Received: 18 May 2020 | Accepted: 1 February 2021 | Published on: 6 March 2021

DOI: 10.1002/nav.21982

RESEARCH ARTICLE

WILEY

Markov additive processes for degradation with jumps under dynamic environments

Yin Shu[1] | **Qianmei Feng**[2] | **Edward P. C. Kao**[3] | **David W. Coit**[4] | **Hao Liu**[5]

[1]Science and Technology Division, Corning Incorporated, Corning, New York, USA
[2]Department of Industrial Engineering, University of Houston, Houston, Texas, USA
[3]Department of Mathematics, University of Houston, Houston, Texas, USA
[4]Department of Industrial and Systems Engineering, Rutgers University, Piscataway, New Jersey, USA
[5]Department of Biostatistics, Indiana University, Indianapolis, Indiana, USA

Correspondence
Qianmei Feng, Department of Industrial Engineering, University of Houston, E206 Engineering Building 2, Houston, TX 77204, USA.
Email: qfeng@central.uh.edu

Funding information
Texas Norman Hackerman Advanced Research Program, Grant/Award Number: 003652-0122-2009. USA National Science Foundation, Grant/Award Numbers: 0969423, 0970140, 1728321.

Abstract

We use general Markov additive processes (Markov modulated Lévy processes) to integrally handle the complexity of degradation including internally-induced and externally-induced stochastic properties with complex jump mechanisms. The background component of the Markov additive process is a Markov chain defined on a finite state space; the additive component evolves as a Lévy subordinator under a certain background state, and may have instantaneous nonnegative jumps occurring at the time the background state switches. We derive the Fokker–Planck equations for such Markov modulated processes, based on which we derive Laplace expressions for reliability function and lifetime moments, represented by the infinitesimal generator matrices of Markov chain and the Lévy measure of Lévy subordinator. The superiority of our models is their flexibility in modeling degradation data with jumps under dynamic environments. Numerical experiments are used to demonstrate that our general models perform well.

KEYWORDS

dynamic environments, Fokker–Planck equations, infinitesimal generator, Lévy measures, Lévy processes, Markov additive processes, reliability

1 | INTRODUCTION

Reliability of systems is one of the major concerns in many fields including energy, health, aerospace, national defense, and so forth. In investigating reliability, unavoidable degradation is one of the major failure mechanisms of the systems, taking the form of damage, corrosion, erosion, fatigue crack, deterioration or wear, and so forth. During the life of many critical systems (e.g., wind turbines, drilling equipment, power/smart grids, mechanical devices, etc.), there are some external time-varying variables/factors that continuously govern the progress of the stochastic degradation of the systems. Such variables are called stochastic covariates (e.g., dynamic environments such as temperature, humidity, or vibration). Incorporating this externally-induced uncertainty together with internally-induced uncertainty in modeling degradation is a challenging research work, especially when there are many complex jumps stemming from both internal features (mechanical, thermal, electrical, or chemical) of the system and instantaneous state changes of external variables/factors. The majority of published research in stochastic degradation modeling has assumed that the degradation evolves under a deterministic environment. Considering external factors, Wiener-based stochastic covariate models in Ebrahimi (2001), Markov modulated linear processes and Markov modulated compound Poisson processes in Kharoufeh et al. (Kharoufeh, 2003; Kharoufeh et al., 2006, 2013; Kharoufeh & Cox, 2005; Kharoufeh & Mixon, 2009) were studied recently. Poisson process is a special case of Lévy process, thus the linear-based and Poisson-based stochastic models are not flexible in general cases (Shu et al., 2015, 2016, 2019).

To integrally handle the complexities of degradation including both internally-induced and externally-induced stochastic properties with complex jump mechanisms, we propose to develop degradation models under dynamic environments using a broad class of general Markov additive processes (Markov modulated Lévy processes), where the background component is a Markov chain with finite states, the additive component evolves as a Lévy subordinator under a certain background state, and may have instantaneous non-negative jumps occurring at the time the background state switches. We develop the Fokker–Planck equations of such analytically appealing stochastic processes in order to derive reliability characteristics. We also develop systematic procedures for deriving and obtaining the explicit and powerful results, represented by infinitesimal generator matrices and Lévy measures. Using Markov modulated Lévy processes, the superiority of our general models stems from their flexibility in modeling stylized features of degradation data series under dynamic environments such as jumps fluctuation, symmetry/asymmetry, and light/heavy tails. Our results are expected to provide accurate reliability prediction and estimation, by realizing multiple uncertainty sources of degradation mechanisms.

Without considering external factors, stochastic processes such as Wiener processes, gamma processes and compound Poisson processes are directly used to represent degradation processes when the degradation is observable (see Esary et al., 1973; Lawless & Crowder, 2004; Si et al., 2013; Tang & Su, 2008; Tsai et al., 2011). To conduct reliability analysis, the failure time is defined as the first passage time of the degradation process. When the degradation is unobservable, it is treated as a latent process, measured and tracked by internal stochastic covariates that are observable marker processes (see Jewell & Kalbfleisch, 1996; Lee et al., 2000; Shi et al., 1996; Singpurwalla, 2006; Whitmore et al., 1998). These markers (e.g., diagnostic factors such as mileage traveled of an auto) provide information about the progress of degradation processes that can be used to infer the reliability function or the hazard function. To conduct reliability/survival analysis, Lee et al. (2000) and Whitmore et al. (1998) used a bivariate Wiener process to describe the correlation of the degradation process and the marker process, and then formulated the reliability function based on the first passage time of the Wiener process. Some models directly defined the hazard function as an explicit function of the marker process (see Jewell & Kalbfleisch, 1996; Shi et al., 1996; Singpurwalla, 2006).

In biostatistics, the marker processes are stochastic processes representing time-varying covariates that track the health of a system under study in the language of Kalbfleisch and Prentice (2002). Jewell et al. (Jewell & Kalbfleisch, 1996; Jewell & Nielsen, 1993) considered the marker processes as associated variables that continuously measure the progress of an individual toward the final expression of the disease (failure). Assuming a simple additive model for the relationship between the marker process and the hazard function, the survival distribution of time to failure was expressed, where the Poisson process was used to represent the marker process. Yashin and Manton (1997) reviewed models in survival analysis under the framework that the hazard function explicitly represents the effects of markers. Typically they discussed the model where the marker processes are Wiener-based diffusion processes, where the relationship between the hazard function and the markers is quadratic. Fusaro et al. (1993) constructed the model using a nonparametric frame to describe the dependency of the hazard on marker variables. Regarding the efficient use of marker information, Malani (1995) proposed a heuristic approach in estimating parameters of survival functions. Shi et al. (1996) studied the distributions of the residual time in acquired immune deficiency syndrome diagnosis based on markers that carry valuable information about disease progression. They derived the residual time distribution for several combinations of marker processes and marker-dependent hazard functions. However, all these stochastic models just represent internally-induced uncertainty with temporal variability.

Considering the effects of external factors, Ebrahimi (2001) presented a stochastic covariate failure model for assessing system reliability, where external stochastic covariates were modeled by Wiener-based diffusion processes. The life distribution was assumed to be explicitly related to such stochastic covariates. However, this work cannot handle the random jumps in degradation.

Markov additive processes are a class of binary stochastic processes with one component as an additive process (e.g., Lévy process) that is modulated by the other component, which is a standard Markov process (see Çınlar, 1972a, 1972b, 1977). They can integrally handle the complexities of degradation processes under dynamic environments. Special Markov additive processes, including Markov modulated linear processes and Markov modulated compound Poisson processes, have been used to represent the linear deterministic degradation with Poisson-type jumps under discrete and finite state Markov environments (see Kharoufeh, 2003; Kharoufeh et al., 2006, 2013; Kharoufeh & Cox, 2005; Kharoufeh & Mixon, 2009). The explicit results were derived based on the nature of the Poisson process. We propose an extension of such models using Markov modulated Lévy processes.

The organization of this paper is as follows. In Section 2, we describe the model construction. In Section 3, we derive the Fokker–Planck equations of general Markov additive processes. In Section 4, we derive the explicit expressions of reliability function and lifetime moments for systems subject to degradation under the dynamic environment. Numerical examples are illustrated in Section 5, and conclusions are given in Section 6.

2 | PRELIMINARIES

In this section, we introduce some mathematical fundamentals related to Lévy processes, followed by model construction for degradation phenomenon under dynamic environments based on Markov modulated Lévy processes.

