Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows

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Abstract

A coupled cell system is a network of dynamical systems, or ‘cells’, coupled together. The architecture of a coupled cell network is a graph that indicates how cells are coupled and which cells are equivalent. Stewart, Golubitsky, and Pivato presented a framework for coupled cell systems that permits a classification of robust synchrony in terms of network architecture. They also studied the existence of other robust dynamical patterns using a concept of quotient network. There are two difficulties with their approach. First, there are examples of networks with robust patterns of synchrony that are not included in their class of networks; and second, vector fields on the quotient do not in general lift to vector fields on the original network, thus complicating genericity arguments. We enlarge the class of coupled systems under consideration by allowing two cells to be coupled in more than one way, and we show that this approach resolves both difficulties. The theory that we develop, the ‘multiarrow formalism’, parallels that of Stewart et al. We use quotient networks to prove that the only robust patterns of synchrony for equilibria are those given by balanced relations, and to discuss Hopf bifurcation in homogeneous cell systems with two-color balanced equivalence relations.

1 Introduction

Stewart, Golubitsky, and Pivato [11] formalize the definition of a coupled cell system in terms of the symmetry groupoid of an associated coupled cell network, and prove three general
theorems about such networks. First, a set of cells can be robustly synchronous if and only if they are in the same equivalence class of some balanced equivalence relation. Second, every balanced relation leads to a new coupled cell network, called a quotient network, that is formed by identifying equivalent cells. Third, the restriction of a coupled cell system to a synchrony subspace (or polydiagonal) is a coupled cell system associated to the quotient network. The approach in [11] has two difficulties:

1) Not every coupled cell system of ODEs corresponding to the quotient network is the restriction of a coupled cell system corresponding to the original network. This fact makes it difficult to prove genericity statements about dynamics in the original network based only on genericity statements about dynamics of the quotient network. (Dias and Stewart [4] obtain necessary and sufficient conditions, on a network with a balanced relation, for every quotient system to be a restriction of a cell system corresponding to the original network.)

2) Reasonable networks that are not included in the theory developed in [11] can exhibit patterns of robust synchrony. Examples are linear chains with Neumann boundary conditions considered in Epstein and Golubitsky [6] and square arrays of cells with Neumann boundary conditions considered in Gillis and Golubitsky [7]. In this paper we show that both of these difficulties can be resolved if the class of coupled cell networks is enlarged to permit multiple couplings between cells, and self-coupling. We call this the multiarrow formalism for coupled cell networks. Although the abstract definition of this enlarged class of coupled cell networks is more complicated than the more restrictive definition in [11], the multiarrow formalism has the side benefit that quotient systems are more easily defined in the enlarged class, and have more convenient properties.

We first motivate the generalization by considering two examples in the important case of a homogeneous network, which we now define. A cell is a system of ODEs, and a coupled cell system is a collection of $N$ cells with couplings. As discussed in [11] a class of coupled cell systems is defined by a coupled cell network, which is a (directed, labeled) graph that specifies, among other information, which cells are coupled to which. Two cells of the network are input isomorphic (see [11]) if the dynamics of the cells are specified by the same differential equations, up to a permutation of the variables. More precisely, if cells 1 and 2 with internal state variables $x_1, x_2 \in \mathbb{R}^k$ are input isomorphic, then the relevant components of the system of ODEs take the form

$$\begin{align*}
\dot{x}_1 &= f(x_1, y_1, \ldots, y_l) \\
\dot{x}_2 &= f(x_2, z_1, \ldots, z_l)
\end{align*}$$

(1.1)

where the $y_j$ (resp. $z_j$) are internal state variables of the cells connected to cell 1 (resp. cell 2). In particular, the two cells receive inputs from the same number $l$ of cells, the input variables are of the same type $y_j, z_j \in \mathbb{R}^{k_j}$, and the dependence of the corresponding components of $\dot{x}$ is specified using the same function $f$ of the relevant internal variables and input variables. The phase space of the coupled cell system is

$$P = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_N} \}$$

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We call a coupled cell network *homogeneous* if all cells are input isomorphic (in which case \( k_1 = \cdots = k_N \)). Homogeneous coupled cell systems are determined by a single function \( f \), as illustrated in (1.1). For the remainder of this introduction we focus on homogeneous coupled cell networks.

We can visualize an equivalence relation \( \bowtie \) on cells by coloring all equivalent cells with the same color. This equivalence relation is *balanced* (in the homogeneous case) if the sets of colors of input cells for two equivalent cells consist of the same colors with the same multiplicities. Theorem 6.5 of [11] states that the subspace

\[
\Delta_\bowtie = \{ x \in P : x_i = x_j \text{ if } i \bowtie j \}
\]

is flow-invariant for all \( f \) if and only if \( \bowtie \) is balanced. A solution in \( \Delta_\bowtie \) is *synchronous* in the strong sense that the time series from cells of the same color are identical; the synchrony is *robust* in the sense that it holds for any choice of \( f \). We call \( \Delta_\bowtie \) the *polydiagonal* or *synchrony subspace* corresponding to \( \bowtie \).

**Quotients Lead to Multiple Arrows**

We describe circumstances in which multiple arrows are natural and useful. Consider the homogeneous five-cell coupled cell network pictured in Figure 1 (left). A balanced coloring of this network is given in the right panel of that figure.

![Figure 1: (Left) Homogeneous five-cell network. (Right) Balanced coloring of the network.](image)

The differential equations corresponding to this five-cell network have the form

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_5) \\
\dot{x}_2 &= f(x_2, x_3, x_5) \\
\dot{x}_3 &= f(x_3, x_4, x_5) \\
\dot{x}_4 &= f(x_4, x_1, x_5) \\
\dot{x}_5 &= f(x_5, x_1, x_3)
\end{align*}
\]
where \( f(a, b, c) = f(a, c, b) \) since all couplings are assumed to be identical. It is straightforward to check that the subspace \( \Delta \) defined by \( x_1 = x_3 \) and \( x_2 = x_4 \) is flow-invariant. The restricted system on \( \Delta \) has the form

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_5) \\
\dot{x}_2 &= f(x_2, x_1, x_5) \\
\dot{x}_5 &= f(x_5, x_1, x_1)
\end{align*}
\]

The quotient cell construction in [11] leads to the coupled cell network of Figure 2 (left). The coupled cell system corresponding to that quotient network, which is not homogeneous, has the form:

\[
\begin{align*}
\dot{w} &= f(w, p, c) \\
\dot{p} &= f(p, w, c) \\
\dot{c} &= g(c, w)
\end{align*}
\]

Therefore, a coupled cell system corresponding to the quotient network is the restriction of a coupled cell system corresponding to the five-cell network if and only if \( g(b, w) = f(b, w, w) \) where \( f(a, b, c) = f(a, c, b) \). In this paper we remove such conditions from consideration by allowing multiple couplings between cells. With multiple couplings, the quotient network is the homogeneous one of Figure 2 (right). Quotient coupled cell systems for the new quotient have the form

\[
\begin{align*}
\dot{w} &= f(w, p, c) \\
\dot{p} &= f(p, w, c) \\
\dot{c} &= f(c, w, w)
\end{align*}
\]

and each of these systems is the restriction to \( \Delta \) of a five-cell system. Homogeneous three-cell networks with each cell having at most two input arrows are classified in [10]. There are 38 such networks.

Figure 2: (Left) Three-cell quotient from [11] of five-cell network in Figure 1 (right). (Right) Three-cell quotient using multiarrows.

**Neumann Boundary Conditions Lead to Self-Coupling**

We now provide a reason for permitting self-coupling. Epstein and Golubitsky [6] consider patterns of synchrony in \( N \)-cell bidirectional linear arrays with Neumann boundary condi-
tions. The systems of ODEs have the form

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_1, x_2) \\
\dot{x}_j &= f(x_j, x_{j-1}, x_{j+1}) & 1 < j < N \\
\dot{x}_N &= f(x_N, x_{N-1}, x_N)
\end{align*}
\]

where \( f(a, b, c) = f(a, c, b) \). When self-coupling of a cell to itself is allowed, the network architecture is the one pictured in Figure 3.

