

Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows

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Abstract

A coupled cell system is a network of dynamical systems, or ‘cells’, coupled together. The architecture of a coupled cell network is a graph that indicates how cells are coupled and which cells are equivalent. Stewart, Golubitsky, and Pivato presented a framework for coupled cell systems that permits a classification of robust synchrony in terms of network architecture. They also studied the existence of other robust dynamical patterns using a concept of quotient network. There are two difficulties with their approach. First, there are examples of networks with robust patterns of synchrony that are not included in their class of networks; and second, vector fields on the quotient do not in general lift to vector fields on the original network, thus complicating genericity arguments. We enlarge the class of coupled systems under consideration by allowing two cells to be coupled in more than one way, and we show that this approach resolves both difficulties. The theory that we develop, the ‘multiarrow formalism’, parallels that of Stewart *et al.*

1 Introduction

Stewart, Golubitsky, and Pivato [9] formalize the definition of a coupled cell system in terms of the symmetry groupoid of an associated coupled cell network, and prove three general theorems about such networks. First, a set of cells can be robustly synchronous if and only if they are in the same equivalence class of some balanced equivalence relation. Second, every balanced relation leads to a new coupled cell network, called a quotient network, that is formed by identifying equivalent cells. Third, the restriction of a coupled cell system to a synchrony subspace (or polydiagonal) is a coupled cell system associated to the quotient network. The approach in [9] has two difficulties:

- 1) Not every coupled cell system of ODEs corresponding to the quotient network is the restriction of a coupled cell system corresponding to the original network. This fact makes it difficult to prove genericity statements about dynamics in the original network based only on genericity statements about dynamics of the quotient network. (Dias and Stewart [3] obtain necessary and sufficient conditions, on a network with a balanced relation, for every quotient system to be a restriction of a cell system corresponding to the original network.)
- 2) Reasonable networks that are not included in the theory developed in [9] can exhibit patterns of robust synchrony. Examples are linear chains with Neumann boundary conditions considered in Epstein and Golubitsky [4] and square arrays of cells with Neumann boundary conditions considered in Gillis and Golubitsky [5].

In this paper we show that both of these difficulties can be resolved if the class of coupled cell networks is enlarged to permit multiple couplings between cells, and self-coupling. We call this the *multiarrow formalism* for coupled cell networks. Although the abstract definition of this enlarged class of coupled cell networks is more complicated than the more restrictive definition in [9], the multiarrow formalism has the side benefit that quotient systems are more easily defined in the enlarged class, and have more convenient properties.

We first motivate the generalization by considering two examples in the important case of a homogeneous network, which we now define. A *cell* is a system of ODEs, and a *coupled cell system* is a collection of N cells with couplings. As discussed in [9] a class of coupled cell systems is defined by a *coupled cell network*, which is a (directed, labeled) graph that specifies, among other information, which cells are coupled to which. Two cells of the network are *input isomorphic* (see [9]) if the dynamics of the cells are specified by the same differential equations, up to a permutation of the variables. More precisely, if cells 1 and 2 with internal state variables $x_1, x_2 \in \mathbf{R}^k$ are input isomorphic, then the relevant components of the system of ODEs take the form

$$\begin{aligned}\dot{x}_1 &= f(x_1, y_1, \dots, y_l) \\ \dot{x}_2 &= f(x_2, z_1, \dots, z_l)\end{aligned}\tag{1.1}$$

where the y_j (resp. z_j) are internal state variables of the cells connected to cell 1 (resp. cell 2). In particular, the two cells receive inputs from the same number l of cells, the input variables are of the same type $y_j, z_j \in \mathbf{R}^{k_j}$, and the dependence of the corresponding components of \dot{x} is specified using the same function f of the relevant internal variables and input variables. The phase space of the coupled cell system is

$$P = \{x = (x_1, \dots, x_n) \in \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_N}\}$$

We call a coupled cell network *homogeneous* if all cells are input isomorphic (in which case $k_1 = \dots = k_N$). Homogeneous coupled cell systems are determined by a single function f , as illustrated in (1.1). For the remainder of this introduction we focus on homogeneous coupled cell networks.

We can visualize an equivalence relation \bowtie on cells by coloring all equivalent cells with the same color. This equivalence relation is *balanced* (in the homogeneous case) if the sets

of colors of input cells for two equivalent cells consist of the same colors with the same multiplicities. Theorem 6.5 of [9] states that the subspace

$$\Delta_{\bowtie} = \{x \in P : x_i = x_j \text{ if } i \bowtie j\}$$

is flow-invariant for all f if and only if \bowtie is balanced. A solution in Δ_{\bowtie} is *synchronous* in the strong sense that the time series from cells of the same color are identical; the synchrony is *robust* in the sense that it holds for any choice of f . We call Δ_{\bowtie} the *polydiagonal* or *synchrony subspace* corresponding to \bowtie .

Quotients Lead to Multiple Arrows

We describe circumstances in which multiple arrows are natural and useful. Consider the homogeneous five-cell coupled cell network pictured in Figure 1 (left). A balanced coloring of this network is given in the right panel of that figure.

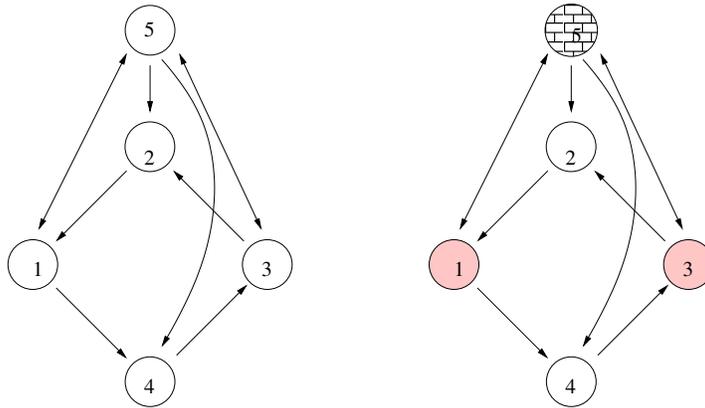


Figure 1: (Left) Homogeneous five-cell network. (Right) Balanced coloring of the network.