2.1 | Lévy–Itô decomposition

Lévy processes provide a potential candidate to describe a broad class of degradation with random jumps. The theories of Lévy processes have been well introduced in Applebaum (2009) and Sato (1999), and they have been widely applied in the fields of economics and finance (see Cont & Tankov, 2004; Fusai & Kyriakou, 2016). Abdel-Hameed (1984) studied the life distribution properties of devices subject to Lévy degradation. Under deterministic environments, Shu et al. (2015, 2016) gave explicit results of reliability function for degradation described by Lévy subordinators and their functional extensions as a class of nondecreasing processes. Their results demonstrated the advantage of using Lévy subordinators as a realistic model for many physical degradation phenomena.

The stochastic processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a standard, right-continuous and augmented filtration $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$. Let $\mathbb{R}^d$ denote the Euclidean space of dimension d, $\mathbb{R}^+$ denote $[0, \infty)$ and $|x| = \sqrt{x \cdot x}$ denote the Euclidean norm for $x \in \mathbb{R}^d$. We begin with the definition of Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with mean $\text{Leb} \times \nu$, where Leb is the Lebesgue measure and ν is a Lévy measure, that is, $\nu\{0\} = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. Let $\theta = \{\theta_t, t \geq 0\}$ be a semigroup of time-shift operator $\theta_t : \omega \mapsto \theta_t \omega$ from Ω to Ω such that $\theta_0 \omega = \omega$ and $\theta_u(\theta_t \omega) = \theta_{u+t}\omega$.

Definition 1 (Çınlar, 2011). A random measure N on $\mathbb{R}^+ \times \mathbb{R}^d$ is called a Poisson random measure with Lévy measure ν if.

- For every Borel subset A of $[0, t] \times \mathbb{R}^d$, $N(A)$ is $\mathcal{F}_t$ measurable;
- $N(\theta_t \omega, B) = N(\omega, B_t)$ for every $\omega \in \Omega$, $t \geq 0$ and Borel subset B of $\mathbb{R}^+ \times \mathbb{R}^d$, where $B_t = \{(t+u) : (u, x) \in B\}$; and
- N is Poisson with mean $Leb \times \nu$.

The Poisson random measure N is said to have the intensity measure $\text{Leb} \times \nu$ with values in $\overline{Z}_+ = \{0,1,2, \dots, +\infty\}$. Let $\mathbb{B} = \{x \in \mathbb{R}^d : |x| \leq 1\}$ be the closed unit ball in $\mathbb{R}^d$, and $\mathbb{B}^c = \{x \in \mathbb{R}^d : |x| > 1\}$ be its complement. The following theorem describes the celebrated Lévy–Itô decomposition (Çınlar, 2011):

Theorem 1 (The Lévy–Itô decomposition Çınlar, 2011)**.** *A process X on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a Lévy process if and only if for every* $t \in \mathbb{R}^+$,

$$X(t) = bt + aW(t) + \int_{[0,t]\times\mathbb{B}} x\{N(s, dx) - ds\nu(dx)\} + \int_{[0,t]\times\mathbb{B}^c} xN(s, dx),$$

for some $b \in \mathbb{R}^d$, *some* $d \times d'$ *covariance matrix* a, *some* d'*-dimensional Wiener process* W, *and a Poisson random measure* N *on* $\mathbb{R}^+ \times \mathbb{R}^d$ *with some Lévy measure* ν *that is independent of* W.

A Lévy subordinator is a one-dimensional Lévy process that is nondecreasing almost surely. Using Lévy–Khintchine formula (Sato, 1999), a Lévy subordinator has the following property:

Corollary 1 (Sato, 1999)**.** *Let* $d = 1$. *A Lévy process is a subordinator if and only if* $a = 0$, $\nu(-\infty, 0] = 0$,

$$\int_0^\infty (x \wedge 1)\nu(dx) < \infty, \qquad \textit{and the drift}$$

$$\overline{b} \equiv b - \int_0^1 x\nu(dx) \geq 0.$$

By Theorem 1 (the Lévy–Itô decomposition) and Corollary 1, a Lévy subordinator $X(t)$ can be written as

$$X(t) = \overline{b}\ t + \int_{[0,1]\times(0,\infty)} xN(ds, dx).$$

2.2 | Model construction

We consider a system subject to degradation with random jumps, which is a process of stochastically continuous degradation with sporadic jumps that occur at random times and have random sizes. In addition, the degradation process is modulated by the environment process. To model the evolution of this type of degradation process, we use Markov additive processes $\{X(t), E(t)\}$ as follows. The cumulative degradation by time t is represented by a nondecreasing continuous time càdlàg (right continuous with left limits) Markov modulated Lévy process $X(t)$, and the modulating process is the environment process, represented by a temporally homogeneous continuous time càdlàg Markov jump process $E(t)$ with finite state space $\mathcal{E} = \{0, 1, \dots, n\}$. Let $G = (r_{ij})$, $r_{ii} = -\sum_{j\neq i} r_{ij}$, $i, j \in \mathcal{E}$ denote the transition rate matrix (infinitesimal generator matrix) of $E(t)$.

More precisely, the bivariate process $\{X(t), E(t)\}$ is a Markov additive process, where conditional on $E(t)$, the conditional law of $X(t)$ evolves as a nondecreasing Lévy process, that is, a Lévy subordinator. Given $E(t) = i \in \mathcal{E}$ during an interval $[t, t+s)$, the characteristics of $X(t)$ are functions of $E(t)$, modeled as

$$b_{E(t)} = b(E(t)) = b(i), \qquad \nu_{E(t)}(x) = \nu_i(x) = \nu(i, x).$$

In practice, the changes of environment states, such as instantaneous temperature increase or decrease, can induce certain damages to the system, modeled by the jumps in the degradation process. Therefore, we assume there is an additional random nonnegative jump in $X(t)$ when the state of $E(t)$ changes. When $E(t)$ changes from state i to state j, the distribution of the jump is denoted as $D_{ij}(z)$, defined on R^+. For $i = j$, $D_{ij}(dz) = \delta_z(0)$, which is a Dirac delta function. When the state space $\mathcal{E}$ is finite, the class of Markov additive process $\{X(t), E(t)\}$ is well understood (see Asmussen, 2003). Without the loss of generalization, assume the initial state $X(0) = 0$, $E(0) = 0$ a.s., and it is easy to extend the results to the case when $X(0) = c$, $E(0) = k$, $c \in R^+$, $k \in \mathcal{E}$.

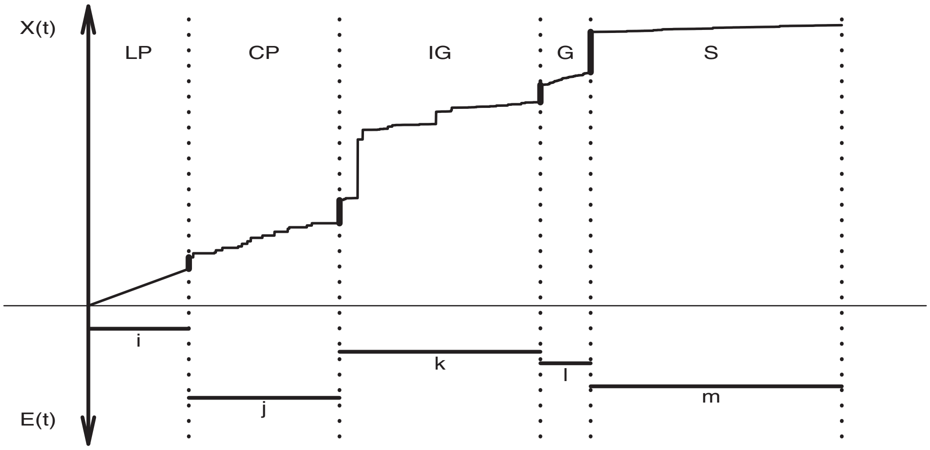

FIGURE 1 A sample path of Markov additive process with a random jump when the environment states changes

To integrally handle internally-induced and externally-induced stochastic properties with complex jump mechanisms, $X(t)$ can be expressed as:

$$X(t) = \int_0^t b(E(\xi-))d\xi + \int_0^t \int_{0<x\leq 1} x(N(E(\xi-), d\xi, dx) - \nu(E(\xi-), dx)d\xi) + \int_0^t \int_{y>1} xN(E(\xi-), d\xi, dx) + \sum_{\xi\in[0,t]} M_{E(\xi-),E(\xi)},$$

where $M_{E(\xi-),\,E(\xi)}$ is a random variable following the distribution $D_{E(\xi-),\,E(\xi)}(z)$, and independent of $E(\xi)$, for all $\xi \in [0,\, t]$. In $X(t)$, under a certain state of $E(t)$, the internally-induced stochastic properties is modeled by a certain Lévy process. One of the most important advantages of using Lévy processes is that their jump parts represented by Lévy measures can model a great deal of jump mechanisms in degradation. At different states of $E(t)$, $X(t)$ may evolve in different patterns with different jump mechanisms that can be modeled by different Lévy processes, representing externally-induced stochastic properties. In addition, instantaneous nonnegative jumps induced by the change in $E(t)$ are also properly modeled by a random distribution (see Figure 1). As illustrated in Figure 1, when $E(t) = i$, $X(t)$ evolves as a linear process (LP); when $E(t) = j$, $X(t)$ evolves as a compound Poisson process (CP); when $E(t) = k$, $X(t)$ evolves as an inverse Gaussian process (IG); when $E(t) = l$, $X(t)$ evolves as a gamma process (G); and when $E(t) = m$, $X(t)$ evolves as a stable process (S).