![Figure 3: Linear array network.](image)

The ten-cell array in Figure 4 provides an example of a balanced coloring that cannot be obtained from the results in [11], since self-coupling is not permitted in that theory. To see that this coloring is balanced observe that each pink cell receives an input from one pink cell and one white cell, each cyan cell receives inputs from two white cells, and each white cell receives inputs from one pink cell and one cyan cell. See also [8].

![Figure 4: Linear array network of ten cells with a three-color balanced relation.](image)

**Structure of the Paper**

The paper is structured as follows. The enlarged class of ‘multiarrow’ coupled cell networks, which permits multiple arrows and self-coupling, is defined in Section 2. The associated admissible vector fields are constructed in Section 3. In that section we also show that distinct networks in the enlarged class can correspond to the same space of admissible vector fields. This (undesired) feature is not present in the class of networks considered in [11]: it is a price we have to pay to obtain the more desirable features. The connection between balanced equivalence relations and robust polysynchrony is discussed in Section 4. In this section we prove Theorem 4.3 that states that flow-invariant subspaces correspond to balanced equivalence relations in the multiarrow formalism. Quotient networks are defined in the context of multiple arrows and self-coupling in Section 5. Here we show that all admissible vector fields on a quotient network lift to admissible vector fields on the original network, a property that fails for the quotients defined in [11]. The important special case of identical-edge homogeneous networks (homogeneous networks in which all coupling arrows are equivalent) is considered in Section 7. Proposition 7.2 states that every homogeneous network with multiarrows and/or self-coupling is a quotient of a homogeneous network with neither multiarrows nor self-coupling.
As noted, Theorem 4.3 states that flow-invariant subspaces can be identified with balanced equivalence relations. Theorem 6.6 strengthens this result by showing that if equilibria have patterns of synchrony that do not change under small admissible perturbations, then the subspace corresponding to this pattern of synchrony is flow-invariant and hence corresponds to a balanced equivalence relation. The lifting property of quotients is used to simplify the proof of Theorem 6.6.

In symmetric networks, Hopf bifurcation typically leads to periodic states in which some cells have identical waveforms (hence identical amplitudes) except for a well-defined phase shift. In Section 8 we show that in identical-edge homogeneous networks, Hopf bifurcation can lead to periodic states with well-defined approximate phase shifts and different amplitudes.

The proofs of several of the main theorems in this paper (particularly Theorems 4.3 and 5.2) are straightforward adaptations of corresponding results in [11] to the enlarged category of networks considered here.

2 Coupled Cell Networks

We begin by formally defining a class of coupled cell networks that permits multiple arrows and self-couplings.

Definition 2.1 In the multiarrow formalism, a coupled cell network \( \mathcal{G} \) consists of:

(a) A finite set \( \mathcal{C} = \{1, \ldots, N\} \) of nodes or cells.

(b) An equivalence relation \( \sim_{\mathcal{C}} \) on cells in \( \mathcal{C} \).
   
   The type or cell label of cell \( c \) is the \( \sim_{\mathcal{C}} \)-equivalence class \( [c]_{\mathcal{C}} \) of \( c \).

(c) Associated to each node \( c \) is a finite set of input terminals \( I(c) \). Each input terminal \( i \in I(c) \) is the receptacle for one arrow or edge that begins in tail cell \( \tau(i) \) and ends in terminal \( i \). That arrow is denoted by \((\tau(i), i)\), has head cell \( c \), and head terminal \( i \). Let \( \mathcal{E} \) denote the set of all arrows.

(d) An equivalence relation \( \sim_{\mathcal{E}} \) on edges in \( \mathcal{E} \).

   The type or coupling label of edge \( e \) is the \( \sim_{\mathcal{E}} \)-equivalence class \( [e]_{\mathcal{E}} \) of \( e \).

(e) Equivalent edges have equivalent tails and heads. That is, if \((\tau(i), i) \sim_{\mathcal{E}} (\tau(j), j) \) where \( i \in I(c) \) and \( j \in I(d) \), then \( \tau(i) \sim_{\mathcal{C}} \tau(j) \) and \( c \sim_{\mathcal{C}} d \).

Observe that self-coupling is permitted (since \( \tau(i) = c \) for a terminal \( i \) in cell \( c \) is permitted) and multiple arrows are permitted (since \( \tau(i) = \tau(j) \) for two distinct terminals in cell \( c \) is permitted).
**Definition 2.2** The relation $\sim_I$ of input equivalence on $C$ is defined by $c \sim_I d$ if and only if there exists an arrow type preserving bijection

$$\beta : I(c) \rightarrow I(d)$$

That is, for every input terminal $i \in I(c)$

$$(\tau(i), i) \sim_E (\tau(\beta(i)), \beta(i))$$

Any such bijection $\beta$ is called an input isomorphism from cell $c$ to cell $d$. The set $B(c, d)$ denotes the collection of all input isomorphisms from cell $c$ to cell $d$. The set

$$B_G = \bigcup_{c, d \in C} B(c, d)$$

is a groupoid (Brandt [1], Brown [2], Higgins [9]), which is an algebraic structure rather like a group, except that the product of two elements is not always defined. We call $B_G$ the groupoid of the network.

**Remark 2.3**

(a) Suppose that a cell $c$ has two input terminals $i, j \in I(c)$ whose receiving arrows $(\tau(i), i)$ and $(\tau(j), j)$ are $\sim_E$ equivalent. Then the transposition $\beta = (i j)$ is an input isomorphism in $B(c, c)$.

(b) The reason for introducing an explicit set $I(c)$ of input terminals is to provide a well-defined set for the input isomorphism $\beta$ in (2.1) to act on. Otherwise we must consider ‘sets’ in which elements may occur more than once. This is the main novelty in Definition 2.2 compared to that in [11].

(c) In place of an explicit set of input terminals we can use the associated set of arrows, which is closer to the spirit of [11]. However, this approach becomes very convoluted because an ‘arrow’ can no longer be identified with a pair of cells $(i, c)$.

**Definition 2.4** A homogeneous network is a coupled cell network such that $B(c, d) \neq \emptyset$ for every pair of cells $c, d$.

### 3 Vector Fields on a Coupled Cell Network

We now define the class $\mathcal{F}_G^P$ of admissible vector fields corresponding to a given coupled cell network $G$. This class consists of all vector fields that are ‘compatible’ with the labeled graph structure, or equivalently are ‘symmetric’ under the groupoid $B_G$. It also depends on a choice of ‘total phase space’ $P$, which we assume is fixed throughout the subsequent discussion.

For each cell in $C$ define a cell phase space $P_c$. This must be a smooth manifold of dimension $\geq 1$, which for simplicity we assume is a nonzero finite-dimensional real vector space. We require

$$c \sim_C d \implies P_c = P_d$$
and we employ the same coordinate systems on $P_c$ and $P_d$. Only these identifications of cell phase spaces are canonical; that is, the relation $c \sim_C d$ implies that cells $c$ and $d$ have the same phase space, but not that they have isomorphic (conjugate) dynamics.

Define the corresponding total phase space to be

$$P = \prod_{c \in C} P_c$$

and employ the coordinate system

$$x = (x_c)_{c \in C}$$
on $P$.

More generally, suppose that $D = (d_1, \ldots, d_s)$ is any finite ordered subset of $s$ cells in $C$. In particular, the same cell can appear more than once in $D$. Define

$$P_D = P_{d_1} \times \cdots \times P_{d_s}$$

Further, write

$$x_D = (x_{d_1}, \ldots, x_{d_s})$$

where $x_{d_j} \in P_{d_j}$.

Finally, suppose that $D_1, D_2$ are subsets of $C$, and that there is a bijection $\gamma : D_1 \to D_2$ such that $\gamma(d) \sim_C d$ for all $d \in D_1$. Define the pullback map

$$\gamma^* : P_{D_2} \to P_{D_1}$$

by

$$(\gamma^*(z))_j = z_{\gamma(j)} \quad (3.1)$$

for all $j \in D_1$ and $z \in P_{D_2}$.