The differential equations corresponding to this five-cell network have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_5) \\ \dot{x}_2 &= f(x_2, x_3, x_5) \\ \dot{x}_3 &= f(x_3, x_4, x_5) \\ \dot{x}_4 &= f(x_4, x_1, x_5) \\ \dot{x}_5 &= f(x_5, x_1, x_3) \end{aligned}$$

where $f(a, b, c) = f(a, c, b)$ since all couplings are assumed to be identical. It is straightforward to check that the subspace Δ defined by $x_1 = x_3$ and $x_2 = x_4$ is flow-invariant. The restricted system on Δ has the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_5) \\ \dot{x}_2 &= f(x_2, x_1, x_5) \\ \dot{x}_5 &= f(x_5, x_1, x_1) \end{aligned}$$

The quotient cell construction in [9] leads to the coupled cell network of Figure 2 (left). The coupled cell system corresponding to that quotient network, which is not homogeneous, has the form:

$$\begin{aligned}\dot{w} &= f(w, p, c) \\ \dot{p} &= f(p, w, c) \\ \dot{c} &= g(c, w)\end{aligned}$$

Therefore, a coupled cell system corresponding to the quotient network is the restriction of a coupled cell system corresponding to the five-cell network if and only if $g(b, w) = f(b, w, w)$ where $f(a, b, c) = f(a, c, b)$. In this paper we remove such conditions from consideration by allowing multiple couplings between cells. With multiple couplings, the quotient network is the homogeneous one of Figure 2 (right). Quotient coupled cell systems for the new quotient have the form

$$\begin{aligned}\dot{w} &= f(w, p, c) \\ \dot{p} &= f(p, w, c) \\ \dot{c} &= f(c, w, w)\end{aligned}$$

and each of these systems is the restriction to Δ of a five-cell system. Homogeneous three-cell networks with each cell having at most two input arrows are classified in [8]. There are 35 such networks.

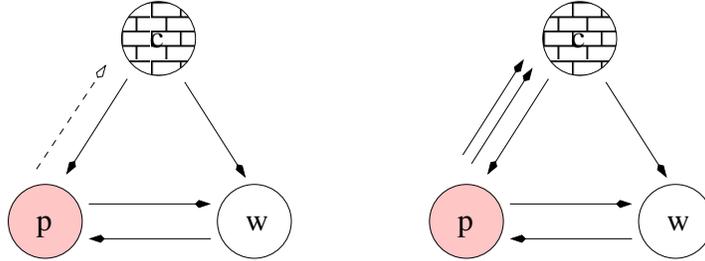


Figure 2: (Left) Five-cell network. (Right) Balanced coloring.

Neumann Boundary Conditions Lead to Self-Coupling

We now provide a reason for permitting self-coupling. Epstein and Golubitsky [4] consider patterns of synchrony in N -cell bidirectional linear arrays with Neumann boundary conditions. The systems of ODEs have the form

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_1, x_2) \\ \dot{x}_j &= f(x_j, x_{j-1}, x_{j+1}) \quad 1 < j < N \\ \dot{x}_N &= f(x_N, x_{N-1}, x_N)\end{aligned}$$

where $f(a, b, c) = f(a, c, b)$. When self-coupling of a cell to itself is allowed, the network architecture is the one pictured in Figure 3.

The ten-cell array in Figure 4 provides an example of a balanced coloring that cannot be obtained from the results in [9], since self-coupling is not permitted in that theory. To see



Figure 3: Linear array network.

that this coloring is balanced observe that each pink cell receives an input from one pink cell and one white cell, each cyan cell receives inputs from two white cells, and each white cell receives inputs from one pink cell and one cyan cell. See also [6].

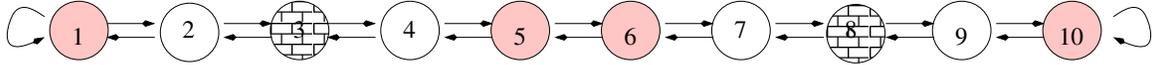


Figure 4: Linear array network of ten cells with a three-color balanced relation.

Structure of the Paper

The remainder of this paper is structured as follows. The enlarged class of ‘multiarrow’ coupled cell networks, which permits multiple arrows and self-coupling, is defined in Section 2. The associated admissible vector fields are constructed in Section 3. In that section we also show distinct networks in the enlarged class can correspond to the same space of admissible vector fields. This (undesired) feature is not present in the class of networks considered in [9]: it is a price we have to pay to obtain the more desirable features. The connection between balanced equivalence relations and robust polysynchrony is discussed in Section 4. Quotient networks are defined in the context of multiple arrows and self-coupling in Section 5. Here we show that all admissible vector fields on a quotient network lift to admissible vector fields on the original network, a property that fails for the quotients defined in [9]. The important special case of identical-edge homogeneous networks (homogeneous networks in which all coupling arrows are equivalent) is considered in Section 6.

In symmetric networks, Hopf bifurcation typically leads to periodic states in which some cells have identical waveforms (hence identical amplitudes) except for a well-defined phase shift. In Section 7 we show that in identical-edge homogeneous networks, Hopf bifurcation can lead to periodic states with well-defined *approximate* phase shifts and *different* amplitudes.

The proofs of several of the main theorems in this paper (particularly Theorems 4.3 and 5.2) are straightforward adaptations of corresponding results in [9] to the enlarged category of networks considered here.

2 Coupled Cell Networks

We begin by formally defining a class of coupled cell network that includes multiple arrows and self-couplings.

Definition 2.1 In the *multiarrow formalism*, a *coupled cell network* \mathcal{G} consists of:

(a) A finite set $\mathcal{C} = \{1, \dots, N\}$ of *nodes* or *cells*.

(b) An equivalence relation $\sim_{\mathcal{C}}$ on cells in \mathcal{C} .

The *type* or *cell label* of cell c is the $\sim_{\mathcal{C}}$ -equivalence class $[c]_{\mathcal{C}}$ of c .

(c) Associated to each node c is a finite set of *input terminals* $I(c)$. Each input terminal $i \in I(c)$ is the receptacle for one *arrow* or *edge* that begins in *tail cell* $\tau(i)$ and ends in terminal i . That arrow is denoted by $(\tau(i), i)$, has *head cell* c , and *head terminal* i . Let \mathcal{E} denote the set of all arrows.

(d) An equivalence relation \sim_E on edges in \mathcal{E} .

The *type* or *coupling label* of edge e is the \sim_E -equivalence class $[e]_E$ of e .

(e) Equivalent edges have equivalent tails and heads. That is, if $(\tau(i), i) \sim_E (\tau(j), j)$ where $i \in I(c)$ and $j \in I(d)$, then $\tau(i) \sim_{\mathcal{C}} \tau(j)$ and $c \sim_{\mathcal{C}} d$.

Observe that self-coupling is permitted (since $\tau(i) = c$ for a terminal i in cell c is permitted) and multiple arrows are permitted (since $\tau(i) = \tau(j)$ for two distinct terminals in cell c is permitted).