3 | FOKKER–PLANCK EQUATIONS FOR MARKOV ADDITIVE PROCESSES

As the partial differential equation of the probability density function, the Fokker–Planck equation describes the time evolution of probability density for stochastic processes, and is thus useful in quantifying random phenomena such as uncertainty propagation (see Risken, 1996; Sun & Duan, 2012). It provides us a way to analyze the probability laws for stochastic processes of interests, especially for those without closed-form distributions. Without an analytical expression of the probability law for $\{X(t), E(t)\}$, the development of the characteristics for such processes and the subsequent reliability function is a nontrivial work, even for simple cases. The difficulty stems from (1) the stochastic evolution of degradation has complex mechanisms such as random jumps, (2) the stochastic nature of environment, and (3) the distributional derivation for the first passage time. We overcome this challenge by deriving the Fokker–Planck equation of $\{X(t), E(t)\}$.

Under the model construction of our Markov additive processes $\{(X(t), E(t))\}$, the environmental process $E(t)$ is a continuous-time homogeneous Markov Chains with finite state space. We further assume that $E(t)$ is a regular jump process so that whenever it jumps to a new state, it can stay at the new state at least for a short random duration. A jump process is a regular jump process if it only has finite many jumps in $[0, t]$ for every $t > 0$. This is a general class of continuous-time Markov Chains that are very practical in applications. A regular jump process is stable and conservative so that it has a density. Conditional on $E(t)$, $X(t)$ is a nondecreasing Lévy process. We further assume a sufficient condition on the Lévy measure $\nu(\cdot)$ to ensure the existence and smoothness of the probability density for the Lévy process

$$\liminf_{\varepsilon\to 0} \frac{\int_{[-\varepsilon,\varepsilon]} x^2 \nu(dx)}{\varepsilon^{2-a}} > 0,$$

for some $0 < a < 2$; more details can be found in Sato (1999, Proposition 28.3, p. 190).

Under these model specifications and assumptions, there exists the joint probability density function $p(x,\ i,\ t)$ of the bivariate stochastic processes $\{X(t), E(t)\}$. We can then derive the Fokker–Planck equation as

$$\frac{\partial p(x,i,t)}{\partial t} = L^* p(x,i,t),$$

where L^* is the adjoint operator of the infinitesimal generator of $\{X(t), E(t)\}$, that is,

$$\int_R Lf(x)g(x)dx = \int_R f(x)L^* g(x)dx.$$

It is important to derive the adjoint operator, and our main result is given in this section. The Fokker–Planck equation is derived and presented in Theorem 2.

Theorem 2 *For the Markov additive process* $\{X(t), E(t)\}$ *described in Section 2.2, the Fokker–Planck equation is*

$$\frac{\partial p(x,i,t)}{\partial t} = -b(i)\frac{\partial}{\partial x}p(x,i,t) + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} p(x-z,j,t)D_{ji}(dz)$$
$$+ \int_{R^+}\left(p(x-y,i,t) - p(x,i,t) + I_{y\in(0,1)}y\frac{\partial}{\partial x}p(x,i,t)\right)\nu(i,dy).$$

Proof *Step* 1: For each $f \in C_0^\infty(R^2)$ (f is a smooth function and compactly supported), and for each $t>0$, we aim to derive $f(X(t+\Delta t), E(t+\Delta t)) - f(X(t), E(t))$.

Both $X(t)$ and $E(t)$ are càdlàg processes. We define $X(\xi-)$ and $E(\xi-)$ as the left limits at the time point ξ, $S = [t, t+\Delta t]$, $S_1 = \{\xi \in S: E(\xi) - E(\xi-) = 0\}$, and $S_2 = \{\xi\in S: E(\xi)-E(\xi-)\neq 0\}$. Then we have

$$\begin{aligned} f(X(t+\Delta t), E(t+\Delta t)) - f(X(t),E(t)) &= \sum_{\xi\in S} f(X(\xi),E(\xi)) - f(X(\xi-),E(\xi-)) \\ &= \sum_{\xi\in S_1} f(X(\xi),E(\xi)) - f(X(\xi-),E(\xi)) \\ &\quad + \sum_{\xi\in S_2} f(X(\xi),E(\xi)) - f(X(\xi-),E(\xi-)). \end{aligned} \tag{1}$$

In a continuous time interval $s\subseteq S_1$, if $E(\xi)=e$, $\xi\in s$, $e\in\mathcal{E}$, $dX(\xi)$ has a constant part $dX_C(\xi) = (b(e) - \int_{0<y<1} y\nu(e,dy))d\xi$, and a random jump part $dX_J(\xi) = \int_{R^+} yN(e,d\xi,dy)$. For X_J, we define τ_m, $0\le m\le M$, $m\in N$, $M\in N$ as the time of the m^{th} jump, $\tau_0 = inf\{\xi: \xi\in s\}$, $\tau_m = inf\{\xi: \xi > \tau_{m-1} \ \&\ \Delta X_J(\xi)>0\}$, where $\Delta X_J(\xi) = X_J(\xi) - X_J(\xi-)$, and $\tau = sup\{\xi:\xi\in s\}$. Then

$$\begin{aligned} \sum_{\xi\in s} f(X(\xi),E(\xi)) - f(X(\xi-),E(\xi)) &= f(X(\max\{\tau_M,\tau-\}),e) - f(X(\tau_M),e) \\ &\quad + \sum_{m=1}^{M}(f(X(\tau_m),e) - f(X(\tau_{m-1}),e)) \\ &= f(X(\max\{\tau_M,\tau-\}),e) - f(X(\tau_M),e) \\ &\quad + \sum_{m=1}^{M}(f(X(\tau_m-)+\Delta X_J(\tau_m),e) - f(X(\tau_m-),e)) \\ &\quad + \sum_{m=1}^{M}(f(X(\tau_m-),e) - f(X(\tau_{m-1}),e)). \end{aligned}$$

Based on the stochastic integration (see Chapter 4 in Applebaum, 2009), we have

$$\begin{aligned} \sum_{\xi\in S_1} f(X(\xi),E(\xi)) - f(X(\xi-),E(\xi)) &= \int_{\xi\in S_1} b(E(\xi))\frac{\partial f}{\partial x}(X(\xi-),E(\xi))d\xi \\ &\quad - \int_{\xi\in S_1}\int_{0<y<1} y\frac{\partial f}{\partial x}(X(\xi-),E(\xi))\nu(E(\xi),dy)d\xi \\ &\quad + \int_{\xi\in S_1}\int_{R^+}(f(X(\xi-)+y,E(\xi)) - f(X(\xi-),E(\xi))) \\ &\quad N(E(\xi),d\xi,dy). \end{aligned}$$

Then (1) becomes

$$\begin{aligned} f(X(t+\Delta t),E(t+\Delta t)) - f(X(t),E(t)) &= \int_{\xi\in S_1} b(E(\xi))\frac{\partial f}{\partial x}(X(\xi-),E(\xi))d\xi \\ &\quad - \int_{\xi\in S_1}\int_{0<y<1} y\frac{\partial f}{\partial x}(X(\xi-),E(\xi))\nu(E(\xi),dy)d\xi \\ &\quad + \int_{\xi\in S_1}\int_{R^+}(f(X(\xi-)+y,E(\xi)) - f(X(\xi-),E(\xi))) \\ &\quad N(E(\xi),d\xi,dy) \\ &\quad + \sum_{\xi\in S_2}(f(X(\xi-)+M_{E(\xi-),E(\xi)},E(\xi)) - f(X(\xi-),E(\xi-))). \end{aligned}$$

Notice that for $\xi\in S_1$, both $E(\xi)$ and $X(\xi-)$ are predictable. Our calculus is in the Itô form.