We use pullback maps to relate different components of a vector field associated with a given coupled cell network. Specifically, the class of vector fields that is encoded by a coupled cell network is given in Definition 3.1. For a given cell $c$ the internal phase space is $P_c$ and the coupling phase space is

$$P_{\tau(I(c))} = P_{\tau(i_1)} \times \cdots \times P_{\tau(i_s)}$$

where $\tau(I(c))$ denotes the ordered set of cells $(\tau(i_1), \ldots, \tau(i_s))$.

**Definition 3.1** A vector field $f : P \to P$ is $B_G$-equivariant or $G$-admissible if:

(a) For all $c \in C$ the component $f_c(x)$ depends only on the internal phase space variables $x_c$ and the coupling phase space variables $x_{\tau(I(c))}$; that is, there exists $\hat{f}_c : P_c \times P_{\tau(I(c))} \to P_c$ such that

$$f_c(x) = \hat{f}_c(x_c, x_{\tau(I(c))}) \quad (3.2)$$
(b) For all \(c, d \in C\) and \(\beta \in B(c, d)\)

\[
\hat{f}_d(x_d, x_{\tau(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\tau(I(d))}))
\] (3.3)

for all \(x \in P\).

For brevity, we write (3.3) as

\[
f_{\beta(c)}(x) = f_c(\beta^*(x))
\] (3.4)

for all \(x \in P\). However, when using (3.4) it is necessary to remember that \(f_d(x)\) depends only on the internal phase space variables \(x_d\) and the coupling phase space variables in \(x_{\tau(I(d))}\). Otherwise, \(\beta^*(x)\) is not defined.

Observe that self-coupling is allowed (that is, \(P_c\) can be one of the factors in \(P_{\tau(I(c))}\)) and multiple arrows between two cells are allowed (since the tail of two arrows terminating in \(I(c)\) can be the same cell). However, when repetition occurs, the repeated coordinates are always identical.

It follows that \(F\) is determined if we specify one mapping \(f\) for each input equivalence class of cells. Indeed, each admissible vector field on a homogeneous cell system is uniquely determined by a single mapping \(f_c\) at some node \(c\). In general, each component \(f_c\) of \(F\) is invariant under the vertex group \(B(c, c)\). Indeed, every such invariant function determines a unique admissible vector field.

ODE-Equivalent Networks. In the enlarged class of coupled cell networks, it is possible for two different coupled cell systems \(G_1\) and \(G_2\) to generate the same space of admissible vector fields. For instance, consider the two two-cell systems in Figure 5. Their corresponding systems of admissible vector fields are

\[
\begin{align*}
\dot{x}_1 &= g(x_1, x_1, x_2) \\
\dot{x}_2 &= g(x_2, x_2, x_1)
\end{align*}
\quad\text{and}\quad
\begin{align*}
\dot{x}_1 &= f(x_1, x_2) \\
\dot{x}_2 &= f(x_2, x_1)
\end{align*}
\]

These two-cell networks clearly define the same spaces of admissible vector fields. Indeed, given \(f\) we can set \(g(x, y, z) = f(x, z)\), and given \(g\) we can set \(f(a, b) = g(a, a, b)\). The issue of when two networks are ODE-equivalent is considered by Dias and Stewart [5].

![Figure 5: Two ODE-equivalent networks.](image-url)
4 Balanced Equivalence Relations

We now extend the key concept of a balanced equivalence relation to the multiaarrow formalism, and generalize its properties.

**Definition 4.1** An equivalence relation $\bowtie\bowtie$ on $\mathcal{C}$ is balanced if for every $c, d \in \mathcal{C}$ with $c \bowtie\bowtie d$, there exists an input isomorphism $\beta \in B(c, d)$ such that $\tau(i) \bowtie\bowtie \tau(\beta(i))$ for all $i \in I(c)$. $\Diamond$

In particular, $B(c, d) \neq \emptyset$ implies $c \sim_I d$. Hence, balanced equivalence relations refine $\sim_I$.

In the important special case where all pairs of arrows connecting the same two cells are $\sim_E$-equivalent, there is a graphical way to test whether a given equivalence relation $\bowtie\bowtie$ is balanced. Color the cells in a network so that two cells have the same color precisely when they are in the same $\bowtie\bowtie$-equivalence class. Since each terminal is the head terminal of an arrow, we color the terminal by the color of the tail of that arrow. Then $\bowtie\bowtie$ is balanced if and only if every pair of identically colored cells is connected by a color-preserving input isomorphism. For example, consider the balanced relation in the network in Figure 1. We add terminals to that figure to obtain Figure 6 (left). The quotient network of Figure 6 (right) is discussed in Section 5.

![Figure 6](image)

**Figure 6:** (Left) Five-cell network with balanced coloring. (Right) Three-cell quotient.

Choose a total phase space $P$, and let $\bowtie\bowtie$ be an equivalence relation on $\mathcal{C}$. We assume that $\bowtie\bowtie$ is a refinement of $\sim_I$; that is, if $c \bowtie\bowtie d$, then $c$ and $d$ have the same cell labels. It follows that the polydiagonal subspace

$$\Delta_{\bowtie\bowtie} = \{ x \in P : x_c = x_d \text{ whenever } c \bowtie\bowtie d \quad \forall c, d \in \mathcal{C} \}$$

is well-defined, since $x_c$ and $x_d$ lie in the same space $P_c = P_d$. The polydiagonal $\Delta_{\bowtie\bowtie}$ is a linear subspace of $P$.

**Definition 4.2** Let $\bowtie\bowtie$ be an equivalence relation on $\mathcal{C}$. Then $\bowtie\bowtie$ is robustly polysynchronous if $\Delta_{\bowtie\bowtie}$ is invariant under every vector field $f \in \mathcal{F}_G^P$. That is,

$$f(\Delta_{\bowtie\bowtie}) \subseteq \Delta_{\bowtie\bowtie}$$
for all $f \in \mathcal{F}_G$. Equivalently, if $x(t)$ is a trajectory of any $f \in \mathcal{F}_G$, with initial condition $x(0) \in \Delta_\infty$, then $x(t) \in \Delta_\infty$ for all $t \in \mathbb{R}$.

We now generalize Theorem 6.5 of [11] to the multiarrow formalism.

**Theorem 4.3** Let $\bowtie$ be an equivalence relation on a coupled cell network. Then $\bowtie$ is robustly polysynchronous if and only if $\bowtie$ is balanced.

**Proof** The proof is essentially the same as that of Theorem 6.5 of [11]. The main points are that it is easy to check directly that $\bowtie$ being balanced is sufficient for $\equiv$ to be robustly polysynchronous, while necessity can be established by considering admissible linear vector fields. We take these points in turn.

First, suppose that $\equiv$ is balanced, and let $f \in \mathcal{F}_G$. Suppose that $c \bowtie d$. By Definition 4.1 the set $B(c,d)$ is non-empty, so there exists $\beta \in B(c,d)$. We have $\beta(c) = d$.

We know that for all $c \in \mathcal{C}$ the component $f_c(x)$ is symmetric under all permutations of the input set $I(c)$ that preserve $\equiv$-equivalence classes. Therefore for any $x \in \Delta_\equiv$

$$f_d(x) = f_d(x_d, x_{\tau(I(d))}) = f_{\beta(c)}(x_d, x_{\tau(I(d))}) = f_c(x_c, \beta^*(x_{\tau(I(c))})) = f_c(x_c, x_{\tau(I(c))}) = f_c(x)$$

because $\beta$ preserves the $\bowtie$-equivalence classes. Therefore $f$ leaves $\Delta_\equiv$ invariant.

For the converse, suppose that $\equiv$ is invariant under all $f \in \mathcal{F}_G$. Then in particular $\equiv$ is invariant under all linear $f \in \mathcal{F}_G$. Let $c \not\equiv d \in \mathcal{C}$ with $c \bowtie d$. We first show that $c \sim_I d$. If not, we can define an admissible linear vector field $f$ such that $f_c = 0, f_d \neq 0$. This contradicts invariance of $\equiv$. Therefore $c \bowtie d$ implies that $c, d$ are input-equivalent as claimed.