Definition 2.2 The relation \sim_I of *input equivalence* on \mathcal{C} is defined by $c \sim_I d$ if and only if there exists an arrow type preserving bijection

$$\beta : I(c) \rightarrow I(d) \tag{2.1}$$

That is, for every input terminal $i \in I(c)$

$$(\tau(i), i) \sim_E (\tau(\beta(i)), \beta(i)) \tag{2.2}$$

Any such bijection β is called an *input isomorphism* from cell c to cell d . The set $B(c, d)$ denotes the collection of all input isomorphisms from cell c to cell d . The set

$$\mathcal{B}_{\mathcal{G}} = \bigcup_{c, d \in \mathcal{C}} B(c, d)$$

is a groupoid (Brandt [1], Brown [2], Higgins [7]), which is an algebraic structure rather like a group, except that the product of two elements is not always defined. We call $\mathcal{B}_{\mathcal{G}}$ the *groupoid of the network*. \diamond

Remark 2.3 (a) Suppose that a cell c has two input terminals $i, j \in I(c)$ whose receiving arrows $(\tau(i), i)$ and $(\tau(j), j)$ are \sim_E equivalent. Then the transposition $\beta = (i \ j)$ is an input isomorphism in $B(c, c)$.

(b) The reason for introducing an explicit set $I(c)$ of input terminals is to provide a well-defined set for the input isomorphism β in (2.1) to act on. Otherwise we must consider ‘sets’ in which elements may occur more than once. This is the main novelty in Definition 2.2 compared to that in [9].

(c) In place of an explicit set of input terminals we can use the associated set of arrows, which is closer to the spirit of [9]. However, this approach becomes very convoluted because an ‘arrow’ can no longer be identified with a pair of cells (i, c) .

Definition 2.4 A *homogeneous* network is a coupled cell network such that $B(c, d) \neq \emptyset$ for every pair of cells c, d .

3 Vector Fields on a Coupled Cell Network

We now define the class $\mathcal{F}_{\mathcal{G}}^P$ of admissible vector fields corresponding to a given coupled cell network \mathcal{G} . This class consists of all vector fields that are ‘compatible’ with the labeled graph structure, or equivalently are ‘symmetric’ under the groupoid $\mathcal{B}_{\mathcal{G}}$. It also depends on a choice of ‘total phase space’ P , which we assume is fixed throughout the subsequent discussion.

For each cell in \mathcal{C} define a *cell phase space* P_c . This must be a smooth manifold of dimension ≥ 1 , which for simplicity we assume is a nonzero finite-dimensional real vector space. We require

$$c \sim_{\mathcal{C}} d \implies P_c = P_d$$

and we employ the same coordinate systems on P_c and P_d . Only these identifications of cell phase spaces are canonical; that is, the relation $c \sim_{\mathcal{C}} d$ implies that cells c and d have the same phase space, but not that they have isomorphic (conjugate) dynamics.

Define the corresponding *total phase space* to be

$$P = \prod_{c \in \mathcal{C}} P_c$$

and employ the coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P .

More generally, suppose that $\mathcal{D} = \{d_1, \dots, d_s\}$ is any finite ordered subset of cells in \mathcal{C} . In particular, the same cell can appear more than once in \mathcal{D} . Define

$$P_{\mathcal{D}} = P_{d_1} \times \dots \times P_{d_s}$$

Further, write

$$x_{\mathcal{D}} = (x_{d_1}, \dots, x_{d_s})$$

where $x_{d_j} \in P_{d_j}$.

Finally, suppose that $\mathcal{D}_1, \mathcal{D}_2$ are subsets of \mathcal{C} , and that there is a bijection $\gamma : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ such that $\gamma(d) \sim_{\mathcal{C}} d$ for all $d \in \mathcal{D}_1$. Define the *pullback map*

$$\gamma^* : P_{\mathcal{D}_2} \rightarrow P_{\mathcal{D}_1}$$

by

$$(\gamma^*(z))_j = z_{\gamma(j)} \tag{3.1}$$

for all $j \in \mathcal{D}_1$ and $z \in P_{\mathcal{D}_2}$

We use pullback maps to relate different components of a vector field associated with a given coupled cell network. Specifically, the class of vector fields that is encoded by a coupled cell network is given in Definition 3.1. For a given cell c the *internal phase space* is P_c and the *coupling phase space* is

$$P_{\tau(I(c))} = P_{\tau(i_1)} \times \cdots \times P_{\tau(i_s)}$$

Definition 3.1 A vector field $f : P \rightarrow P$ is $\mathcal{B}_{\mathcal{G}}$ -equivariant or \mathcal{G} -admissible if:

- (a) For all $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variables x_c and the coupling phase space variables $x_{\tau(I(c))}$; that is, there exists $\hat{f}_c : P_c \times P_{\tau(I(c))} \rightarrow P_c$ such that

$$f_c(x) = \hat{f}_c(x_c, x_{\tau(I(c))}) \tag{3.2}$$

- (b) For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$

$$\hat{f}_d(x_d, x_{\tau(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\tau(I(d))})) \tag{3.3}$$

for all $x \in P$. ◇

For brevity, we write (3.3) as

$$f_{\beta(c)}(x) = f_c(\beta^*(x)) \tag{3.4}$$

for all $x \in P$. However, when using (3.4) it is necessary to remember that $f_d(x)$ depends only on the internal phase space variables x_d and the coupling phase space variables in $x_{\tau(I(d))}$. Otherwise, $\beta^*(x)$ is not defined.

Observe that self-coupling is allowed (that is, P_c can be one of the factors in $P_{\tau(I(c))}$) and multiple arrows between two cells are allowed (since the tail of two arrows terminating in $I(c)$ can be the same cell). However, when repetition occurs, the repeated coordinates are always identical.

It follows that F is determined if we specify one mapping f for each input equivalence class of cells. Indeed, each admissible vector field on a homogeneous cell system is uniquely determined by a single mapping f_c at some node c . In general, each component f_c of F is invariant under the vertex group $B(c, c)$. Indeed, every such invariant function determines a unique admissible vector field.