Step 2: For each $f\in C_0^\infty(R^2)$, we aim to derive the infinitesimal generator L of $\{X(t),E(t)\}$:

$$Lf(x,i) = \lim_{\Delta t\to 0}\frac{E(f(X(t+\Delta t),E(t+\Delta t))|X(t)=x, E(t)=i) - f(x,i)}{\Delta t}. \tag{2}$$

As $E(t)$ has the transition rate matrix (infinitesimal generator matrix) $G = (r_{ij}), r_{ii} = -\sum_{j\neq i} r_{ij}$, defining $P(E(t+\Delta t)=j|\,E(t)=i) = P_{ij}(\Delta t)$, we have

$$\begin{aligned} &\lim_{\Delta t\to 0}\frac{E\left(\sum_{\xi\in S_2}(f(X(\xi)+M_{E(\xi-),E(\xi)},E(\xi)) - f(X(\xi-),E(\xi-)))|X(t)=x,E(t)=i\right)}{\Delta t} \\ &= \lim_{\Delta t\to 0}\frac{\sum_{j\neq i}\int_{R^+}(f(x+z,j)-f(x,i))D_{ij}(dz)P_{ij}(\Delta t)}{\Delta t} \\ &= \sum_{j\neq i} r_{ij}\int_{R^+}(f(x+z,j)-f(x,i))D_{ij}(dz) \\ &= \sum_{j\in\mathcal{E}} r_{ij}\int_{R^+} f(x+z,j)D_{ij}(dz). \end{aligned}$$

Since the Poisson random measure $N(dt,dy)$ has a Poisson distribution with mean $\nu(dy)dt$, we have

$$\begin{aligned} Lf(x,i) &= b(i)\frac{\partial}{\partial x}f(x,i) - \int_{0<y<1} y\frac{\partial}{\partial x}f(x,i)\nu(i,dy) \\ &\quad + \int_{R^+}(f(x+y,i) - f(x,i))\nu(i,dy) \\ &\quad + \sum_{j\in\mathcal{E}} r_{ij}\int_{R^+} f(x+z,j)D_{ij}(dz). \end{aligned} \tag{3}$$

Step 3: We aim to derive L^*, the adjoint operator corresponding to the infinitesimal generator L in (3):

$$\sum_{i\in\mathcal{E}}\int_{R^+} Lf(x,i)p(x,i,t)dx = \sum_{i\in\mathcal{E}}\int_{R^+} f(x,i)L^*p(x,i,t)dx. \quad (4)$$

Using integration by parts, as $p(0, i, t) = 0$, and $p(\infty, i, t) = 0$, we have

$$\sum_{i\in\mathcal{E}}\int_{R^+} b(i)\frac{\partial}{\partial x}f(x,i)p(x,i,t)dx = \sum_{i\in\mathcal{E}}\int_{R^+} b(i)p(x,i,t)df(x,i)$$
$$= \sum_{i\in\mathcal{E}} b(i)p(x,i,t)f(x,i)\Big|_{R^+} - \sum_{i\in\mathcal{E}}\int_{R^+} b(i)f(x,i)dp(x,i,t)$$
$$= \sum_{i\in\mathcal{E}}\int_{R^+} -b(i)\frac{\partial}{\partial x}p(x,i,t)f(x,i)dx, \quad (5)$$

and

$$\sum_{i\in\mathcal{E}}\int_{R^+}\int_{R^+}(f(x+y,i)-f(x,i))\nu(i,dy)p(x,i,t)dx$$
$$-\sum_{i\in\mathcal{E}}\int_{R^+}\int_{R^+} I_{y\in(0,1)}y\frac{\partial}{\partial x}f(x,i)\nu(i,dy)p(x,i,t)dx$$
$$= \sum_{i\in\mathcal{E}}\int_{R^+}\int_{R^+}(p(x-y,i,t)-p(x,i,t))\nu(i,dy)f(x,i)dx$$
$$+\sum_{i\in\mathcal{E}}\int_{R^+}\int_{R^+} I_{y\in(0,1)}y\frac{\partial}{\partial x}p(x,i,t)\nu(i,dy)f(x,i)dx. \quad (6)$$

By swapping i and j, we have

$$\sum_{i\in\mathcal{E}}\int_{R^+}\sum_{j\in\mathcal{E}} r_{ij}\int_{R^+} f(x+z,j)D_{ij}(dz)p(x,i,t)dx$$
$$= \sum_{i\in\mathcal{E}}\int_{R^+}\sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} p(x-z,j,t)D_{ji}(dz)f(x,i)dx. \quad (7)$$

Then from (5), (6) and (7), we have

$$L^*p(x,i,t) = -b(i)\frac{\partial}{\partial x}p(x,i,t) + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} p(x-z,j,t)D_{ji}(dz)$$
$$+\int_{R^+}\left(p(x-y,i,t)-p(x,i,t)+I_{y\in(0,1)}y\frac{\partial}{\partial x}p(x,i,t)\right)\nu(i,dy).$$

For each $f \in C_0^\infty(R^2)$, we denote $u(t) = E(f(X(t),E(t)))$. Then based on (2), we have $Lu(t) = \frac{\partial}{\partial t}u(t)$ (see Sun & Duan, 2012), and

$$\sum_{i\in\mathcal{E}}\int_{R^+} Lf(x,i)p(x,i,t)dx = \sum_{i\in\mathcal{E}}\int_{R^+}\frac{\partial}{\partial t}(f(x,i)p(x,i,t))dx.$$

From (4), we have

$$\sum_{i\in\mathcal{E}}\int_{R^+} f(x,i)L^*p(x,i,t)dx = \sum_{i\in\mathcal{E}}\int_{R^+}\frac{\partial}{\partial t}(f(x,i)p(x,i,t))dx,$$

then

$$\frac{\partial p(x,i,t)}{\partial t} = L^*p(x,i,t). \quad \blacksquare$$

To make comparison, Corollary 2 provides the Fokker–Planck equation of $\{X(t),E(t)\}$ for the case that there is no jump in $X(t)$ when the state of $E(t)$ changes.

Corollary 2 *For the Markov additive process $\{X(t),E(t)\}$, assuming there is no jump in $X(t)$ when the state of $E(t)$ changes, the Fokker–Planck equation is*

$$\frac{\partial p(x,i,t)}{\partial t} = -b(i)\frac{\partial}{\partial x}p(x,i,t) + \sum_{j\in\mathcal{E}} r_{ji}p(x,j,t)$$
$$+\int_{R^+}\left(p(x-y,i,t)-p(x,i,t)+I_{y\in(0,1)}y\frac{\partial}{\partial x}p(x,i,t)\right)\nu(i,dy).$$

4 | RELIABILITY FUNCTION AND LIFETIME MOMENTS

A system fails when the degradation process $X(t)$ exceeds a failure threshold x. To simplify the formula, we assume x is a constant, and it is straightforward to extend the model when the failure threshold is a random variable. The lifetime of the system and its moments are defined respectively as

$$T_x = inf\{t : X(t) > x\}, M(T^n, x) = E(T_x^n).$$

Since $X(t)$ is nondecreasing, we have

$$\{T_x \ge t\} \equiv \{X(t) \le x\},$$

then the reliability function can be defined as

$$R(x,t) = P(T_x \ge t) = P(X(t) \le x) = F_{X(t)}(x).$$

In this section, for a degradation process under the dynamic environment described by the Markov additive process $\{X(t),E(t)\}$, we derive the explicit expressions of $R(x, t)$ and lifetime moments $M(T^n, x)$ in terms of Laplace transform, represented by the infinitesimal generator matrix and the Lévy measure.

Laplace transform of $p(x, i, t)$ with respect to (w.r.t.) t is defined to be

$$p^L(x,i,\omega) = \int_{R^+} e^{-\omega t}p(x,i,t)dt, \quad \omega > 0.$$

Laplace transform of $p^L(x, i, \omega)$ w.r.t. x is

$$p^{LL}(u,i,\omega) = \int_{R^+} e^{-ux}p^L(x,i,\omega)dx, \quad u > 0.$$

The results are presented in Theorems 3 and 4.