Next, we construct a class of admissible linear vector fields as follows. For each pair of $\sim_C$-equivalence classes of cells $([c], [d])$ choose representatives $c, d \in \mathcal{C}$. Choose some linear map

$$\lambda_{dc} : P_d \to P_c$$

If $c' \sim_C c$ and $d' \sim_C d$, use the canonical identifications of $P_{c'}$ with $P_c$ and $P_{d'}$ with $P_d$ to pull back $\lambda_{dc}$ to a linear map

$$\lambda_{d'c'} : P_{d'} \to P_{c'}$$

That is, we ensure that $\lambda_{dc}$ remains ‘the same’ map when cells are replaced by canonically identified cells.

Now choose a transversal $\mathcal{T}$ to the set of $\sim_I$-equivalence classes. That is, arrange for $\mathcal{T}$ to contain precisely one member of each $\sim_I$-equivalence class. For each $t \in \mathcal{T}$ define

$$\Lambda_t(x) = \sum_{i \in I(t)} \lambda_{\tau(i)t}(x_{\tau(i)})$$

If $i, j \in I(t)$ and $(\tau(i), t) \sim_E (\tau(j), t)$ impose the extra condition

$$\lambda_{\tau(i)t} = \lambda_{\tau(j)t}$$

(4.1)
where we canonically identify \( P_{\tau(i)} \) with \( P_{\tau(j)} \). Condition (4.1) ensures that \( \Lambda_t \) is \( B(t,t) \)-invariant.

Any \( c \in C \) is \( \sim_t \)-equivalent to precisely one \( t(c) \in T \). Let \( \beta \in B(t(c),c) \) and use the pullback \( \beta^* \) to define

\[
\Lambda_c(x) = \Lambda_{\beta(t(c))}(x) = \Lambda_t(c)(\beta^*(x))
\]

The \( B(t,t) \)-invariance of \( \Lambda_t(c) \) implies that all \( \beta \in B(t(c),c) \) lead to the same \( \Lambda_c \). Lemma 4.5 of [11], trivially extended to the multiple-arrow formalism, implies that \( \Lambda \) is \( B_{\equiv} \)-equivariant, that is, admissible.

The final preparatory step is to partition the input sets \( I(c) \) according to the \( \sim_E \)-equivalence classes of arrows \( (\tau(i),c) \). Full details (which easily generalize to the multiple-arrow formalism) are at the end of Section 3 of [11] under the heading ‘Structure of \( B(c,d) \)’. Introduce an equivalence relation \( \equiv_c \) on \( I(c) \) for which

\[
j_1 \equiv_c j_2 \iff (\tau(j_1),c) \sim_E (\tau(j_2),c)
\]

and let the \( \equiv_c \)-equivalence classes be \( K^c_0, \ldots, K^c_r \) for \( r = r(c) \). By convention \( K^c_0 = \{c\} \). By Section 3 of [11] the vertex group \( B(c,c) \) is isomorphic to the direct product of symmetric groups \( S_{k^c_j} \) acting on the sets \( K^c_j \), where \( k^c_j = |K^c_j| \).

Let the \( \bowtie \)-equivalence classes be \( A_1, \ldots, A_m \). Let \( X_t \) denote the common value of the components \( x_i \) for \( i \in A_t \). Let \( \mu^c_s \) denote the common value of the \( \lambda_{\tau(j)t(c)} \) for \( j \in K^c_s \). Restrict \( \Lambda \) to \( \Delta_{\bowtie} \). If \( c \in C \) then

\[
\Lambda_c(x) = \sum_{j \in I(c)} \lambda_{\tau(j)t(c)}(x_{\tau(j)})
\]

\[
= \sum_{s=0}^{r(c)} \sum_{j \in K^c_s} \lambda_{\tau(j)t(c)}(x_{\tau(j)})
\]

\[
= \sum_{s=0}^{r(c)} \sum_{i=1}^{m} \sum_{j \in K^c_s \cap A_i} \lambda_{\tau(j)t(c)}(x_{\tau(j)})
\]

\[
= \sum_{s=0}^{r(c)} \sum_{i=1}^{m} \mu^c_s(X_i)
\]

\[
= \sum_{s=0}^{r(c)} \sum_{i=1}^{m} |K^c_s \cap A_i| \mu^c_s(X_i)
\]

Now suppose that \( c \bowtie d \) with \( c \neq d \). Since \( \bowtie \) is robustly synchronous, \( \Lambda_c \) and \( \Lambda_d \) must agree on \( \Delta_{\bowtie} \). Therefore

\[
|K^c_s \cap A_i| = |K^d_s \cap A_i|
\]

whenever \( 0 \leq s \leq r(c) = r(d) \) and \( 1 \leq l \leq m \). This is the ‘cardinality condition’ (6.2) of [11], and it clearly implies that \( \bowtie \) is balanced (use the fact that \( B(c,c) \cong S_{k^c_1} \times \cdots \times S_{k^c_{r(c)}} \), as in the proof of Theorem 6.5 of [11]). \( \square \)
5 Quotient Networks

In this section we show that each balanced equivalence relation \(\bowtie\) of a coupled cell network \(\mathcal{G}\) induces a unique canonical coupled cell network \(\mathcal{G}_{\bowtie}\) on \(\Delta_{\bowtie}\), called the quotient network. This was not the case in the setting of [11] where quotient networks always existed but where there was not always a unique canonical choice. It was shown in [11] in the context of coupled cell systems without self-coupling and multiple arrows that every admissible vector field on the original network restricts to an admissible vector field on \(\Delta_{\bowtie}\) in every quotient network. However, in general admissible vector fields on a quotient network could not be extended to an admissible vector field on the original network.

In the present context admissible vector fields restrict to admissible vector fields and every admissible vector field on the canonical quotient \(\mathcal{G}_{\bowtie}\) lifts to an admissible vector field on \(\mathcal{G}\). We begin by defining the (canonical) quotient network.

(A) Let \(\bar{c}\) denote the \(\bowtie\)-equivalence class of \(c \in \mathcal{C}\). The cells in \(\mathcal{C}_{\bowtie}\) are the \(\bowtie\)-equivalence classes in \(\mathcal{C}\); that is,
\[
\mathcal{C}_{\bowtie} = \{\bar{c} : c \in \mathcal{C}\}
\]
Thus we obtain \(\mathcal{C}_{\bowtie}\) by forming the quotient of \(\mathcal{C}\) by \(\bowtie\), that is, \(\mathcal{C}_{\bowtie} = \mathcal{C}/\bowtie\).

(B) Define
\[
\bar{c} \sim_{\mathcal{C}_{\bowtie}} \bar{d} \iff c \sim_{\mathcal{C}} d
\]
The relation \(\sim_{\mathcal{C}_{\bowtie}}\) is well-defined since \(\bowtie\) refines \(\sim_{\mathcal{C}}\).

(C) The number of input terminals in a quotient cell \(\bar{c}\) is the same as the number of input terminals in cell \(c\), that is, \(|I(\bar{c})| = |I(c)|\). The arrows in the quotient network are the projection of arrows in the original network, that is,
\[
\mathcal{E}_{\bowtie} = \{(\bar{\tau}(i), i) : (\tau(i), i) \in \mathcal{E}\} \quad (5.1)
\]
There is an overenumeration of arrows in the quotient defined by (5.1) since any two distinct \(\bowtie\)-equivalent cells \(d_1\) and \(d_2\) contribute a full complement of arrows to the terminals in \(I(\bar{d}_j)\). We claim that the assignment of arrows is consistent because \(\bowtie\) is balanced. More precisely, since \(\bowtie\) is balanced, there is an input isomorphism \(\beta : I(d_1) \to I(d_2)\). Let \(i \in I(d_1)\) be a terminal head; then \(\beta(i)\) is a terminal head in \(I(d_2)\), and \(\tau(i) \bowtie \tau(\beta(i))\). It follows that \(\tau(i) = \tau(\beta(i))\) and that the arrows \((\tau(i), i)\) and \((\tau(\beta(i)), \beta(i))\) are \(\sim_E\)-equivalent. So the same type of arrow is attached to terminals in \(I(\bar{c})\) regardless of which cell in \(\mathcal{C}\) is used to define those arrows.

