length of 4 and a vertical length of 4 and one vertex over in the upper right of the graph paper...what is the hypotenuse length?

In TCG: 8 in EG: $4\sqrt{2}$

Is this triangle congruent to the TCG equilateral triangle of side length 4 that is to the right on the x axis? Why or why not?
Can you say the triangles have the quality SAS?
Is SAS an axiom in this geometry?
Do you see why this geometry is non-euclidean?

We'll have to use the old fashioned definition of congruence – 3 corresponding congruent sides and 3 corresponding congruent angles for congruence in Taxicab Geometry.

SAS will NOT do as a shortcut to declaring congruence at all. Note that it is NOT on the axiom list – I stopped short of this axiom. Taxicab Geometry and Euclidean geometry have only the axioms up to SAS in common.
Circles:

A circle is the set of all points that are equidistant from a given point called the center of the circle.

Happily, we do have circles in TCG. Let's figure out what they look like!

Sketch the TCG circle centered at the origin with radius 4:

(Check back on page 3 for several points that measure 4 from the origin before you get started!)
How many times have you told someone that you'll pick them up downtown right in front of their building and that you'll "circle" the block til you see them? Does this language reflect Euclidean Geometry or Taxicab Geometry?