Theorem 3 *For a degradation process under the dynamic environment that is described by the Markov additive process $\{X(t),E(t)\}$ in Section 2.2, the Laplace expression of reliability function is*

$$R^{LL}(u,\omega) = u^{-1}[1,0,\dots,0][\mathbf{A}-\mathbf{B}]^{-1}[1,1,\dots,1]^T,$$

where $\mathbf{A}$ *is a diagonal matrix with diagonal entries* $\omega + b^*(i)u - \int_{R^+}(e^{-uy}-1)\nu(i,dy)$, *and* $\mathbf{B} = [r_{ij}d_{ij}^L]$, $i,j\in\mathcal{E}$. *In addition,* $b^*(i) \ge 0$, ν *is the Lévy measure,* r_{ij}, $i,j\in\mathcal{E}$ *are entries of the infinitesimal generator matrix of* $E(t)$, $d_{ji}^L(u) = \int_{R^+} e^{-uz}D_{ji}(dz)$, $[1, 0, \dots, 0]$ *is a vector of size*

$n+1$, *where the first element is* 1 *and all others are* 0, *and* $[1, 1, \dots, 1]$ *is a vector of size* $n+1$, *where all the elements are* 1.

Proof Based on Theorem 2, the Fokker–Planck equation for $\{X(t), E(t)\}$ is

$$\frac{\partial p(x,i,t)}{\partial t} = -b(i)\frac{\partial}{\partial x}p(x,i,t) + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} p(x-z,j,t)D_{ji}(dz) + \int_{R^+}\left(p(x-y,i,t) - p(x,i,t) + I_{y\in(0,1)}y\frac{\partial}{\partial x}p(x,i,t)\right)\nu(i,dy). \quad (8)$$

For (8), we do Laplace transform of $p(x, i, t)$ w.r.t. t for both sides,

$$\omega p^L(x,i,\omega) - p(x,i,0) = -b(i)\frac{\partial p^L(x,i,\omega)}{\partial x} + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} p^L(x-z,j,\omega)D_{ji}(dz) + \int_{R^+}\left(p^L(x-y,i,\omega) - p^L(x,i,\omega) + I_{y\in(0,1)}y\frac{\partial p^L(x,i,\omega)}{\partial x}\right)\nu(i,dy). \quad (9)$$

For (9), we do Laplace transform of $p^L(x, i, \omega)$ w.r.t. x for both sides, then

$$\omega p^{LL}(u,i,\omega) - I_{i=0} = -b(i)up^{LL}(u,i,\omega) + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} e^{-uz}p^{LL}(u,j,\omega)D_{ji}(dz) + \int_{R^+}(e^{-uy}p^{LL}(u,i,\omega) - p^{LL}(u,i,\omega) + I_{y\in(0,1)}yup^{LL}(u,i,\omega))\nu(i,dy).$$

Let $b^*(i) = b(i) - \int_{0<y<1} y\nu(i,dy)$,

$$\omega p^{LL}(u,i,\omega) - I_{i=0} = \left(-b^*(i)u + \int_{R^+}(e^{-uy}-1)\nu(i,dy)\right)p^{LL}(u,i,\omega) + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} e^{-uz}D_{ji}(dz)p^{LL}(u,j,\omega).$$

Let $d_{ji}^L(u) = \int_{R^+} e^{-uz}D_{ji}(dz)$, and then the matrix form is

$$\mathbf{p}^{LL}(u,\omega)[\mathbf{A}-\mathbf{B}] = [1,0,\dots,0],$$

where $\mathbf{p}^{LL}(u,\omega) = [p^{LL}(u,0,\omega), p^{LL}(u,1,\omega), \dots, p^{LL}(u,n,\omega)]$, $\mathbf{A}$ is a diagonal matrix with diagonal entries $\omega + b^*(i)u - \int_{R^+}(e^{-uy}-1)\nu(i,dy)$, and $\mathbf{B} = [r_{ij}d_{ij}^L], i,j\in\mathcal{E}$.

We have

$$R^{LL}(u,\omega) = u^{-1}[1,0,\dots,0][\mathbf{A}-\mathbf{B}]^{-1}[1,1,\dots,1]^T.$$

■

Remark 1 For (8), we do Laplace transform of $p(x, i, t)$ w.r.t. x for both sides,

$$p^L(u,i,t) = \int_{R^+} e^{-ux}p(x,i,t)dx, \quad u>0,$$

then

$$\frac{\partial p^L(u,i,t)}{\partial t} = -b(i)up^L(u,i,t) + \sum_{j\in\mathcal{E}} r_{ji}\int_{R^+} e^{-uz}p^L(u,j,t)D_{ji}(dz) + \int_{R^+}(e^{-uy}p^L(u,i,t) - p^L(u,i,t) + I_{y\in(0,1)}yup^L(u,i,t))\nu(i,dy)$$
$$= \left(-b^*(i)u + \int_{R^+}(e^{-uy}-1)\nu(i,dy)\right)p^L(u,i,t) + \sum_{j\in\mathcal{E}} r_{ji}d_{ji}^L p^L(u,j,t).$$

Solving this ordinary differential equation, we have the solution in the matrix form:

$$\mathbf{p}^L(u,t) = [1,0,\dots,0]\exp\{t[\mathbf{B}-\mathbf{A_0}]\},$$

where $\mathbf{A_0}$ is a diagonal matrix with diagonal entries $b^*(i)u - \int_{R^+}(e^{-uy}-1)\nu(i,dy), i\in\mathcal{E}$.

We use Theorem 3 to derive the Laplace expression for the moments of lifetime T_x as Theorem 4.

Theorem 4 *For a degradation process under the dynamic environment that is described by the Markov additive process* $\{X(t), E(t)\}$ *in Section* 2.2, *the Laplace expression of lifetime moments is*

$$M^L(T^n,u) = n!u^{-1}[1,0,\dots,0][\mathbf{A_0}-\mathbf{B}]^{-n}[1,1,\dots,1]^T,$$

where $\mathbf{A_0}$ *is a diagonal matrix with diagonal entries* $b^*(i)u - \int_{R^+}(e^{-uy}-1)\nu(i,dy)$, *and* $\mathbf{B} = [r_{ij}d_{ij}^L], i,j\in\mathcal{E}$. *In addition,* $b^*(i)\geq 0$, ν *is the Lévy measure,* r_{ij}, $i, j\in\mathcal{E}$ *are entries of the infinitesimal generator matrix of* $E(t)$, $d_{ji}^L(u) = \int_{R^+} e^{-uz}D_{ji}(dz)$, $[1, 0, \dots, 0]$ *is a vector of size* $n+1$, *where the first element is* 1 *and all others are* 0, *and* $[1, 1, \dots, 1]$ *is a vector of size* $n+1$, *where all the elements are* 1.

Proof We have $P(T_x < t) \equiv \widetilde{P}(x,t) = 1 - R(x,t)$. Then $\widetilde{P}(dt,x) = -R(x,dt)$. The Laplace transform of $\widetilde{P}(dt,x)$ w.r.t t is

$$\widetilde{p}^L(x,\omega) = -\omega R^L(x,\omega) + u(x), \quad (10)$$

where $u(x)$ is the unit step function. For (10), we do Laplace transform w.r.t. x for both sides, then

$$\widetilde{p}^{LL}(u,\omega) = -\omega R^{LL}(u,\omega) + u^{-1}.$$

From Theorem 3, we have

$$\widetilde{p}^{LL}(u,\omega) = -\omega u^{-1}[1,0,\dots,0][\mathbf{A}-\mathbf{B}]^{-1}[1,1,\dots,1]^T + u^{-1}.$$

We denote

$$\widetilde{p}_n^{LL}(u,\omega) = (-1)^n \frac{\partial^n \widetilde{p}^{LL}(u,\omega)}{\partial \omega^n},$$

and then the Laplace expression of lifetime moments is

$$M^L(T^n,u) = \widetilde{p}_n^{LL}(u,0) = (-1)^n \left[\frac{\partial^n \widetilde{p}^{LL}(u,\omega)}{\partial \omega^n}\right]_{\omega=0},$$

where

$$\left[\frac{\partial^n \widetilde{p}^{LL}(u,\omega)}{\partial \omega^n}\right]_{\omega=0} = -u^{-1} n \left[\frac{\partial^{n-1}\{[1,0,\ldots,0][\mathbf{A}-\mathbf{B}]^{-1}[1,1,\ldots,1]^T\}}{\partial \omega^{n-1}}\right]_{\omega=0},$$

and

$$\left[\frac{\partial^{n-1}\{[1,0,\ldots,0][\mathbf{A}-\mathbf{B}]^{-1}[1,1,\ldots,1]^T\}}{\partial \omega^{n-1}}\right]_{\omega=0} = (-1)^{n-1}(n-1)![1,0,\ldots,0][\mathbf{A_0}-\mathbf{B}]^{-n}[1,1,\ldots,1]^T.$$

Therefore, we have

$$M^L(T^n,u) = n!u^{-1}[1,0,\ldots,0][\mathbf{A_0}-\mathbf{B}]^{-n}[1,1,\ldots,1]^T.$$

■

To make comparison, Corollaries 3 and 4 provide the Laplace expressions of reliability function and lifetime moments, respectively, assuming there is no jump in $X(t)$ at the time the state of $E(t)$ changes.