Note that when \(d_1 \bowtie d_2\) any arrows with head terminal in \(I(d_2)\) and tail cell \(d_1\) lead to self-coupling arrows in the quotient. Finally, if \(c_1\) and \(c_2\) are distinct \(\bowtie\)-equivalent cells having equivalent arrows with the same terminal cell \(d\), then multiarrows will be present in the quotient network, where a multiarrow is a set of several edge-equivalent arrows between two given cells. See Figure 6.
(D) Two quotient arrows are equivalent when the original arrows are equivalent. That is,

\[
(\tau(i_1), i_1) \sim_{E_{\infty}} (\tau(i_2), i_2) \iff (\tau(i_1), i_1) \sim_E (\tau(i_2), i_2)
\]

(5.2)

where \(i_j\) is a terminal head in cell \(c_j\). The remark in Paragraph (C) on overenumeration of arrows shows that \(\sim_{E_{\infty}}\) is well defined. Now (5.2) implies that when two arrows in \(E_{\infty}\) are \(\sim_{E_{\infty}}\)-equivalent, their head cells \(\overline{c}_1\) and \(\overline{c}_2\) satisfy \(c_1 \sim_C c_2\), and their tail cells \(\overline{\tau}(i_1)\) and \(\overline{\tau}(i_2)\) satisfy \(\tau(i_1) \sim_C \tau(i_2)\). Therefore, \(\overline{c}_1 \sim_{C_{\infty}} \overline{c}_2\) and \(\overline{\tau}(i_1) \sim_{C_{\infty}} \overline{\tau}(i_2)\).

(E) Input isomorphisms on \(G\) project onto input isomorphisms of \(G_{\infty}\).

Let \(\beta : I(c) \to I(d)\) be an input isomorphism between input sets of cells \(c\) and \(d\). Then \(\beta : I(\overline{c}) \to I(\overline{d})\) is also a bijection since \(I(c) = I(\overline{c})\) and \(I(d) = I(\overline{d})\). Identity (5.2) guarantees that (2.2) is valid for \(E_{\infty}\)-equivalence and \(\beta\) is an input isomorphism for \(G_{\infty}\). Identity (5.2) also guarantees the converse — every input equivalence on \(G_{\infty}\) lifts to one on \(G\).

Remark 5.1 Since input isomorphisms project, we see that any quotient of a homogeneous network is also a homogeneous cell network. The quotient of the balanced relation of the five-cell example in Figure 1 and Figure 2 (left) shows that this remark is not valid for quotients in the class of networks considered in [11].

We can now generalize Theorem 9.2 of [11] to the multiarrow formalism:

**Theorem 5.2** Let \(\Delta\) be a balanced relation on a coupled cell network \(G\). The restriction of a \(G\)-admissible vector field to \(\Delta_{\infty}\) is \(G_{\infty}\)-admissible.

**Proof** The proof of Theorem 5.2 is identical to the proof of Theorem 9.2 of [11]. □

Admissible Vector Fields on a Quotient Lift

Having defined the quotient network, we next discuss why the multiarrow formalism implies that every \(G_{\infty}\)-admissible vector field on the quotient lifts to a \(G\)-admissible vector field on the original network. Let \(\tau\) be a quotient cell and suppose that the dynamics on that cell are prescribed by the ODE

\[
\dot{x}_{\tau} = \hat{f}_{\tau}(x_{\tau}, x_{\tau(I(\tau))})
\]

where \(x_{\tau} \in P_{\tau} = P_c\) are the internal state space variables and \(x_{\tau(I(\tau))} \in P_{\tau(I(\tau))} = P_{\tau(I(c))}\) are the coupling variables. We can lift this ODE to each cell \(c\) that quotients onto \(\tau\) by

\[
\dot{x}_c = \hat{f}_c(x_c, x_{\tau(I(c))})
\]

(5.3)

Observe that if \(c \propto d\) (or \(\overline{c} = \overline{d}\)), then there exists an input isomorphism \(\beta : I(c) \to I(d)\). Now \(G_{\infty}\)-admissibility and the fact that input isomorphisms on \(G\) project onto input isomorphisms on \(G_{\infty}\) imply that

\[
\hat{f}_{\tau}(x_{\tau}, x_{\tau(I(\tau))}) = \hat{f}_{\tau}(x_{\tau}, \beta^*(x_{\tau(I(c))}))
\]

Note that if \(c = d\), then (5.3) is consistent since \(f_{\tau}\) is invariant under \(B(\overline{c}, \overline{c})\). Therefore

\[
\hat{f}_c(x_c, x_{\tau(I(d))}) = \hat{f}_c(x_c, \beta^*(x_{\tau(I(d))}))
\]

and the lift (5.3) is \(G\)-admissible.
6 Hyperbolic Equilibria and Balanced Relations

Let \( x_0 = (x_0^1, \ldots, x_0^n) \in P \). Define the equivalence relation \( \equiv_{x_0} \) by \( c \equiv_{x_0} d \) if and only if \( e \sim_{C} d \) and \( x_0^c = x_0^d \). Suppose that we color two cells \( c \) and \( d \) the same color if and only if \( c \equiv_{x_0} d \). Then this coloring is the pattern of synchrony associated to \( x_0 \). Note that

\[
\Delta_{\equiv_{x_0}} = \{ x \in P : x_c = x_d \text{ if } c \equiv_{x_0} d \}
\]

is the smallest subspace of \( P \) that contains of all points with the same pattern of synchrony.

**Definition 6.1** Let \( x_0 \in P \) be a hyperbolic equilibrium of an admissible cell system. The equivalence relation \( \equiv_{x_0} \) is rigid if in each perturbed admissible system the hyperbolic equilibrium near \( x_0 \) remains in \( \Delta_{\equiv_{x_0}} \). We also say that the pattern of synchrony defined by \( x_0 \) is rigid.

**Strong Admissibility.** In Theorem 6.6 we prove that only those patterns of synchrony that are generated by balanced relations are rigid. We prove this theorem by showing that rigid patterns of synchrony lead to flow-invariant subspaces. The following is a key idea in the proof.

**Definition 6.2** A map \( G : P \to P \) is strongly admissible if \( G_c(x) = G_c(x_c) \) for every cell \( c \) and \( G_c = G_d \) for every pair of cells where \( e \sim_C d \).

A strongly admissible map \( G \) is admissible since \( e \sim_I d \) implies \( e \sim_C d \) and hence \( G_c = G_d \).

**Lemma 6.3** Let \( F : P \to P \) be admissible and let \( G : P \to P \) be strongly admissible. Then \( F \circ G \) and \( G \circ F \) are admissible.

**Proof** Both \( (F \circ G)_c \) and \( (G \circ F)_c \) are functions defined on \( P_c \times P_{\tau(I(c))} \). That is,

\[
\begin{align*}
(G \circ F)_c(x_c, x_{\tau(I(c))}) &= G_c(F_c(x_c, x_{\tau(I(c))})) \\
(F \circ G)_c(x_c, x_{\tau(I(c))}) &= F_c(G_c(x_c), G_{\tau(i_1)}(x_{\tau(i_1)}), \ldots, G_{\tau(i_s)}(x_{\tau(i_s)}))
\end{align*}
\]

where \( I(c) = \{i_1, \ldots, i_s\} \).

Let \( \beta : I(c) \to I(d) \) be an input isomorphism in \( B(c, d) \). Order \( I(d) = \{j_1, \ldots, j_s\} \) so that \( \beta(i_k) = j_k \). It follows from the definition of input isomorphism that \( e \sim_I d \) and \( \tau(i_k) \sim_C \tau(j_k) \) for each \( k \). Hence, \( F_c = F_d \), \( G_c = G_d \), and \( G_{\tau(i_k)} = G_{\tau(j_k)} \).