ODE-Equivalent Networks

In the enlarged class of coupled cell networks, it is possible for two different coupled cell systems \mathcal{G}_1 and \mathcal{G}_2 to generate the same space of admissible vector fields. For instance, consider the two two-cell systems in Figure 5. Their corresponding systems of admissible vector fields are

$$\begin{aligned} \dot{x}_1 &= g(x_1, x_1, x_2) & \text{and} & & \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= g(x_2, x_2, x_1) & & & \dot{x}_2 &= f(x_2, x_1) \end{aligned}$$

These clearly define the same spaces of admissible vector fields. Indeed, given f we can set $g(x, y, z) = f(x, z)$, and given g we can set $f(a, b) = g(a, a, b)$. We call a coupled cell network *redundant* if the number of coupling terminals can be reduced without changing the set of admissible vector fields. The cell system is *minimal* if it is not redundant.

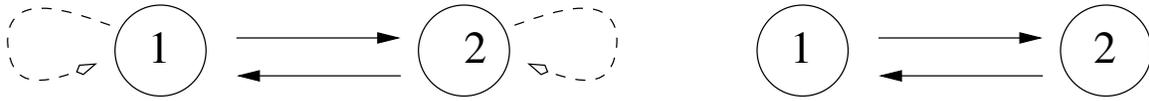


Figure 5: (Left) Two-cell redundant network. (Right) Corresponding minimal network.

Let \mathcal{G} be a coupled cell network and fix the phase space \mathcal{P} of \mathcal{G} . Denote the vector space of admissible vector fields on \mathcal{G} by $\mathcal{F}_{\mathcal{G}}(\mathcal{P})$. Suppose that $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a bijection that preserves cell equivalence, and that $\mathcal{P}_2 = P_1 \times \cdots \times P_N$ is a phase space for \mathcal{G}_2 . Define the *pullback phase space* on \mathcal{G}_1 to be

$$\mathcal{P}_1 = \gamma^*(\mathcal{P}_2) = P_{\gamma(1)} \times \cdots \times P_{\gamma(N)}$$

The bijection γ also induces a push-forward map γ_* from vector fields on \mathcal{P}_1 to those on \mathcal{P}_2 by viewing γ as a (linear) map from \mathcal{P}_1 to \mathcal{P}_2 .

Definition 3.2 Two coupled cell networks \mathcal{G}_1 and \mathcal{G}_2 are *ODE-equivalent* if

- (a) There is a bijection $\gamma : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ that preserves cell equivalence
- (b) For every phase space \mathcal{P}_2 for \mathcal{G}_2 ,

$$\gamma_*(\mathcal{F}_{\mathcal{G}_1}(\gamma^*(\mathcal{P}_2))) = \mathcal{F}_{\mathcal{G}_2}(\mathcal{P}_2)$$

Note that if two coupled cell networks are ODE-equivalent, then we can number the cells in each network so that the admissible vector fields on the two networks are identical. The two networks in Figure 5 are ODE-equivalent.

Definition 3.3 Two coupled cell networks \mathcal{G}_1 and \mathcal{G}_2 are *isomorphic* if

- (a) There is a cell equivalence preserving bijection $\beta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$

(b) For each cell $c \in \mathcal{C}_1$ there is an *input isomorphism* $\beta_c : I(c) \rightarrow I(\beta(c))$, that is,

$$(\tau(i), i) \sim_{E_1} (\tau(j), j) \iff (\tau(\beta_c(i)), \beta_c(i)) \sim_{E_2} (\tau(\beta_c(j)), \beta_c(j))$$

for every $i, j \in I(c)$.

◇

Isomorphic networks are always ODE-equivalent. Figure 5 shows that ODE-equivalent networks are not necessarily isomorphic.

Theorem 3.4 *Two minimal ODE-equivalent coupled cell networks are isomorphic.*

Proof ODE-equivalence implies that we can renumber the cells in \mathcal{C}_2 so that the bijection γ is the identity and $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}$. Hence the admissible vector fields satisfy $\mathcal{F}_{\mathcal{G}_1}(\mathcal{P}) = \mathcal{F}_{\mathcal{G}_2}(\mathcal{P})$.

Since $\mathcal{F}_{\mathcal{G}_j}(\mathcal{P})$ is the sum of vector fields supported on input equivalences of \mathcal{G}_j , the input equivalence classes of \mathcal{G}_1 and \mathcal{G}_2 are identical. It follows that the identity is a bijection from \mathcal{C}_1 to \mathcal{C}_2 that preserves cell-equivalence, so Definition 3.3(a) is satisfied.

Next we focus on one input equivalence class, which we can assume, after renumbering, consists of cells $\{1, \dots, K\}$. Input equivalence implies that the number of terminals attached to each of these K cells in is the same in \mathcal{G}_1 and \mathcal{G}_2 . Call this number s_j . Minimality implies that s_j cannot be reduced without changing the class of admissible vector fields. Hence $s_1 = s_2 = s$, say. In the two networks the general component on cell 1 of an admissible vector field has the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_{\tau_1(1)}, \dots, x_{\tau_1(s)}) \\ \dot{x}_1 &= g(x_1, x_{\tau_2(1)}, \dots, x_{\tau_2(s)}) \end{aligned}$$

where τ_j is the terminal map for network \mathcal{G}_j . The map f is invariant under the vertex group $B(1, 1)$ in \mathcal{G}_1 . Since the two spaces of admissible vector fields are identical, the map g must also be invariant under $B(1, 1)$. In particular, if edges $(\tau_1(i), i) \sim_{E_1} (\tau_1(j), j)$, then edges $(\tau_2(i), i) \sim_{E_2} (\tau_2(j), j)$. So Definition 3.3(b) is satisfied when $\beta : \mathcal{C}_1 \rightarrow \mathcal{C}_2$, and each $\beta_c : I(c) \rightarrow I(c)$ is the identity map on $I(c)$. Thus, the two networks are isomorphic. □

4 Balanced Equivalence Relations

We now extend the key concept of a balanced equivalence relation to the multiarrow formalism, and generalize its properties.

Definition 4.1 An equivalence relation \bowtie on \mathcal{C} is *balanced* if for every $c, d \in \mathcal{C}$ with $c \bowtie d$, there exists an input isomorphism $\beta \in B(c, d)$ such that $\tau(i) \bowtie \tau(\beta(i))$ for all $i \in I(c)$. ◇

In particular, $B(c, d) \neq \emptyset$ implies $c \sim_I d$. Hence, balanced equivalence relations refine \sim_I .

In the important special case where all pairs of arrows connecting the same two cells are \sim_E -equivalent, there is a graphical way to test whether a given equivalence relation \bowtie is

balanced. Color the cells in a network so that two cells have the same color precisely when they are in the same \bowtie -equivalence class. Since each terminal is the head terminal of an arrow, we color the terminal by the color of the tail of that arrow. Then \bowtie is balanced if and only if every pair of identically colored cells is connected by a color-preserving input isomorphism. For example, consider the balanced relation in the network in Figure 1. We add terminals to that figure to obtain Figure 6 (left). The quotient network of Figure 6 (right) is discussed in Section 5.