Corollary 3 *For a degradation process under the dynamic environment that is described by the Markov additive process $\{X(t),E(t)\}$, assuming there is no jump in $X(t)$ when the state of $E(t)$ changes, the Laplace expression of reliability function is*

$$R^{LL}(u,\omega) = u^{-1}[1,0,\ldots,0][\mathbf{A}-\mathbf{G}]^{-1}[1,1,\ldots,1]^T,$$

where $\mathbf{A}$ is a diagonal matrix with diagonal entries $\omega + b^*(i)u - \int_{R^+}(e^{-uy}-1)\nu(i,dy), i \in \mathcal{E}$, and $\mathbf{G}$ is the infinitesimal generator matrix of $E(t)$.

Corollary 4 *For a degradation process under the dynamic environment that is described by the Markov additive process $\{X(t),E(t)\}$, assuming there is no jump in $X(t)$ when the state of $E(t)$ changes, the Laplace expression of lifetime moments is*

$$M^L(T^n,u) = n!u^{-1}[1,0,\ldots,0][\mathbf{A_0}-\mathbf{G}]^{-n}[1,1,\ldots,1]^T,$$

where $\mathbf{A_0}$ is a diagonal matrix with diagonal entries $b^*(i)u - \int_{R^+}(e^{-uy}-1)\nu(i,dy), i \in \mathcal{E}$, and $\mathbf{G}$ is the infinitesimal generator matrix of $E(t)$.

5 | NUMERICAL EXAMPLES

To illustrate our models, we consider two cases of $\{X(t),E(t)\}$:

Case 1 There are no jumps in degradation $X(t)$ when the states of environment $E(t)$ changes;

Case 2 There are random jumps in degradation $X(t)$ when the states of environment $E(t)$ changes.

We use a Markov process with two states {0, 1} to model the environment, and its infinitesimal generator matrix is

$$\mathbf{G} = \begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix}.$$

In Case 2, we use a Lévy distribution to model the jumps when the environment switches from state 0 to state 1:

$$D_{01}(dz) = \begin{cases} \frac{\sqrt{\frac{\xi}{2\pi}}\exp\left(-\frac{\xi}{2(z-\varpi)}\right)}{(z-\varpi)^{\frac{3}{2}}} & \textit{for } z > \varpi > 0 \\ 0 & \textit{otherwise}, \end{cases}$$

and then $d_{01}^L(u) = e^{-u\varpi-\sqrt{2u\xi}}$. A gamma distribution is used to model the jumps when the environment switches from state 1 to state 0:

$$D_{10}(dz) = \frac{\beta^\alpha z^{\alpha-1}e^{-\beta z}}{\Gamma(\alpha)}, \quad z > 0,$$

and then $d_{10}^L(u) = \left(\frac{\beta}{\beta+u}\right)^\alpha$.

We use a Lévy measure to model the Lévy degradation under the environment state 0:

$$\nu(0,dy) = \frac{\delta\gamma^{-2\kappa_1}\kappa_1 y^{-\kappa_1-1}\exp\left(-\frac{1}{2}\gamma^2 y\right)}{\Gamma(\kappa_1)\Gamma(1-\kappa_1)}dy,$$

where $y, \delta > 0$, $0 < \kappa_1 < 1$, $\gamma \geq 0$, which represents a positive tempered stable process $PTS(\kappa_1,\delta,\gamma)$ (Barndorff-Nielsen & Shephard, 2012).

We use another Lévy measure to model the Lévy degradation under the environment state 1:

$$\nu(1,dy) = \frac{\kappa_2}{\Gamma(1-\kappa_2)}\frac{1}{y^{\kappa_2+1}}dy,$$

where $y > 0$, $0 < \kappa_2 < 1$, which represents a positive stable process $PS(\kappa_2)$ (Barndorff-Nielsen & Shephard, 2012). When κ_1, κ_2 are close to 0, the corresponding stable processes propagate with big jumps; when κ_1, κ_2 are close to 1, the stable processes evolve with small jumps. Of note, in the case of stable Lévy process, it can be easily checked that the conditions for Theorem 2 are satisfied.

For Case 1, the Laplace expression of reliability function based on Corollary 3 is

$$R^{LL}(u,\omega)$$
$$= u^{-1}[1,0]\begin{bmatrix} \omega + b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00} & -r_{01} \\ -r_{10} & \omega + b^*(1)u + u^{\kappa_2} - r_{11} \end{bmatrix}^{-1}[1,1]^T$$
$$= u^{-1}\frac{\omega + b^*(1)u + u^{\kappa_2} - r_{11} + r_{01}}{\left(\omega + b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}\right)(\omega + b^*(1)u + u^{\kappa_2} - r_{11}) - r_{01}r_{10}}.$$

The Laplace expression of lifetime moments based on Corollary 4 is

$$M^L(T^n,u) = n!u^{-1}[1,0]\begin{bmatrix} b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00} & -r_{01} \\ -r_{10} & b^*(1)u + u^{\kappa_2} - r_{11} \end{bmatrix}^{-n}[1,1]^T.$$

The first and second moments of lifetime for Case 1 are

$$M^L(T^1,u) = u^{-1}\frac{b^*(1)u+u^{\kappa_2}-r_{11}+r_{01}}{\left(b^*(0)u-\delta\gamma+\delta\left(\gamma^{\frac{1}{\kappa_1}}+2u\right)^{\kappa_1}-r_{00}\right)(b^*(1)u+u^{\kappa_2}-r_{11})-r_{01}r_{10}},$$

$$M^L(T^2,u) = 2u^{-1}\frac{[b^*(1)u+u^{\kappa_2}-r_{11}]^2+r_{01}r_{10}}{\left[\left(b^*(0)u-\delta\gamma+\delta\left(\gamma^{\frac{1}{\kappa_1}}+2u\right)^{\kappa_1}-r_{00}\right)(b^*(1)u+u^{\kappa_2}-r_{11})-r_{01}r_{10}\right]^2} + 2u^{-1}\frac{r_{01}[b^*(1)u+u^{\kappa_2}-r_{11}]+r_{01}\left[b^*(0)u-\delta\gamma+\delta\left(\gamma^{\frac{1}{\kappa_1}}+2u\right)^{\kappa_1}-r_{00}\right]}{\left[\left(b^*(0)u-\delta\gamma+\delta\left(\gamma^{\frac{1}{\kappa_1}}+2u\right)^{\kappa_1}-r_{00}\right)(b^*(1)u+u^{\kappa_2}-r_{11})-r_{01}r_{10}\right]^2}.$$

For Case 2, the Laplace expression of reliability function based on Theorem 3 is

$$R^{LL}(u,\omega)$$
$$= u^{-1}[1,0]\begin{bmatrix} \omega + b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L & -r_{01}d_{01}^L \\ -r_{10}d_{10}^L & \omega + b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L \end{bmatrix}^{-1}[1,1]^T$$
$$= u^{-1}\frac{\omega + b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L + r_{01}d_{01}^L}{\left(\omega + b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L\right)(\omega + b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L) - r_{01}d_{01}^L r_{10}d_{10}^L}.$$

The Laplace expression of lifetime moments based on Theorem 4 is

$$M^L(T^n,u) = n!u^{-1}[1,0]\begin{bmatrix} b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L & -r_{01}d_{01}^L \\ -r_{10}d_{10}^L & b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L \end{bmatrix}^{-n}[1,1]^T.$$

The first and second moments of lifetime for Case 2 are

$$M^L(T^1,u)$$
$$= u^{-1}\frac{b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L + r_{01}d_{01}^L}{\left(b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L\right)(b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L) - r_{01}d_{01}^L r_{10}d_{10}^L},$$

$$M^L(T^2,u)$$
$$= 2u^{-1}\frac{[b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L]^2 + r_{01}d_{01}^L r_{10}d_{10}^L}{\left[\left(b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L\right)(b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L) - r_{01}d_{01}^L r_{10}d_{10}^L\right]^2}$$
$$+ 2u^{-1}\frac{r_{01}d_{01}^L[b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L] + r_{01}d_{01}^L\left[b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L\right]}{\left[\left(b^*(0)u - \delta\gamma + \delta\left(\gamma^{\frac{1}{\kappa_1}} + 2u\right)^{\kappa_1} - r_{00}d_{00}^L\right)(b^*(1)u + u^{\kappa_2} - r_{11}d_{11}^L) - r_{01}d_{01}^L r_{10}d_{10}^L\right]^2}.$$