We claim that both \( (F \circ G)_c \) and \( (G \circ F)_c \) are \( \beta \) related to \( (F \circ G)_d \) and \( (G \circ F)_d \). To verify this point for \( G \circ F \), compute

\[
\begin{align*}
(G \circ F)_c(x_c, x_{\tau(I(c))}) &= G_c(F_c(x_c, x_{\tau(I(c))})) \\
&= G_c(F_d(x_d, x_{\tau(I(d))})) \\
&= G_d(F_d(x_d, x_{\tau(I(d))})) \\
&= (G \circ F)_d(x_d, x_{\tau(I(d))})
\end{align*}
\]
Thus \( G \circ F \) is admissible. It also follows that

\[
(F \circ G)_d(x_d, x_{\tau(j_1)}, \ldots, x_{\tau(j_s)}) = F_d(G_d(x_d), G_{\tau(j_1)}(x_{\tau(j_1)}), \ldots, G_{\tau(j_s)}(x_{\tau(j_s)}))
\]

\[
= F_c(G_d(x_d), G_{\tau(j_1)}(x_{\tau(j_1)}), \ldots, G_{\tau(j_s)}(x_{\tau(j_s)}))
\]

\[
= F_c(G_c(x_d), G_{\tau(j_1)}(x_{\tau(j_1)}), \ldots, G_{\tau(j_s)}(x_{\tau(j_s)}))
\]

\[
= (F \circ G)_c(x_d, x_{\tau(j_1)}, \ldots, x_{\tau(j_s)})
\]

Thus \( F \circ G \) is also admissible.

**Definition 6.4** Let \( \bowtie \) be an equivalence relation. A point \( x = (x_1, \ldots, x_N) \in \Delta_{\bowtie} \) is **generic** if \( x_i = x_j \) for \( i \sim_C j \) implies that \( i \sim_{\bowtie} j \).

Observe that generic points are open and dense in \( \Delta_{\bowtie} \).

**Lemma 6.5** Let \( \bowtie \) be an equivalence relation on cells. Let \( x_0 \) be a generic point in \( \Delta_{\bowtie} \) and let \( y_0 \) be any point in \( \Delta_{\bowtie} \). Then there exists a strongly admissible map \( G : P \to P \) such that \( G(x_0) = y_0 \).

**Proof** Let \( x_0 = (x_1, \ldots, x_N) \) and \( y_0 = (y_1, \ldots, y_N) \). We need to choose functions \( G_c \) (where \( G_c = G_d \) whenever \( c \sim_C d \)) so that

\[
G_c(x_c) = y_c \quad (6.1)
\]

Conditions (6.1) are incompatible only when \( c \sim_C d, x_c = x_d, \) and \( y_c \neq y_d \). When these conditions are compatible we can always choose strongly admissible interpolation functions \( G_c \) to satisfy (6.1). The facts \( x_0, y_0 \in \Delta_{\bowtie} \) and \( x_0 \) is generic ensures that conditions (6.1) are compatible because \( x_c = x_d \) implies \( y_c = y_d \).

**Perturbation Spaces and Hyperbolic Equilibria.** Let \( x_0 \in P \). Form the subspace

\[
W_{x_0} = \{ p(x_0) : p \text{ is admissible} \}
\]

Let \( \bowtie_{x_0} \) be the coarsest balanced equivalence relation for which \( x_0 \in \Delta_{\bowtie_{x_0}} \). See Definition 9.1 in the appendix. In particular, \( \Delta_{\bowtie_{x_0}} \) is the smallest robust polysynchronous subspace containing \( x_0 \). Since \( \Delta_{\bowtie_{x_0}} \) is flow-invariant, \( x_0 \in \Delta_{\bowtie_{x_0}} \), and \( p \) is admissible, it follows that \( p(x_0) \in \Delta_{\bowtie_{x_0}} \). Thus, \( W_{x_0} \subseteq \Delta_{\bowtie_{x_0}} \). Equality need not hold, in general. However, equality does hold when the pattern of synchrony defined by a hyperbolic equilibrium \( x_0 \) is rigid.

**Theorem 6.6** The equivalence relation \( \equiv_{x_0} \) determined by the hyperbolic equilibrium \( x_0 \) is rigid if and only if \( \equiv_{x_0} \) is balanced. Moreover, in this case, \( \equiv_{x_0} = \bowtie_{x_0} \) and

\[
W_{x_0} = \Delta_{\equiv_{x_0}} = \Delta_{\bowtie_{x_0}}. \quad (6.2)
\]
Proof  Let $x_0 \in P$ be a hyperbolic equilibrium for an admissible vector field $f$ and assume that $\equiv_{x_0}$ is a balanced equivalence relation. It is straightforward to show that $\equiv_{x_0}$ is rigid. Hyperbolicity implies that every small admissible perturbation $g$ of $f$ will have a unique hyperbolic equilibrium $y_0$ near $x_0$. Since $\Delta_{\equiv_{x_0}}$ is flow-invariant, uniqueness implies that $y_0 \in \Delta_{\equiv_{x_0}}$. Just restrict $f$ and $g$ to $\Delta_{\equiv_{x_0}}$ and use hyperbolicity on this subspace. So the pattern of synchrony of the equilibrium $x_0$ is rigid.

To prove the converse, we assume that $\equiv_{x_0}$ is a rigid equivalence relation. By the definition of $\equiv_{x_0}$, $x_0$ is generic in $\Delta_{\equiv_{x_0}}$. It follows from Lemma 6.5 that $\Delta_{\equiv_{x_0}} = W_{x_0}$ (6.3)

We claim that $\Delta_{\equiv_{x_0}} = W_{x_0}$. To verify this claim, let $p$ be an admissible vector field. Consider the perturbation $f_\varepsilon = f + \varepsilon p$ and denote by $x_\varepsilon$ the perturbed hyperbolic equilibrium for $f_\varepsilon$. So $f_\varepsilon(x_\varepsilon) = 0$ (6.4)

Since rigidity implies $x_\varepsilon \in \Delta_{\equiv_{x_0}}$, it follows that

$$\left. \frac{d}{d\varepsilon} x_\varepsilon \right|_{\varepsilon=0} \in \Delta_{\equiv_{x_0}}$$

Differentiating (6.4) with respect to $\varepsilon$ and evaluating at $\varepsilon = 0$ yields

$$0 = \left. \frac{d}{d\varepsilon} (f(x_\varepsilon) + \varepsilon p(x_\varepsilon)) \right|_{\varepsilon=0} = \left. (Df)_{x_0} \frac{d}{d\varepsilon} x_\varepsilon \right|_{\varepsilon=0} + p(x_0)$$

Thus $p(x_0) \in (Df)_{x_0}(\Delta_{\equiv_{x_0}})$; that is, $W_{x_0} \subset (Df)_{x_0}(\Delta_{\equiv_{x_0}})$. In view of (6.3), we obtain

$$\Delta_{\equiv_{x_0}} \subset W_{x_0} \subset (Df)_{x_0}(\Delta_{\equiv_{x_0}})$$

(6.5)

Since the vector space $\Delta_{\equiv_{x_0}}$ is finite dimensional, (6.5) implies that the inclusions above are all equalities, in particular $W_{x_0} = \Delta_{\equiv_{x_0}}$.

Next we show that $\Delta_{\equiv_{x_0}}$ is flow invariant. It then follows from Theorem 4.3 that $\equiv_{x_0}$ is a balanced relation, as desired. Let $y \in \Delta_{\equiv_{x_0}}$ and $q$ be an admissible vector field. We must show that $q(y) \in \Delta_{\equiv_{x_0}}$. By Lemma 6.5, $y = G(x_0)$ for some strongly admissible vector field $G$, and therefore $q(y) = q \circ G(x_0)$. By Lemma 6.3, $q \circ G$ is an admissible field, and therefore $q(y) \in W_{x_0} = \Delta_{\equiv_{x_0}}$.