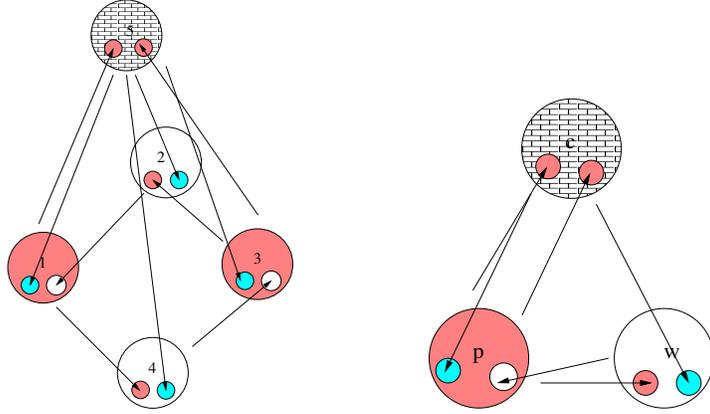


Figure 6: (Left) Five-cell network with balanced coloring. (Right) Three-cell quotient.

Choose a total phase space P , and let \bowtie be an equivalence relation on \mathcal{C} . We assume that \bowtie is a refinement of $\sim_{\mathcal{C}}$; that is, if $c \bowtie d$, then c and d have the same cell labels. It follows that the *polydiagonal subspace*

$$\Delta_{\bowtie} = \{x \in P : x_c = x_d \text{ whenever } c \bowtie d \quad \forall c, d \in \mathcal{C}\}$$

is well-defined, since x_c and x_d lie in the same space $P_c = P_d$. The polydiagonal Δ_{\bowtie} is a linear subspace of P .

Definition 4.2 Let \bowtie be an equivalence relation on \mathcal{C} . Then \bowtie is *robustly polysynchronous* if Δ_{\bowtie} is invariant under every vector field $f \in \mathcal{F}_G^P$. That is,

$$f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$$

for all $f \in \mathcal{F}_G^P$. Equivalently, if $x(t)$ is a trajectory of any $f \in \mathcal{F}_G^P$, with initial condition $x(0) \in \Delta_{\bowtie}$, then $x(t) \in \Delta_{\bowtie}$ for all $t \in \mathbf{R}$. \diamond

We now generalize Theorem 6.5 of [9] to the multiarrow formalism.

Theorem 4.3 *Let \bowtie be an equivalence relation on a coupled cell network. Then \bowtie is robustly polysynchronous if and only if \bowtie is balanced.*

Proof The proof is essentially the same as that of Theorem 6.5 of [9]. The main points are that it is easy to check directly that \bowtie being balanced is sufficient for Δ_{\bowtie} to be robustly polysynchronous, while necessity can be established by considering admissible *linear* vector fields. We take these points in turn.

First, suppose that Δ_{\bowtie} is balanced, and let $f \in \mathcal{F}_G^P$. Suppose that $c \bowtie d$. By Definition 4.1 the set $B(c, d)$ is non-empty, so there exists $\beta \in B(c, d)$. We have $\beta(c) = d$.

We know that for all $c \in \mathcal{C}$ the component $f_c(x)$ is symmetric under all permutations of the input set $I(c)$ that preserve \bowtie -equivalence classes. Therefore for any $x \in \Delta_{\bowtie}$

$$f_d(x) = f_d(x_{I(d)}) = f_{\beta(c)}(x_{I(d)}) = f_c(\beta^*(x_{I(d)})) = f_c(x_{I(c)}) = f_c(x)$$

because β preserves the \bowtie -equivalence classes. Therefore f leaves Δ_{\bowtie} invariant.

For the converse, suppose that Δ_{\bowtie} is invariant under all $f \in \mathcal{F}_G^P$. Then in particular Δ_{\bowtie} is invariant under all linear $f \in \mathcal{F}_G^P$. Let $c \neq d \in \mathcal{C}$ with $c \bowtie d$. We first show that $c \sim_I d$. If not, we can define an admissible linear vector field f such that $f_c = 0, f_d \neq 0$. This contradicts invariance of Δ_{\bowtie} . Therefore $c \bowtie d$ implies that c, d are input-equivalent as claimed.

Next, we construct a class of admissible linear vector fields as follows. For each pair of \sim_C -equivalence classes of cells ($[c], [d]$) choose representatives $c, d \in \mathcal{C}$. Choose some linear map

$$\lambda_{dc} : P_d \rightarrow P_c$$

If $c' \sim_C c$ and $d' \sim_C d$, use the canonical identifications of $P_{c'}$ with P_c and $P_{d'}$ with P_d to pull back λ_{dc} to a linear map

$$\lambda_{d'c'} : P_{d'} \rightarrow P_{c'}$$

That is, we ensure that λ_{dc} remains ‘the same’ map when cells are replaced by canonically identified cells.

Now choose a transversal \mathcal{T} to the set of \sim_I -equivalence classes. That is, arrange for \mathcal{T} to contain precisely one member of each \sim_I -equivalence class. For each $\tau \in \mathcal{T}$ define

$$\Lambda_{\tau}(x) = \sum_{i \in I(\tau)} \lambda_{i\tau}(x_i)$$

If $i, j \in I(\tau)$ and $(i, \tau) \sim_E (j, \tau)$ impose the extra condition

$$\lambda_{i\tau} = \lambda_{j\tau} \tag{4.1}$$

where we canonically identify P_i with P_j . Condition (4.1) ensures that Λ_{τ} is $B(\tau, \tau)$ -invariant.

Any $c \in \mathcal{C}$ is \sim_I -equivalent to precisely one $\tau(c) \in \mathcal{T}$. Let $\beta \in B(\tau(c), c)$ and use the pullback β^* to define

$$\Lambda_c(x) = \Lambda_{\beta(\tau(c))}(x) = \Lambda_{\tau(c)}(\beta^*(x))$$

The $B(\tau, \tau)$ -invariance of $\Lambda_{\tau(c)}$ implies that all $\beta \in B(\tau(c), c)$ lead to the same Λ_c . Lemma 4.5 of [9], trivially extended to the multiple-arrow formalism, implies that Λ is \mathcal{B}_G -equivariant, that is, admissible.