TABLE 1 Parameter values for the models

x	[0,30]	α	0.2	κ_2	0.9
$r_{00} = -r_{01}$	−10	β	50	$b^*(0)$	0.02
$r_{10} = -r_{11}$	15	δ	0.6	$b^*(1)$	0.01
ξ	0.0001	κ_1	0.8		
ϖ	0.01	γ	0.9		

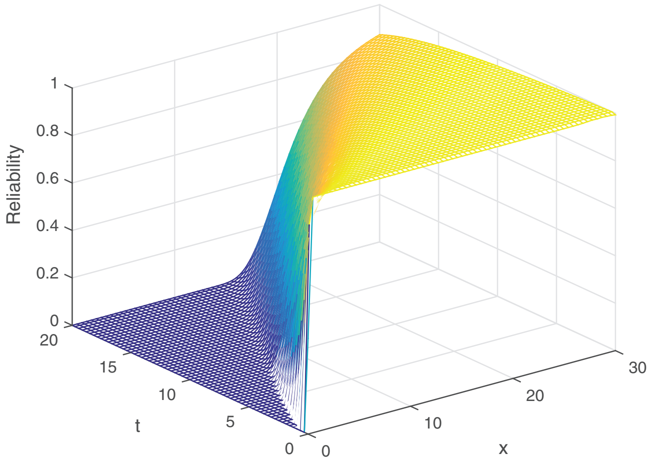

FIGURE 2 Reliability function w.r.t. time t and failure threshold x for Case 1

The system fails when $X(t)$ exceeds the threshold x. The inversion algorithms for Laplace transform (Abate & Whitt, 1995; Brancik, 2007) were implemented to invert the Laplace expressions in Theorems 3, 4 and Corollaries 3, 4 in order to compute the values of reliability and lifetime moments.

The values for the parameters are given in Table 1. The parameters of a Markov additive process can be estimated when a real degradation data set from dynamic environments is available. Since the probability density function of a general Lévy subordinator is not available in a closed-form, the traditional maximum likelihood estimation and Bayesian estimation are not convenient for such general jump processes and their functional extensions. Based on the characteristic function of Lévy subordinator we can use the cumulant M-estimator (CME) (Jongbloed & van der Meulen, 2006) to estimate the parameters. The statistical inference method will be presented in a separate manuscript.

Figures 2 and 3 show the reliability w.r.t. time t and failure threshold x based on general Markov additive processes. For both cases, the reliability decreases as the time increases, and it increases as the threshold increases. Figure 4 shows the reliability w.r.t. time t when $x = 15$ and $x = 20$ for both cases. The reliability in Case 2 decreases faster than that in Case 1 at the same threshold. Figure 5 and Figure 6 illustrate the first moments and the second moments of lifetime with respect to failure threshold x for both cases. Both the first and the second moments of lifetime in Case 2 are less than that in Case 1 at the same threshold. Besides the Lévy measures used in this section, we can specify different Lévy measures to fit the corresponding degradation data, and evaluate their reliability function and lifetime moments.

6 | CONCLUSIONS

In this paper, we developed new systematic procedures to derive powerful and compact results for reliability analysis based on the degradation process under the dynamic environment:

Step 1: Derive the infinitesimal generator of the stochastic process of interests;
Step 2: Derive the adjoint operator corresponding to the infinitesimal generator, based on which the

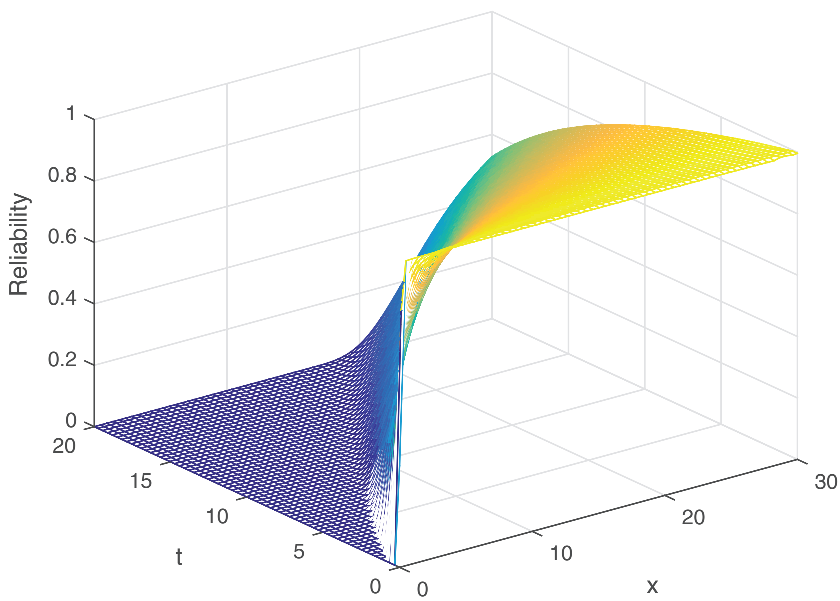

FIGURE 3 Reliability function w.r.t. time t and failure threshold x for Case 2

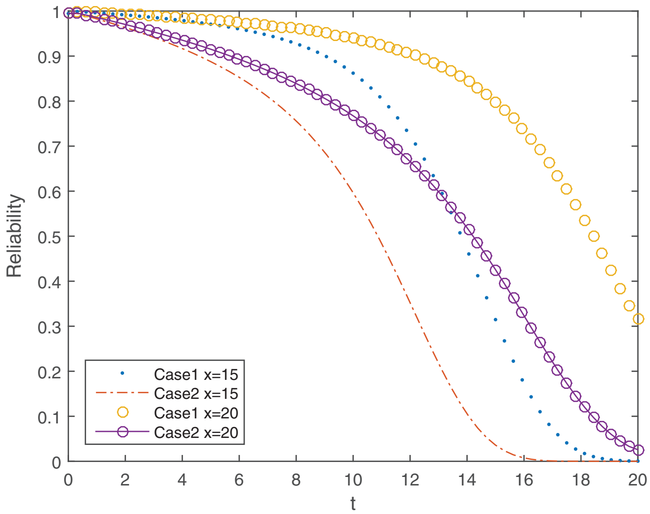

FIGURE 4 Reliability functions w.r.t. time t when $x = 15$ and $x = 20$ for both Case 1 and Case 2

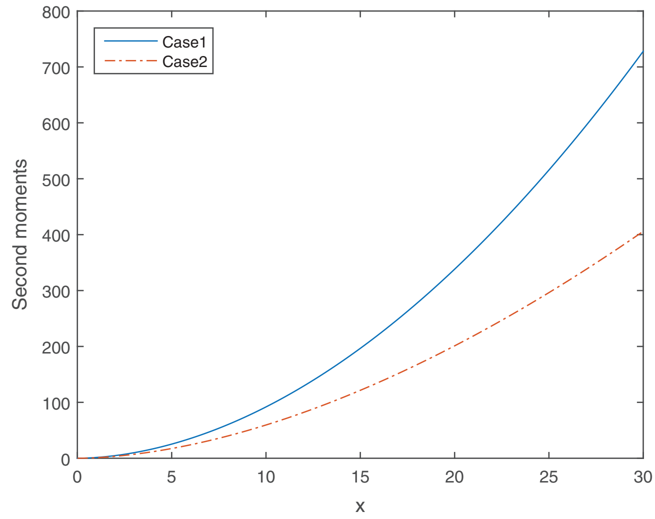

FIGURE 6 Second moments of lifetime w.r.t. failure threshold x for both Case 1 and Case 2

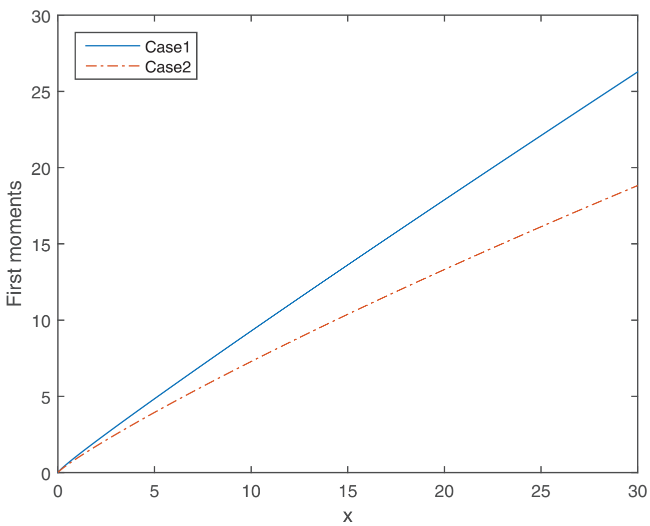

FIGURE 5 First moments of lifetime w.r.t. failure threshold x for both Case 1 and Case 2

Fokker–Planck equation of such stochastic process is developed;

Step 3: Derive the reliability characteristics of the system in terms of Laplace transform.