Finally we verify the moreover part of the theorem. Since $\Delta_{\equiv_{x_0}}$ is flow-invariant, it follows that $\Delta_{\equiv_{x_0}} \subset \Delta_{\equiv_{x_0}}$. Since $W_{x_0} \subset \Delta_{\equiv_{x_0}}$, (6.2) follows. Hence $\equiv_{x_0} = \sim_{x_0}$.

\[\square\]

7 Identical Edge Homogeneous Networks

An identical-edge homogeneous cell network $G$ is a homogeneous network in which all edges in $E$ are equivalent.
Proposition 7.1 Every quotient of an identical-edge homogeneous network is an identical-edge homogeneous network.

Proof This statement follows directly from Section 5 (D) and Remark 5.1.

Proposition 7.2 Every identical-edge homogeneous network is the quotient of an identical-edge homogeneous network without multiple edges or self-coupling.

Proof We begin by showing that if some cell in an \(N\)-cell network, say cell 1, has \(m\) self-couplings, then we can enlarge the network to an \((N + m)\)-cell network, having the original network as a quotient, so that the enlarged network has no self-couplings in cells residing in the pullback of cell 1. Add arrows and cells to the enlarged network as follows.

1) Replace cell 1 with \(m + 1\) input isomorphic cells.

2) For each arrow in the original network with head cell 1 and tail cell \(i\), where \(i \neq 1\), add \(m\) edge-equivalent arrows with tail cell \(i\), where one of the new arrows terminates in each of the \(m\) new cells.

3) Each pair of the \(m + 1\) cells in the preimage of cell 1 has identical arrows with head terminal in the first cell of the pair and tail cell in the second cell of the pair.

Note that all arrows starting from one of the \(m\) new cells terminate in cell 1. In particular, none of the \(m + 1\) cells in the preimage of cell 1 have self-coupling arrows. In the new network, assign all cells in the preimage of cell 1 the same color, and all other cells different colors. This coloring is balanced, and yields the original network as a quotient network. See Figure 7. Therefore we can enlarge the original network so that it has no self-coupling arrows.

![Figure 7: (Left) Three-cell network with self-coupling. (Right) Four-cell enlargement of original system.](image)

Next, we assume that the network has no self-coupling and that there are \(m\) identical arrows from cell 1 to terminals in cell 2. There is an extended coupled cell network with \(N + m - 1\) cells formed by replacing cell 1 with \(m\) identical cells and changing arrows as follows:
1) Each of the $m$ cells replacing cell 1 connects to cell 2 with one arrow. Note that cell 2 receives the same number of arrows from the $m$ copies of cell 1 that it received previously from the single cell 1 in the original network.

2) Add arrows so that every cell that was connected to cell 1 in the original network is now connected to each of the $m$ cell 1s in the new network.

Note that there are no arrows starting from one of the $m-1$ new cells that terminate in a cell in the original network not equal to cell 2. In the new network, color all cells in the preimage of cell 1 the same color, and all other cells different colors. This coloring is balanced, and yields the original network as a quotient network. See Figure 8. Proceeding inductively, we can eliminate multiple arrows between cells.

![Figure 8: (Left) Three-cell network with multiple arrows. (Right) Four-cell enlargement of original system.](image)

8 Hopf Bifurcation in Two-Color Networks

We now specialize our results to equivalence relations with two colors. We prove a Hopf bifurcation theorem in the case of an identical-edge homogeneous network, with the feature that well-defined approximate phase shifts and approximate amplitude relations hold near bifurcation.

**Proposition 8.1** Suppose that an identical-edge homogeneous network has a balanced relation with two colors. Then there is a unique type of synchrony-breaking Hopf bifurcation from a synchronous equilibrium that leads to periodic solutions that are synchronous on all cells of the same color, and that are approximately one-half period out of phase with all cells of the opposite color. The amplitudes of these periodic signals need not be equal.

**Proof** The two-color balanced relation leads to a two-cell quotient network (one cell for each color). Moreover, Proposition 7.1 implies that the two cells are input isomorphic and all edges are identical. Such two-cell networks are determined by the number of self-coupling arrows $l_j$ on cell $j$ and the number of edges $m_1 \geq 0$ coupling cell 2 to cell 1. Let $m_2 \geq 0$ be the number of edges coupling cell 1 to cell 2; then $l_1 + m_1 = l_2 + m_2 \equiv p$ and $m_2 = m_1 + l_1 - l_2$. 

The coupled cell systems have the form

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_1, \ldots, x_1, x_2, \ldots, x_2) \\
\dot{x}_2 &= f(x_2, x_2, \ldots, x_2, x_1, \ldots, x_1)
\end{align*}
\] 

(8.1)

where \(x_1, x_2 \in \mathbb{R}^k\). Since \(\{x : x_1 = x_2\}\) is flow-invariant we can arrange for the robust existence of an equilibrium in this subspace. Moreover, by a change of coordinates, we can assume that the equilibrium is at the origin. Let \(J\) be the Jacobian matrix of this equilibrium. By (8.1)

\[
J = \begin{bmatrix} A + l_1 B & m_1 B \\ m_2 B & A + l_2 B \end{bmatrix}
\]

where \(A\) is the linearization of the internal dynamics and \(B\) is the coupling matrix. Assume that \(x_1, x_2 \in \mathbb{R}^k\). Let \(v \in \mathbb{R}^k\) and observe that

\[
J \begin{bmatrix} v \\ v \end{bmatrix} = \begin{bmatrix} (A + pB)v \\ (A + pB)v \end{bmatrix} \quad \text{and} \quad J \begin{bmatrix} m_1 v \\ -m_2 v \end{bmatrix} = \begin{bmatrix} (A + (l_2 - m_1)B)m_1 v \\ -(A + (l_2 - m_1)B)m_2 v \end{bmatrix}
\]

Thus, the eigenvalues of \(J\) are given by eigenvalues of the \(k \times k\) matrices \(A + pB\) and \(A + (l_2 - m_1)B\). Either matrix can have purely imaginary eigenvalues when \(k \geq 2\). Critical eigenvalues in the matrix \(A + pB\) leads to periodic solutions that are synchronous on all cells, since the synchrony subspace \(x_1 = x_2\) is flow-invariant.

Synchrony-breaking Hopf bifurcations occur if the matrix \(A + (l_2 - m_1)B\) has (simple) purely imaginary eigenvalues \(\pm \omega i\). Let \(v_0 \in \mathbb{C}^k\) be an eigenvector associated to the eigenvalue \(\omega i\). Then Hopf bifurcation can lead to a branch of periodic solutions that to first order in the bifurcation parameter has the form

\[
x_1(t) = m_1 \text{Re}(e^{i\omega t}v_0) \quad x_2(t) = -m_2 \text{Re}(e^{i\omega t}v_0)
\]

The amplitudes of the time series \(x_1(t)\) and \(x_2(t)\) are different (unless \(m_1 = m_2\)). Indeed to first order they are in the ratio \(m_1 : m_2\) near the bifurcation point. The minus sign in \(x_2\) shows that the time series are (to first order) a half-period out of phase.

\[\Box\]

Example 8.2 Consider the two-cell system in Figure 9 (left). This network can be obtained as a two-color quotient network of the five-cell network in Figure 6 (left) by identifying the four pink and white cells as one color and the cyan cell as the other color. The time series of a periodic state obtained by Hopf bifurcation in this network is shown in Figure 9 (right). Note that the time series from cells 1 and 2 are approximately one-half period out of phase even though the amplitudes of these signals are quite different. The amplitude ratio here is convincingly close to \(|m_1/m_2| = 2\).

When \(m_1 = m_2\) (so \(l_1 = l_2\)) we can say more:
Corollary 8.3 Suppose that an identical-edge homogeneous network has a balanced equivalence relation with two colors, black and white. If the number of white cells coupled to a white cell is equal to the number of black cells coupled to a black cell, then the synchrony-breaking Hopf bifurcation in Proposition 8.1 leads to robust periodic solutions that are synchronous on cells of the same color, and exactly one-half period out of phase with cells of the opposite color.