The final preparatory step is to partition the input sets $I(c)$ according to the \sim_E -equivalence classes of arrows (i, c) . Full details (which easily generalize to the multiple-arrow formalism) are at the end of Section 3 of [9] under the heading ‘Structure of $B(c, d)$ ’. Introduce an equivalence relation \equiv_c on $I(c)$ for which

$$j_1 \equiv_c j_2 \iff (j_1, c) \sim_E (j_2, c)$$

and let the \equiv_c -equivalence classes be K_0^c, \dots, K_r^c for $r = r(c)$. By convention $K_0^c = \{c\}$. By Section 3 of [9] the vertex group $B(c, c)$ is isomorphic to the direct product of symmetric groups $\mathbf{S}_{k_j^c}$ acting on the sets K_j^c , where $k_j^c = |K_j^c|$.

Let the \bowtie -equivalence classes be A_1, \dots, A_m . Let X_l denote the common value of the components x_i for $i \in A_l$. Let μ_s^c denote the common value of the $\lambda_{j\tau(c)}$ for $j \in K_s^c$. Restrict Λ to Δ_{\bowtie} . If $c \in \mathcal{C}$ then

$$\begin{aligned} \Lambda_c(x) &= \sum_{j \in I(c)} \lambda_{j\tau(c)}(x_j) \\ &= \sum_{s=0}^{r(c)} \sum_{j \in K_s^c} \lambda_{j\tau(c)}(x_j) \\ &= \sum_{s=0}^{r(c)} \sum_{l=1}^m \sum_{j \in K_s^c \cap A_l} \lambda_{j\tau(c)}(x_j) \\ &= \sum_{s=0}^{r(c)} \sum_{l=1}^m \sum_{j \in K_s^c \cap A_l} \mu_s^c(X_l) \\ &= \sum_{s=0}^{r(c)} \sum_{l=1}^m |K_s^c \cap A_l| \mu_s^c(X_l) \end{aligned}$$

Now suppose that $c \bowtie d$ with $c \neq d$. Since \bowtie is robustly synchronous, Λ_c and Λ_d must agree on Δ_{\bowtie} . Therefore

$$|K_s^c \cap A_l| = |K_s^d \cap A_l|$$

whenever $0 \leq s \leq r(c) = r(d)$ and $1 \leq l \leq m$. This is the ‘cardinality condition’ (6.2) of [9], and it clearly implies that \bowtie is balanced (use the fact that $B(c, c) \cong \mathbf{S}_{k_1^c} \times \dots \times \mathbf{S}_{k_{r(c)}^c}$, as in the proof of Theorem 6.5 of [9]). \square

5 Quotient Networks

In this section we show that each balanced equivalence relation \bowtie of a coupled cell network \mathcal{G} induces a quotient network \mathcal{G}_{\bowtie} , and that every admissible vector field on the quotient network \mathcal{G}_{\bowtie} lifts to an admissible vector field on \mathcal{G} .

We begin by defining the quotient network.

(A) Let \bar{c} denote the \bowtie -equivalence class of $c \in \mathcal{C}$. The cells in \mathcal{C}_{\bowtie} are the \bowtie -equivalence classes in \mathcal{C} ; that is,

$$\mathcal{C}_{\bowtie} = \{\bar{c} : c \in \mathcal{C}\}$$

Thus we obtain \mathcal{C}_{\bowtie} by forming the *quotient* of \mathcal{C} by \bowtie , that is, $\mathcal{C}_{\bowtie} = \mathcal{C} / \bowtie$.

(B) Define

$$\bar{c} \sim_{\mathcal{C}_{\bowtie}} \bar{d} \iff c \sim_{\mathcal{C}} d$$

The relation $\sim_{\mathcal{C}_{\bowtie}}$ is well-defined since \bowtie refines $\sim_{\mathcal{C}}$.

(C) The number of input terminals in a quotient cell \bar{c} is the same as the number of input terminals in cell c , that is, $|I(\bar{c})| = |I(c)|$. The arrows in the quotient network are the projection of arrows in the original network, that is,

$$\mathcal{E}_{\bowtie} = \{\overline{(\tau(i), i)} : (\tau(i), i) \in \mathcal{E}\} \quad (5.1)$$

There is an overenumeration of arrows in the quotient defined by (5.1) since any two distinct \bowtie -equivalent cells d_1 and d_2 contribute a full complement of arrows to the terminals in $I(\bar{d}_j)$. We claim that the assignment of arrows is consistent because \bowtie is balanced. More precisely, since \bowtie is balanced, there is an input isomorphism $\beta : I(d_1) \rightarrow I(d_2)$. Let $i \in I(d_1)$ be a terminal head; then $\beta(i)$ is a terminal head in $I(d_2)$, and $\tau(i) \bowtie \tau(\beta(i))$. It follows that $\overline{(\tau(i), i)} = \overline{(\tau(\beta(i)), \beta(i))}$ and that the arrows $\overline{(\tau(i), i)}$ and $\overline{(\tau(\beta(i)), \beta(i))}$ are \sim_E -equivalent. So the same type of arrow is attached to terminals in $I(\bar{c})$ regardless of which cell in \mathcal{C} is used to define those arrows.

Note that when $d_1 \bowtie d_2$ any arrows with head terminal in $I(d_2)$ and tail cell d_1 lead to self-coupling arrows in the quotient. Finally, if c_1 and c_2 are distinct \bowtie -equivalent cells having equivalent arrows with the same terminal cell d , then multiarrows will be present in the quotient network, where a *multiarrow* is a set of several edge-equivalent arrows between two given cells. See Figure 6.

(D) Two quotient arrows are equivalent when the original arrows are equivalent. That is,

$$\overline{(\tau(i_1), i_1)} \sim_{\mathcal{E}_{\bowtie}} \overline{(\tau(i_2), i_2)} \iff (\tau(i_1), i_1) \sim_E (\tau(i_2), i_2) \quad (5.2)$$

where i_j is a terminal head in cell c_j . The remark in Paragraph (C) on overenumeration of arrows shows that $\sim_{\mathcal{E}_{\bowtie}}$ is well defined. Now (5.2) implies that when two arrows in \mathcal{E}_{\bowtie} are $\sim_{\mathcal{E}_{\bowtie}}$ -equivalent, their head cells \bar{c}_1 and \bar{c}_2 satisfy $c_1 \sim_{\mathcal{C}} c_2$, and their tail cells $\tau(i_1)$ and $\tau(i_2)$ satisfy $\tau(i_1) \sim_{\mathcal{C}} \tau(i_2)$. Therefore, $\bar{c}_1 \sim_{\mathcal{C}_{\bowtie}} \bar{c}_2$ and $\overline{(\tau(i_1), i_1)} \sim_{\mathcal{E}_{\bowtie}} \overline{(\tau(i_2), i_2)}$.