Our work in this paper is summarized as: (1) we model the degradation process under the dynamic environment using the Markov additive process, while most models in the literature were constructed for the deterministic environment; (2) we use the Lévy subordinator to model the degradation under a certain environment state, and the corresponding Lévy measure can represent different complex jump mechanisms including infinite activities and finite activities in degradation; (3) our models are general to fit more types of degradation data than those based on gamma/Poisson processes; and (4) we derive the Fokker–Planck equation for a class of general Markov additive processes, and obtain the explicit expressions for reliability function and lifetime moments, which provide a new methodology to deal with multiple dependent degradation processes under dynamic environments.

ACKNOWLEDGMENTS

This paper is based upon work supported by the Texas Norman Hackerman Advanced Research Program under Grant No. 003652-0122-2009, and by USA National Science Foundation (NSF) under Grant Nos. 0970140, 1728321 and 0969423.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated during the current study.

ORCID

Qianmei Feng https://orcid.org/0000-0001-5873-8372

REFERENCES

Abate, J., & Whitt, W. (1995). Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal on Computing*, *7*(1), 36–43.

Abdel-Hameed, M. (1984). Life distribution properties of devices subject to a Lévy wear process. *Mathematics of Operations Research*, *9*, 606–614.

Applebaum, D. (2009). Lévy processes and stochastic calculus (2nd ed.). Cambridge University Press.

Asmussen, S. (2003). Applied probability and queues. Springer Science & Business Media.

Barndorff-Nielsen, O. E., & Shephard, N. (2012). Basics of Lévy processes. Unpublished chapter of the authors' book on *Lévy Driven Volatility Models*. Retrieved from https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.309.3802&rep=rep1&type=pdf

Brancik, L. (2007). Numerical inversion of two-dimensional Laplace transforms based on partial inversions. *Proceedings of the 17th International Conference Radioelektronika*, Brno, Czech Republic (pp. 451–454).

Çınlar, E. (1972a). Markov additive processes II. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, *24*, 84–121.

Çınlar, E. (1972b). Markov additive processes I. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, *24*, 85–93.

Çınlar, E. (1977). *Shock and wear models and Markov additive processes*. In I. N. Shimi & C. P. Tsokos (Eds.), The theory and applications of reliability (pp. 193–214). Academic Press.

Çınlar, E. (2011). Probability and stochastics. Springer New York.

Cont, R., & Tankov, P. (2004). Financial modelling with jump processes. Chapman & Hall/CRC.

Ebrahimi, N. (2001). A stochastic covariate failure model for assessing system reliability. *Journal of Applied Probability*, *38*(3), 761–767.

Esary, J. D., Marshall, A. W., & Proshan, F. (1973). Shock models and wear processes. *Annals of Probability*, *1*, 627–649.

Fusai, G., & Kyriakou, I. (2016). General optimized lower and upper bounds for discrete and continuous arithmetic Asian options. *Mathematics of Operations Research*. Published online in Articles in Advance.

Fusaro, R. E., Nielsen, J. P., & Scheike, T. H. (1993). Marker-dependent hazard estimation: An application to AIDS. *Statistics in Medicine*, *12*, 813–865.

Jewell, N. P., & Kalbfleisch, J. D. (1996). Marker processes in survival analysis. *Lifetime Data Analysis*, *2*, 1529.

Jewell, N. P., & Nielsen, J. P. (1993). A framework for consistent prediction rules based on markers. *Biometrika*, *80*(1), 153–164.

Jongbloed, G., & van der Meulen, F. H. (2006). Parametric estimation for subordinators and induced OU processes. *Scandinavian Journal of Statistics*, *33*(4), 825–847.

Kalbfleisch, J. D., & Prentice, R. L. (2002). The statistical analysis of failure time data, (2nd ed.). Wiley & Sons, Inc.

Kharoufeh, J. P. (2003). Explicit results for wear processes in a Markovian environment. *Operations Research Letters*, *31*, 237–244.

Kharoufeh, J. P., & Cox, S. M. (2005). Stochastic models for degradation-based reliability. *IIE Transactions*, *37*(6), 533–542.

Kharoufeh, J. P., Cox, S. M., & Oxley, M. E. (2013). Reliability of manufacturing equipment in complex environments. *Annals of Operations Research*, *209*(1), 231–254.

Kharoufeh, J. P., Finkelstein, D. E., & Mixon, D. G. (2006). Availability of periodically inspected systems with Markovian wear and shocks. *Journal of Applied Probability*, *43*(2), 303–317.

Kharoufeh, J. P., & Mixon, D. G. (2009). On a Markov-modulated shock and wear process. *Naval Research Logistics*, *56*(6), 563–576.

Lawless, J., & Crowder, M. (2004). Covariates and random effects in a gamma process model with application to degradation and failure. *Lifetime Data Analysis*, *10*, 213–227.

Lee, M. T., DeGruttola, V., & Schoenfeld, D. (2000). A model for markers and latent health status. *Journal of the Royal Statistical Society Series B (Statistical Methodology)*, *62*(4), 747–762.

Malani, H. M. (1995). A modification of the redistribution to the right algorithm using disease markers. *Biometrika*, *82*, 515–526.

Risken, H. (1996). The Fokker–Planck equation, (2nd ed.). Berlin, Heidelberg: Springer-Verlag.

Sato, K. (1999). Lévy processes and infinitely divisible distributions. Cambridge University Press.

Shi, M., Taylor, J. M. G., & Munoz, A. (1996). Models for residual time to AIDS. *Lifetime Data Analysis*, *2*, 31–49.

Shu, Y., Feng, Q., & Coit, D. (2015). Life distribution analysis based on Lévy subordinators for degradation with random jumps. *Naval Research Logistics*, *62*(6), 483–492.

Shu, Y., Feng, Q., Kao, E., & Liu, H. (2016). Lévy driven non-Gaussian Ornstein–Uhlenbeck processes for degradation-based reliability analysis. *IIE Transactions*, *48*(11), 993–1003.

Shu, Y., Feng, Q., & Liu, H. (2019). Using degradation-with-jump measures to estimate life characteristics of lithium-ion battery. *Reliability Engineering & System Safety*, *191*, 106515.

Si, X. S., Wang, W. B., Hu, C. H., Chen, M. Y., & Zhou, D. H. (2013). A Wiener-process-based degradation model with a recursive filter algorithm for remaining useful life estimation. *Mechanical Systems and Signal Processing*, *35*, 219–237.

Singpurwalla, N. D. (2006). The hazard potential: Introduction and overview. *Journal of the American Statistical Association*, *101*(476), 1705–1717.

Sun, X., & Duan, J. Q. (2012). Fokker–Planck equations for nonlinear dynamical systems driven by non Gaussian Lévy processes. *Journal of Mathematical Physics*, *53*(7), 1–10.

Tang, J., & Su, T. (2008). Estimating failure time distribution and its parameters based on intermediate data from a Wiener degradation model. *Naval Research Logistics*, *55*, 265–276.

Tsai, C., Tseng, S., & Balakrishnan, N. (2011). Mis-specification analyses of gamma and Wiener degradation processes. *Journal of Statistical Planning and Inference*, *141*, 3725–3735.

Whitmore, G. A., Crowder, M. J., & Lawless, J. F. (1998). Failure inference from a marker process based on a bivariate Wiener model. *Lifetime Data Analysis*, *4*, 229–251.

Yashin, A. I., & Manton, K. G. (1997). Effects of unobserved and partially observed covariate processes on system failure: A review of models and estimation strategies. *Statistical Science*, *12*, 20–34.

How to cite this article: Shu Y, Feng Q, Kao EPC, Coit DW, Liu H. Markov additive processes for degradation with jumps under dynamic environments. *Naval Research Logistics* 2021;68:908–919. https://doi.org/10.1002/nav.21982