Proof When \( m_1 = m_2 \) (and hence \( l_1 = l_2 \)) in Proposition 8.1, the transposition \((x_1, x_2) \mapsto (x_2, x_1)\) is a symmetry of (8.1) and the bifurcating states have an exact spatio-temporal symmetry \( x_2(t) = x(t + \frac{T}{2}) \), where \( T \) is the (minimal) period. \( \square \)

Appendix: Lattice of Balanced Equivalence Relations

Let \( C \) be a coupled cell network with graph \( G \), let \( \Lambda_G \) be the set of all balanced equivalence relations on \( G \), and let \( \triangleright_1, \triangleright_2 \in \Lambda_G \). We say that \( \triangleright_1 \) refines \( \triangleright_2 \), denoted by \( \triangleright_1 \prec \triangleright_2 \), if and only if

\[ c \triangleright_1 d \Rightarrow c \triangleright_2 d \]

That is, the partition of \( C \) defined by \( \triangleright_1 \) is finer than that defined by \( \triangleright_2 \) in the sense that for any \( c \in C \)

\[ [c]_1 \subseteq [c]_2 \]

where \([c]_j\) is the \( \triangleright_j \)-equivalence class of \( c \) for \( j = 1, 2 \). Observe that \( \prec \) is a partial ordering on \( \Lambda_G \). We will show that with this partial ordering, the set \( \Lambda_G \) is a lattice (a partially ordered set in which any two elements have a least upper bound and a greatest lower bound, or ‘join’ and ‘meet’, see Davey and Priestley [3]. We will also establish some of the basic properties of this lattice.

Let \( P_c \) denote the cell phase space of \( c \) and let \( P \) be the total phase space. Forming polydiagonals reverses order, in the following sense:

\[ \triangleright_1 \prec \triangleright_2 \quad \Leftrightarrow \quad \Delta_{\triangleright_1} \supseteq \Delta_{\triangleright_2} \quad (9.1) \]
Finest and Coarsest Balanced Equivalence Relations. The finest element of \( \Lambda_G \) is the minimal relation = of equality. Equality is the finest element in the sense that equivalence classes consist of singletons and are as small as possible. Note that \( \Delta_\ast = P \), so the largest possible polydiagonal corresponds to the finest possible balanced equivalence relation.

There also exists a coarsest balanced equivalence relation. The corresponding polydiagonal is the smallest balanced polydiagonal; that is, the intersection of all \( \Delta_\ast \) over all balanced equivalence relations \( \bowtie \). This intersection is non-trivial because equality is a balanced relation. It is flow-invariant under all \( \mathcal{B}_G \)-equivariant vector fields, so must correspond to some balanced equivalence relation, which we denote by \( \bowtie^* \). The relation \( \bowtie^* \) refines \( \sim_I \), but need not equal \( \sim_I \) because \( \sim_I \) need not be balanced. Indeed, if \( G \) is a chain of three identical cells \( 1 \leftarrow 2 \leftarrow 3 \) where the arrows are identical, then the \( \sim_I \)-equivalence classes are \( \{1, 2\} \), \( \{3\} \). However, the corresponding equivalence relation is not balanced because the input sets of cells 1 and 2 are \( \{1, 2\} \) and \( \{2, 3\} \) respectively, and \( 2 \not\sim_I 3 \).

Coarsest Balanced Equivalence Relations Containing \( x \in P \). We next show that for each \( x \in P \), there is a unique coarsest balanced equivalence relation \( \bowtie_x \) such that \( x \in \Delta_{\bowtie_x} \). This equivalence relations is defined in a way analogous to the definition of \( \bowtie^* \).

Definition 9.1 Given a point \( x \in P \). Define \( \bowtie_x \) to be the balanced equivalence relation corresponding to the smallest flow-invariant subspace containing \( x \).

To show that \( \bowtie_x \) exists observe that the intersection of all \( \Delta_{\bowtie} \) over all balanced equivalence relations \( \bowtie \) where \( x \in \Delta_{\bowtie} \) is the smallest subspace that is flow-invariant under all \( \mathcal{B}_G \)-equivariant vector fields. So it must correspond to some balanced equivalence relation, which we denote by \( \bowtie_x \).

Lattice of Balanced Equivalence Relations. We must define two operations on balanced equivalence relations — join and meet. We begin by constructing the join.

Proposition 9.2 If \( \bowtie_1, \bowtie_2 \in \Lambda_G \), then there exists \( \bowtie_3 \in \Lambda_G \) such that
\[
\Delta_{\bowtie_1} \cap \Delta_{\bowtie_2} = \Delta_{\bowtie_3}
\]

Proof Both \( \Delta_{\bowtie_1} \) and \( \Delta_{\bowtie_2} \) are flow-invariant for all \( f \in \mathcal{F}_G^P \). Therefore \( \Delta_{\bowtie_1} \cap \Delta_{\bowtie_2} \) is flow-invariant for all \( f \in \mathcal{F}_G^P \). By Theorem 4.3 there exists a balanced equivalence relation \( \bowtie_3 \) such that \( \Delta_{\bowtie_1} \cap \Delta_{\bowtie_2} = \Delta_{\bowtie_3} \), which is what we wish to prove.

We call \( \bowtie_3 \) the join of \( \bowtie_1, \bowtie_2 \), and write it as
\[
\bowtie_3 = \bowtie_1 \lor \bowtie_2
\]

It is straightforward to verify from the corresponding polydiagonals that \( \bowtie_3 \)-equivalence can be characterized as follows.
Proposition 9.3 Let $\Join_3 = \Join_1 \lor \Join_2$. Then $c \Join_3 d$ if and only if there exists a finite chain $c = c_1, \ldots, c_k = d$
such that for all $j$ with $1 \leq j \leq k - 1$ either $c_j \Join_1 c_{j+1}$ or $c_j \Join_2 c_{j+1}$.
Equivalently, $\Join_3$ is the finest balanced equivalence relation such that
$$c \Join_1 d \text{ or } c \Join_2 d \Rightarrow c \Join_3 d$$

We note the following two properties of joins. In terms of the refinement ordering, we have $\Join_1 \Join_3$ and $\Join_2 \Join_3$. For polydiagonals we have $\Delta_\Join_1 \Join_2 = \Delta_\Join_1 \cap \Delta_\Join_2$. Also note that $\Join_x$ is the join of all balanced equivalence relations $\Join$ such that $x \in \Delta_\Join_x$.

Next we define the meet of two balanced equivalence relations. Define $\Join_4 = \Join_1 \land \Join_2$, the meet of two balanced equivalence relations $\Join_1$ and $\Join_2$, to be the join of all balanced equivalence relations $\Join$ satisfying
$$c \Join d \implies c \Join_1 d \text{ and } c \Join_2 d$$
Then $\Join_4 \Join_1, \Join_4 \Join_2$, and $\Delta_\Join_1 \cup \Delta_\Join_2 \subseteq \Delta_\Join_1 \land \Join_2$.

Proposition 9.4 The pair $(\Lambda_G, \Join)$ is a lattice.

Proof Since $G$ is finite, so is $\Lambda_G$, and ‘lattice’ is the same as ‘complete lattice’. By Theorem 2.16 of Davey and Priestley [3], since $\Lambda_G$ has a top element ($\Join^*$), it is enough to prove that every subset $S$ of $\Lambda_G$ has a unique meet. This follows by induction because any pair of elements has a meet. (Bear in mind that $\land$ is commutative and associative, Davey and Priestley [3] Theorem 5.2.)

Quotients and Balanced Equivalence Relations. Now suppose that $\Join$ is a balanced equivalence relation on $G$ and form the quotient network $G/\Join$. Let $\Lambda_G^\Join$ be the sublattice of $\Lambda_G$ consisting of all balanced equivalence relations $\Join_1$ for which $\Join \Join_1 \Join_1$. The following is straightforward.

Proposition 9.5 There is a natural lattice isomorphism between $\Lambda_G/\Join$ and $\Lambda_G^\Join$.

In closing we mention that there is a bijection between the set of balanced equivalence relations on $G$ and the set of complete subgroupoids of $B_G$, so $\Lambda_G$ can also be interpreted as a lattice of complete subgroupoids. This relates $\Lambda_G$ to the ‘normalizer’ viewpoint of Dias and Stewart [4].

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References