(E) Input isomorphisms on \mathcal{G} project onto input isomorphisms of \mathcal{G}_{\bowtie} .

Let $\beta : I(c) \rightarrow I(d)$ be an input isomorphism between input sets of cells c and d . Then $\beta : I(\bar{c}) \rightarrow I(\bar{d})$ is also a bijection since $I(c) = I(\bar{c})$ and $I(d) = I(\bar{d})$. Identity (5.2) guarantees that (2.2) is valid for \mathcal{E}_{\bowtie} -equivalence and β is an input isomorphism for \mathcal{G}_{\bowtie} . Identity (5.2) also guarantees the converse — every input equivalence on \mathcal{G}_{\bowtie} lifts to one on \mathcal{G} .

Remark 5.1 Since input isomorphisms project, we see that any quotient of a homogeneous network is also a homogeneous cell network. The quotient of the balanced relation of the five-cell example in Figure 1 and Figure 2 (left) shows that this remark is not valid for quotients in the class of networks considered in [9].

We can now generalize Theorem 9.2 of [9] to the multiarrow formalism:

Theorem 5.2 *Let \bowtie be a balanced relation on a coupled cell network \mathcal{G} . The restriction of a \mathcal{G} -admissible vector field to Δ_{\bowtie} is \mathcal{G}_{\bowtie} -admissible.*

Proof The proof of Theorem 5.2 is identical to the proof of Theorem 9.2 of [9]. \square

Admissible Vector Fields on a Quotient Lift

Having defined the quotient network, we next discuss why the multiarrow formalism implies that every \mathcal{G}_{\bowtie} -admissible vector field on the quotient lifts to a \mathcal{G} -admissible vector field on the original network. Let \bar{c} be a quotient cell and suppose that the dynamics on that cell are prescribed by the ODE

$$\dot{x}_{\bar{c}} = \hat{f}_{\bar{c}}(x_{\bar{c}}, x_{\tau(I(\bar{c}))})$$

where $x_{\bar{c}} \in P_{\bar{c}} = P_c$ are the internal state space variables and $x_{\tau(I(\bar{c}))} \in P_{\tau(I(\bar{c}))} = P_{\tau(I(c))}$ are the coupling variables. We can lift this ODE to each cell c that quotients onto \bar{c} by

$$\dot{x}_c = \hat{f}_{\bar{c}}(x_c, x_{\tau(I(c))}) \tag{5.3}$$

Observe that if $c \bowtie d$ (or $\bar{c} = \bar{d}$), then there exists an input isomorphism $\beta : I(c) \rightarrow I(d)$. Now \mathcal{G}_{\bowtie} -admissibility and the fact that input isomorphisms on \mathcal{G} project onto input isomorphisms on \mathcal{G}_{\bowtie} imply that

$$\hat{f}_{\bar{d}}(x_d, x_{\tau(I(\bar{d}))}) = \hat{f}_{\bar{c}}(x_d, \beta^*(x_{\tau(I(\bar{c}))}))$$

Note that if $c = d$, then (5.3) is consistent since $f_{\bar{c}}$ is invariant under $B(\bar{c}, \bar{c})$. Therefore

$$\hat{f}_d(x_d, x_{\tau(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\tau(I(d))}))$$

and the lift (5.3) is \mathcal{G} -admissible.

6 Identical Edge Homogeneous Networks

An *identical-edge* homogeneous cell network \mathcal{G} is a homogeneous network in which all edges in \mathcal{E} are equivalent.

Proposition 6.1 *Every quotient of an identical-edge homogeneous network is an identical-edge homogeneous network.*

Proof This statement follows directly from Section 5 (D) and Remark 5.1. □

Proposition 6.2 *Every identical-edge homogeneous network is the quotient of an identical-edge homogeneous network without multiple edges or self-coupling.*

Proof We begin by showing that if some cell in an N -cell network, say cell 1, has m self-couplings, then we can enlarge the network to an $(N + m)$ -cell network, having the original network as a quotient, so that the enlarged network has no self-couplings in cells residing in the pullback of cell 1. Add arrows and cells to the enlarged network as follows.

- 1) Replace cell 1 with with $m + 1$ input isomorphic cells.
- 2) For each arrow in the original network with head cell 1 and tail cell i , where $i \neq 1$, add m edge-equivalent arrows with tail cell i , where one of the new arrows terminates in each of the m new cells.
- 3) Each pair of the $m + 1$ cells in the preimage of cell 1 has identical arrows with head terminal in the first cell of the pair and tail cell in the second cell of the pair.

Note that all arrows starting from one of the m new cells terminate in cell 1. In particular, none of the $m + 1$ cells in the preimage of cell 1 have self-coupling arrows. In the new network, assign all cells in the preimage of cell 1 the same color, and all other cells different colors. This coloring is balanced, and yields the original network as a quotient network. See Figure 7. Therefore we can enlarge the original network so that it has no self-coupling arrows.

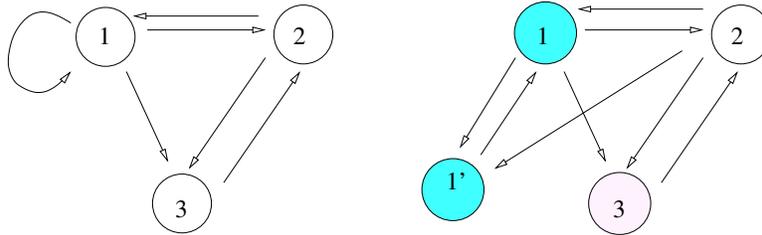


Figure 7: (Left) Three-cell network with self-coupling. (Right) Four-cell enlargement of original system.

Next, we assume that the network has no self-coupling and that there are m identical arrows from cell 1 to terminals in cell 2. There is an extended coupled cell network with $N + m - 1$ cells formed by replacing cell 1 with m identical cells and changing arrows as follows:

- 1) Each of the m cells replacing cell 1 connects to cell 2 with one arrow. Note that cell 2 receives the same number of arrows from the m copies of cell 1 that it received previously from the single cell 1 in the original network.

- 2) Add arrows so that every cell that was connected to cell 1 in the original network is now connected to each of the m cell 1s in the new network.

Note that there are no arrows starting from one of the $m - 1$ new cells that terminate in a cell in the original network not equal to cell 2. In the new network, color all cells in the preimage of cell 1 the same color, and all other cells different colors. This coloring is balanced, and yields the original network as a quotient network. See Figure 8. Proceeding inductively, we can eliminate multiple arrows between cells. \square



Figure 8: (Left) Three-cell network with multiple arrows. (Right) Four-cell enlargement of original system.

7 Hopf Bifurcation in Two-Color Networks

We now specialize our results to equivalence relations with two colors. We prove a Hopf bifurcation theorem in the case of an identical-edge homogeneous network, with the feature that well-defined approximate phase shifts and approximate amplitude relations hold near bifurcation.

Proposition 7.1 *Suppose that an identical-edge homogeneous network has a balanced relation with two colors. Then there is a unique type of synchrony-breaking Hopf bifurcation from a synchronous equilibrium that leads to periodic solutions that are synchronous on all cells of the same color, and that are approximately one-half period out of phase with all cells of the opposite color. The amplitudes of these periodic signals need not be equal.*

Proof The two-color balanced relation leads to a two-cell quotient network (one cell for each color). Moreover, Proposition 6.1 implies that the two cells are input isomorphic and all edges are identical. Such two-cell networks are determined by the number of self-coupling arrows l_j on cell j and the number of edges $m_1 \geq 0$ coupling cell 2 to cell 1. Let $m_2 \geq 0$ be the number of edges coupling cell 1 to cell 2; then $l_1 + m_1 = l_2 + m_2 \equiv p$ and $m_2 = m_1 + l_1 - l_2$. The coupled cell systems have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \underbrace{x_1, \dots, x_1}_{l_1 \text{ times}}, \underbrace{x_2, \dots, x_2}_{m_1 \text{ times}}) \\ \dot{x}_2 &= f(x_2, \underbrace{x_2, \dots, x_2}_{l_2 \text{ times}}, \underbrace{x_1, \dots, x_1}_{m_2 \text{ times}}) \end{aligned} \quad (7.1)$$

where $x_1, x_2 \in \mathbf{R}^k$. Since $\{x : x_1 = x_2\}$ is flow-invariant we can arrange for the robust existence of an equilibrium in this subspace. Moreover, by a change of coordinates, we can assume that the equilibrium is at the origin. Let J be the Jacobian matrix of this equilibrium. By (7.1)

$$J = \begin{bmatrix} A + l_1 B & m_1 B \\ m_2 B & A + l_2 B \end{bmatrix}$$

where A is the linearization of the internal dynamics and B is the coupling matrix. Assume that $x_1, x_2 \in \mathbf{R}^k$. Let $v \in \mathbf{R}^k$ and observe that

$$J \begin{bmatrix} v \\ v \end{bmatrix} = \begin{bmatrix} (A + pB)v \\ (A + pB)v \end{bmatrix} \quad \text{and} \quad J \begin{bmatrix} m_1 v \\ -m_2 v \end{bmatrix} = \begin{bmatrix} (A + (l_2 - m_1)B)m_1 v \\ -(A + (l_2 - m_1)B)m_2 v \end{bmatrix}$$

Thus, the eigenvalues of J are given by eigenvalues of the $k \times k$ matrices $A + pB$ and $A + (l_2 - m_1)B$. Either matrix can have purely imaginary eigenvalues when $k \geq 2$. Critical eigenvalues in the matrix $A + pB$ leads to periodic solutions that are synchronous on all cells, since the synchrony subspace $x_1 = x_2$ is flow-invariant.

Synchrony-breaking Hopf bifurcations occur if the matrix $A + (l_2 - m_1)B$ has (simple) purely imaginary eigenvalues $\pm\omega i$. Let $v_0 \in \mathbf{C}^k$ be an eigenvector associated to the eigenvalue ωi . Then Hopf bifurcation can lead to a branch of periodic solutions that to first order in the bifurcation parameter has the form

$$x_1(t) = m_1 \text{Re}(e^{i\omega t} v_0) \quad x_2(t) = -m_2 \text{Re}(e^{i\omega t} v_0)$$

The amplitudes of the time series $x_1(t)$ and $x_2(t)$ are different (unless $m_1 = m_2$). Indeed to first order they are in the ratio $m_1 : m_2$ near the bifurcation point. The minus sign in x_2 shows that the time series are (to first order) a half-period out of phase. \square

Example 7.2 Consider the two-cell system in Figure 9 (left). This network can be obtained

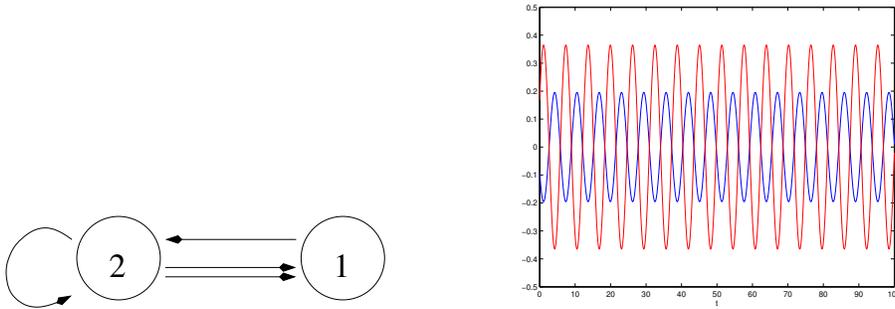


Figure 9: (Left) Two-cell homogeneous network. (Right) Half-period out of phase periodic state with different amplitudes obtained by Hopf bifurcation.

as a two-color quotient network of the five-cell network in Figure 6 (left) by identifying the four pink and white cells as one color and the cyan cell as the other color. The time series

of a periodic state obtained by Hopf bifurcation in this network is shown in Figure 9 (right). Note that the time series from cells 1 and 2 are approximately one-half period out of phase even though the amplitudes of these signals are quite different. The amplitude ratio here is convincingly close to $|m_1/m_2| = 2$.

When $m_1 = m_2$ (so $l_1 = l_2$) we can say more:

Corollary 7.3 *Suppose that an identical-edge homogeneous network has a balanced equivalence relation with two colors, black and white. If the number of white cells coupled to a white cell is equal to the number of black cells coupled to a black cell, then the synchrony-breaking Hopf bifurcation in Proposition 7.1 leads to robust periodic solutions that are synchronous on cells of the same color, and exactly one-half period out of phase with cells of the opposite color.*

Proof When $m_1 = m_2$ (and hence $l_1 = l_2$) in Proposition 7.1, the transposition $(x_1, x_2) \mapsto (x_2, x_1)$ is a symmetry of (7.1) and the bifurcating states have an exact spatio-temporal symmetry $x_2(t) = x_1(t + \frac{T}{2})$, where T is the (minimal) period. \square

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